Eigenvalue collision for matrix Gaussian processes

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Eigenvalues of GOE

Consider a d-dimensional random matrix X of the form

$$X = \begin{pmatrix} \sqrt{2}\xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,d} \\ \xi_{1,2} & \sqrt{2}\xi_{2,2} & & \xi_{2,d} \\ \vdots & & \ddots & \vdots \\ \xi_{1,d} & \xi_{2,d} & \cdots & \sqrt{2}\xi_{d,d} \end{pmatrix},$$

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where the $\xi_{i,j}$ are real centered i.i.d. Gaussian variables.
Denote by $(\lambda_1, \dots, \lambda_d)$, the vector of ordered eigenvalues of X, and let F be its associated distribution function.

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Question

If the $\xi_{i,j}$'s are Gaussian processes instead of Gaussian variables, when can we guarantee that the probability that the eigenvalues of X are "always" different?

Statement of the problem

For $r \in \mathbb{N}$ fixed, consider i.i.d, real centered Gaussian fields, indexed by $(i,j) \in \mathbb{N}^2$,

$$\{\xi_{i,j}(t)\}_{t\in\mathbb{R}^r}, \qquad \text{and} \qquad \{\eta_{i,j}(t)\}_{t\in\mathbb{R}^r},$$

defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
We will assume that there exists a non-negative definite function $R : \mathbb{R}^2 \to \mathbb{R}$, such that

$$\mathbb{E}[\xi_{i,j}(s)\xi_{p,q}(t)] = \mathbb{E}[\eta_{i,j}(s)\eta_{p,q}(t)] = \delta_{i,p}\delta_{j,q}R(s,t),$$

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Consider, for $\beta \in \{1,2\}$ and $d \in \mathbb{N}$ fixed, the matrix-valued process $X^{\beta} = \{X_{i,j}^{\beta}\}_{i,n \in \mathbb{N}}$, by

$$X_{i,j}^{\beta}(t) = \begin{cases} \xi_{i,j}(t) + \mathbb{1}_{\{\beta=2\}} \mathbf{i} \eta_{i,j}(t) & \text{if} \quad i < j \\ (\mathbb{1}_{\{\beta=1\}}\sqrt{2} + \mathbb{1}_{\{\beta=2\}})\xi_{i,i}(t) + \mathbb{1}_{\{\beta=2\}}\eta_{i,i}(t) & \text{if} \quad i = j \\ \xi_{i,j}(t) - \mathbb{1}_{\{\beta=2\}} \mathbf{i} \eta_{i,j}(t) & \text{if} \quad j < i. \end{cases}$$

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Let A^{β} be a fixed Hermitian deterministic matrix, such that A^{β} has real entries in the case $\beta = 1$, and complex entries in the case $\beta = 2$. We are interested in the ordered eigenvalues $\lambda_1^{\beta}(t) \geq \cdots \geq \lambda_d^{\beta}(t)$ of

$$Y^{\beta}(t) := A^{\beta} + X^{\beta}(t).$$

Statement of the problem

Goal:

For a fixed interval $I \subset \mathbb{R}^r$ of the form $I = [a_1, b_1] \times \cdots \times [a_r, b_r]$, we want to determine necessary and sufficient conditions on X^β, under which the following non-collision property holds

$$\mathbb{P}\left[\lambda_i^\beta(t) = \lambda_j^\beta(t) \text{ for some } t \in I, \text{ and } 1 \leq i < j \leq n\right] = 0.$$

A basic example

The fractional Brownian motion of Hurst parameter $H \in (0,1)$, is a centered Gaussian process $\{B_t\}_{t \geq 0}$ with covariance function

$$R(s,t) := \mathbb{E}[B_t B_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

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- For all $0 < \alpha < H$, the trayectories of B are Hölder continuous of order α.
- If $H \neq \frac{1}{2}$, it is not a martingale, doesn't satisfy the Markov property and its increments are not independent.

Previous work

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Question:

What happens when $H < \frac{1}{2}$?

