Non-crossing partitions and free cumulants

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Basic definitions

Denote by $\mathcal{P}(n)$ the set of partitions of $[n] := \{1, \dots, n\}$. For $\pi \in \mathcal{P}(n)$, define the relation $\sim_\pi$ by: $p \sim_\pi q$ if p, q belong to the same block.

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We call a partition $\pi \in \mathcal{P}(n)$ **crossing** if there exist $p_1 < q_1 < p_2 < q_2$ in $[n]$ such that $p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2$.

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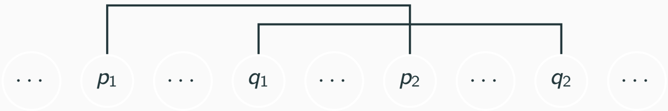

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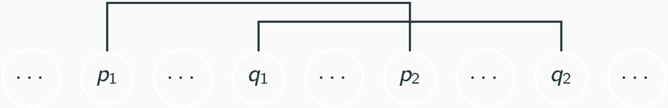
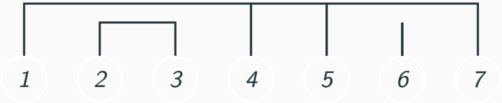


The set of all **non-crossing** partitions of $[n]$ will be denoted by $NC(n)$.

Examples of non-crossing partitions

Example

The partition $\{\{1,4,5,7\},\{2,3\},\{6\}\}$ *of* $[7]$ *is non-crossing, and it's diagram has the following shape*



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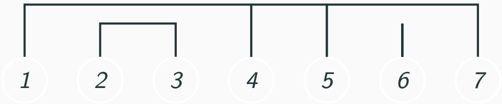

There is an alternative graphical representation (see the draw)...

Examples of non-crossing partitions

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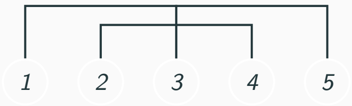
The partition $\{\{1,3,5\},\{2,4\}\}$ *of* $[5]$ *is crossing, and its diagram has the following shape*



Counting $|NC(n)|$

Proposition

For every $n \in \mathbb{N}$, $|NC(n)| = C_n := \frac{1}{n+1}\left(\begin{array}{c} 2n \\ n \end{array} \right)$.

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Proof: Let $D_n := |NC(n)|$ and $D_0 := 1$. It suffices to show that

$$D_n = \sum_{i=1}^{n} D_{i-1} D_{n-i}.$$

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Let $NC^{(i)}(n)$ be the partitions $\pi \in NC(n)$, for which the block containing 1, contains i as its largest element. Then,

$$NC^{(i)}(n) \cong NC^{(i)}(i) \times NC(n-i) \cong NC(i-1) \times NC(n-i)$$

□

Posets and lattices

Let P be a finite partially ordered set (poset). Let $\pi, \sigma \in P$.

1. If the set $\{\tau \in P \mid \tau \geq \pi \text{ and } \tau \geq \sigma\}$ is non-empty and has a unique minimum τ_0, we say that τ_0 is the join of π and τ, denoted $\pi \vee \sigma$.
2. If the set $\{\rho \in P \mid \rho \leq \pi \text{ and } \rho \geq \sigma\}$ is non-empty and has a unique minimum τ_0, we say that ρ_0 is the meet of π and τ, denoted $\pi \vee \sigma$.

If every two elements of P have a meet and a join, we say that P is a lattice.

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Let P be a finite poset. There exists a function $\mu(\cdot,\cdot) : \{(\sigma,\pi) \in P^2 \mid \sigma \leq \pi\}$ such that

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$$f(\pi) = \sum_{\substack{\sigma \in P \\ \sigma \leq \pi}} g(\sigma)$$

for all $\pi \in P$ and

$$g(\pi) = \sum_{\substack{\sigma \in P \\ \sigma \leq \pi}} f(\sigma)\mu(\sigma,\pi)$$

for all $\pi \in P$ are equivalent.

The lattice structure of $NC(n)$

The set $NC(n)$ can be endowed with the "reverse refinement" poset structure, under which $0_n = \{\{1\}, \ldots, \{n\}\}$ and $1_n := \{[n]\}$.