Hypothesis of the Main Theorem

Assume that the there exist $(H_1, \ldots, H_r) \in (0,1)^r$, and $c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4} > 0$ such that for all $s = (s_1, \ldots, s_r), t = (t_1, \ldots, t_r) \in I$,

$$c_{2,1} \leq \mathbb{E}\left[\xi_{1,1}(t)^2\right],$$

$$c_{2,2}\sum_{j=1}^{r}|s_j - t_j|^{2H_j} \leq \mathbb{E}\left[|\xi_{1,1}(s) - \xi_{1,1}(t)|^2\right] \leq c_{2,3}\sum_{j=1}^{r}|s_j - t_j|^{2H_j},$$

$$c_{2,4}\sum_{j=1}^{r}|s_j - t_j|^{2H_j} \leq Var\left[\xi_{1,1}(t) \mid \xi_{1,1}(s)\right],$$

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Theorem (Jaramillo, Nualart)

For $\beta = 1, 2$, we have the following

(i) *If $Q < \beta + 1$,*

$$\mathbb{P}\left[\lambda_i^{\beta}(t) = \lambda_j^{\beta}(t) \quad \textit{for some} \quad t \in I, \ \textit{and}\ 1 \leq i < j \leq n\right] = 0.$$

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(ii) *If $Q > \beta + 1$,*

$$\mathbb{P}\left[\lambda_i^{\beta}(t) = \lambda_j^{\beta}(t) \ \ \textit{for some} \ \ t \in I, \ \textit{and}\ 1 \le i < j \le n\right] > 0.$$

Main Theorem

Corollary

Suppose that $r = 1$ and the $\xi_{i,j}$'s and $\eta_{i,j}$'s are fractional Brownian motions of Hurst parameter H. Then,

- *If $\frac{1}{1+\beta} < H < 1$, the eigenvalues of Y^{β} don't collide,*
- *If $H < \frac{1}{1+\beta}$, the eigenvalues of Y^{β} collide with positive probability.*

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  Moreover, if either $A^{\beta} = 0$ or the spectrum of A^{β} has cardinality $d - 1$, then for every $T > 0$,

$$\mathbb{P}\left[\lambda_i^{\beta}(t) = \lambda_j^{\beta}(t) \;\text{ for some }\; t \in (0, T),\; \text{and } 1 \leq i, j \leq n\right] = 1.$$

Sketch of the proof

Let $V = \{V_1(t), \ldots, V_n(t)\}_{t \in \mathbb{R}^r}$ be any n-dimensional Gaussian field, whose entries are i.i.d and satisfy the same properties as $\xi_{1,1}$.

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Corollary (Biermé, Lacaux, Xiao)

Let $F \subset \mathbb{R}^n$ be a Borel set. Then, if $\dim_H F$ denotes the Hausdorff dimension of F,

- *If $\dim_H F < n - Q$, the set $V^{-1}(F) \cap I$ is empty with probability one.*

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Let $F \subset \mathbb{R}^n$ be a Borel set. Then, if $\dim_H F$ denotes the Hausdorff dimension of F,

- *If $\dim_H F < n - Q$, the set $V^{-1}(F) \cap I$ is empty with probability one.*
- *If $\dim_H F > n - Q$, then*

$$0 < \mathbb{P}\left[V^{-1}(F) \cap I \neq \emptyset \right].$$

Proof of the main theorem

Let $\mathcal{S}_{deg}(d)$ and $\mathcal{H}_{deg}(d)$ denote the set of degenerate real symmetric matrices and complex Hermitian matrices, respectively.

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and

$$\mathbb{P}\left[\lambda_i^2(t) = \lambda_j^2(t) \ \text{ for some } \ t \in I, \text{ and } 1 \leq i < j \leq n\right] \\ = \mathbb{P}\left[Y^2(t) \in \mathcal{H}_{deg}^d \ \text{ for some } \ t \in I\right]$$

Proof of the main theorem

Define $n_1(d) := \frac{d(d+1)}{2}$ and $n_2(d) := d^2$, and identify the real symmetric matrices and the complex Hermitian matrices with $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively.

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Lemma

There exist $\mathcal{S}^d_{in}, \mathcal{S}^d_{out} \subset \mathbb{R}^{n_1(d)}$ and $\mathcal{H}^d_{in}, \mathcal{H}^d_{out} \subset \mathbb{R}^{n_2(d)}$, satisfying

$$\mathcal{S}^d_{in} \subset \mathcal{S}^d_{deg} \subset \mathcal{S}^d_{out} \qquad \text{and} \qquad \mathcal{H}^d_{in} \subset \mathcal{H}^d_{deg} \subset \mathcal{H}^d_{out},$$

and

- *$\mathcal{S}^d_{in}$ and $\mathcal{H}^d_{in}$ are manifolds of dimensions $n_1(d) - 2$ and $n_2(d) - 3$.*
- *$\mathcal{S}^d_{out}$ is locally, the image of smooth function defined in an open subset of $\mathbb{R}^{n_1(d)-2}$ and $\mathcal{H}^d_{in}$ is locally the image of smooth function defined in an open subset of $\mathbb{R}^{n_2(d)-3}$.*

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