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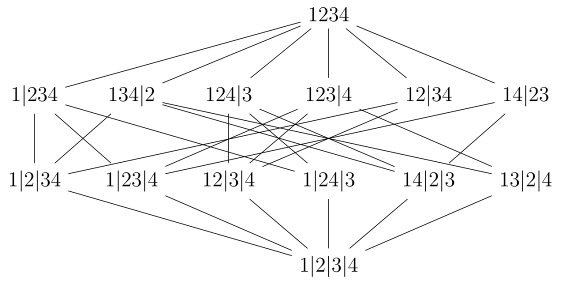
Proof: Since $NC(n)$ has a maximum 1_n, it suffices to show there is a meet $\pi \wedge \sigma$ for $\pi, \sigma \in NC(n)$. Indeed, if $\pi = \{V_1, \ldots, V_r\}$ and $\sigma = \{W_1, \ldots, W_s\}$, then

$$\pi \wedge \sigma := \{V_i \cap W_j \mid i \in [r], j \in [s],\ V_i \cap W_j \neq \emptyset\}$$

defines the largest partition in $NC(n)$, which is smaller than π and σ. $\square$

The lattice structure of $NC(n)$

The following picture shows the Hasse diagram of $NC(4)$.



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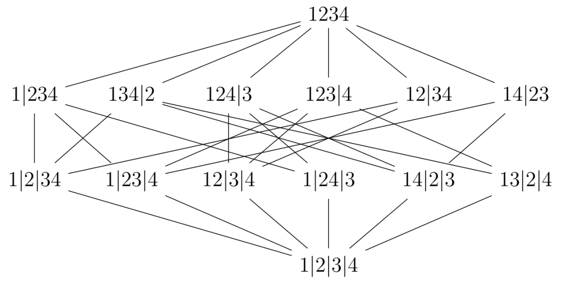

Notice the symmetry on the number of partitions of a given rank.

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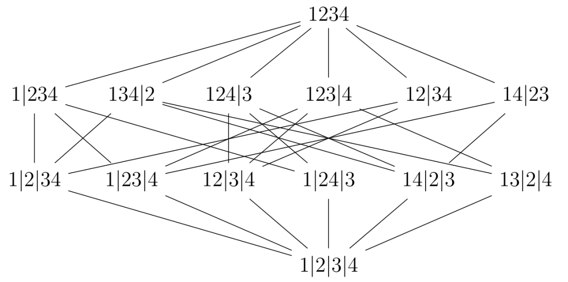

Notice the symmetry on the number of partitions of a given rank. This property reflects the fact that $NC(n)$ is actually **self-dual**. This property doesn't hold for the set of partitions $\mathcal{P}(n)$.

Kreweras complementation

Here we show a particular anti-isomorphism from $NC(n)$ to itself

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Definition
The complementation map $K : NC(n) \to NC(n)$ is defined as follows.
We consider additional numbers $\overline{1}, \dots, \overline{n}$ and interlace them with $1, \dots, n$ in the following alternating way

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Let π be a non-crossing partition of $\{1, \dots, n\}$. Then its **Kreweras complement** $K(\pi) \in NC(\overline{1}, \overline{2}, \dots, \overline{n}) \cong NC(n)$ is defined to be the biggest element among those $\sigma \in NC(\overline{1}, \dots, \overline{n})$ which have the property that

$$\pi \cup \sigma \in NC(1, \overline{1}, \dots, n, \overline{n}).$$

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Example
Consider the partition $\pi := 127|3|46|5|8 \in NC(8)$.

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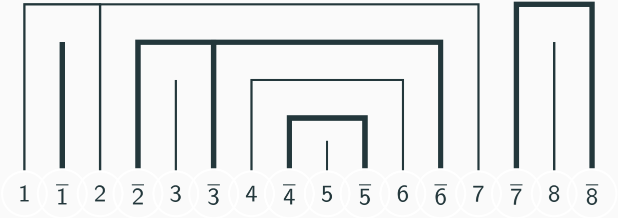
Consider the partition $\pi := 127|3|46|5|8 \in NC(8)$. *For the complement* $K(\pi)$ *we get* $K(\pi) = 1|236|45|78$, *as can be seen from the graphical representation:*



The factorization of intervals in $NC(n)$

Theorem
For any $\pi, \sigma \in NC(n)$ with $\pi \leq \sigma$, there exists a "canonical" sequence $(k_1, \ldots, k_n)$ of non-negative integers such that we have the lattice-decomposition

$$[\pi, \sigma] \cong NC(1)^{k_1} \times NC(2)^{k_2} \times \cdots \times NC(n)^{k_n}.$$

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Proof The proof actually looks like an algorithm. We clearly have

$$[\pi, \sigma] \cong \prod_{V \in \sigma} [\pi|_V, \sigma|_V].$$

How the factorization works:

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Since each $[0_k|_W, K(\tau)|_W] = NC(W) \cong NC(|W|)$, and hence $[\tau, 1_k]$ is anti-isomorphic to $\prod_{W \in K(\tau)} NC(|W|)$. The latter is anti-isomorphic to itself, and we get the desired result. □

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Moreover,

Proposition
For every $n \geq 1$, $\mu_n(0_n, 1_n)$ is a signed Catalan number

$$\mu_n(0_n, 1_n) = (-1)^{n-1} C_{n-1}.$$

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If the variables X, Y are non-commutative random objects, we have more options for deciding what is a "reasonable" notion of independence, i.e. what to put in the right hand side for the equation

$$\mathbb{E}\left[X^2 Y X Y^7 X^6 Y^{10}\right] = \underline{\hspace{3cm}}$$

Non-commutative Probability Spaces

Definition
A non-commutative ***-probability space** is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital *-algebra (namely, a vector space with a multiplication and an involution $a \mapsto a^*$) over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$. An element $a \in \mathcal{A}$ is called a (non-commutative) random variable.

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We define the (algebraic) **distribution** of $a_1, \dots, a_n$, as the linear functional $\mu_{a_1,\dots,a_n} : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C} \to$ given by

$$\mu_{a_1,\dots,a_n}(X_{i_1}^{m_1} \dots X_{i_k}^{m_k}) := \varphi(a_{i_1}^{m_1} \dots a_{i_k}^{m_k}).$$

Definition
Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and I be an index set. Let, for each $i \in I$, $\mathcal{A}_i \subset \mathcal{A}$ be a unital algebra. The subalgebras $\{\mathcal{A}_i\}_{i \in I}$ are **freely independent**, if

$$\varphi(a_1 \cdots a_k) = 0,$$

whenever we have the following

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2. $\varphi(a_j) = 0$ for all $j = 1, \ldots, k$.
3. and neighboring elements are from different subalgebras, i.e. $i(1) \neq i(2) \neq \cdots i(k-1) \neq i(k)$.

Partitioned moments

Definition
Define the sequence of linear functionals $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ via $\varphi_n(a_1,\dots,a_n) := \varphi(a_1\cdots a_n)$. If $\pi \in NC(n)$, define as well the **partitioned moments** by the formula

$$\varphi_\pi[a_1,\dots,a_n] := \prod_{V\in\pi} \varphi(V)[a_1,\dots,a_n],$$

where $\varphi(V)[a_1,\dots,a_n]$ is defined by

$$\varphi(V)[a_1,\dots,a_n] := \varphi_n(a_{i_1},\dots,a_{i_s}), \qquad \text{for } V=\{i_1,\dots,i_s\}.$$

Free cumulants

Definition

We define the **free cumulants** $\{\kappa_\pi\}_{\pi \in NC(n)}$, as the linear functionals $\kappa_\pi : \mathcal{A}^n \to \mathbb{C}$, defined by

$$\kappa_\pi[a_1, \dots, a_n] := \sum_{\substack{\sigma \in NC(n) \\ \sigma \le \pi}} \varphi_\sigma[a_1, \dots, a_n]\mu(\sigma, \pi),$$

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or equivalently, by

$$\varphi(a_1\cdots a_n) = \sum_{\sigma\in NC(n)} \kappa_\pi[a_1,\ldots,a_n].$$

We will also use the notation $\kappa_n := \kappa_{1_n}$.

What is so special about free cumulants?

Theorem
Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $\{\kappa_n\}_{n\in\mathbb{N}}$ be the corresponding cumulants. Then the following two statements are equivalent

1. $\{\mathcal{A}_i\}_{i\in I}$ *are freely independent.*
2. *For all $n \geq 2$ and $a_j \in \mathcal{A}_{i(j)}$ with $(j = 1, \ldots, n)$ and $i(1), \ldots i(n) \in I$, we have $\kappa_n(a_1, \ldots, a_n) = 0$ whenever there exist $1 \leq k, l \leq n$ with $i(l) \neq i(k)$.*

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In particular, if a, b are free random variables, then

$$\kappa_n^{a+b} = \kappa_n^a + \kappa_n^b.$$

So we have a very straightforward method for determining the distribution of $a + b$ if they are freely independent!

Conclusion:

Although there is a lot more to say about $NC(n)$ and about free independence, at least, as very rough conclusion of the talk, we observe that cumulants are easy to handle for free random variables, and the moments of free random variables (which, in principle looked considerably hard to describe) can be written in terms of cumulants, provided that we understand well the structure of the lattice $NC(n)$.

Bibliography

Lectures on the combinatorics of free probability (Cambridge U. Press, 2006), pp. 135-194 (mainly pp. 173-194)