A Random Matrix Approximation for the Non-commutative Fractional Brownian Motion

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Abstract

A functional limit theorem for the empirical measure-valued process of eigenvalues of a matrix fractional Brownian motion is obtained. It is shown that the limiting measure-valued process is the non-commutative fractional Brownian motion recently introduced by Nourdin and Taqqu [7]. Young and Skorohod stochastic integral techniques and fractional calculus are the main tools used.

Key words and phrases: Matrix fractional Brownian motion, free probability, Young integral, fractionl calculus.

1 Introduction and main result

Motivated by the fact that there is often a close correspondence between classical probability and free probability, in a recent work Nourdin and Taqqu [7] introduced the non-commutative fractional Brownian motion (ncfBm). It appears as the limiting process in a central limit theorem for long range-dependence time series in free probability, in analogy to the classical probability case (see [15], for example). A ncfBm of Hurst parameter $H \in (0,1)$ is a centered semicircular process $S^H = \left\{S_t^H\right\}_{t\geq 0}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$ with covariance function

$$\varphi(S_t^H S_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

The case $S^{1/2}$ is the well known free Brownian motion introduced in [2]. The ncfBm S^H is the only standardized semicircular process which is self-similar and has stationary increments. For the study of the ncfBm and the required free probability framework we refer to Section 2 in [7] or Chapter 8 in [6]. In the present work we will be mainly dealing with the law $(\mu_t^H)_{t\geq 0}$ of a ncfBm instead of the non-commutative process.

Since the seminal paper by Voiculescu [16], it is by now well known that free probability is a convenient framework for investigating limits of spectral distributions of random matrices (see Section 5.4 in the book by Anderson, Guionnet and Zeitouni [1]). On the functional asymptotic behavior side, Biane [2] proved that the free Brownian motion $S^{1/2}$ appears as the measure-valued process limit of $d \times d$ Hermitian matrix Brownian motions with size d going to infinity. Roughly speaking, this result gives a realization of the free Brownian motion $S^{1/2}$ as the spectral limit of well known matrix-valued processes.

On the other hand, for a fixed dimension d, the matrix-valued fractional Brownian was recently studied by Nualart and Pérez-Abreu [12]. It was shown that its corresponding eigenvalue process is non-colliding almost surely and a Skorohod stochastic differential equation governing this process was established.

The main purpose of this paper is to show that the ncfBm S^H has a realization as the measure-valued process limit of $d \times d$ matrix fractional Brownian motions, as the size d goes to infinite. This gives a correspondence between classical fractional Brownian motion and non-commutative fractional Brownian motion. Our methodology uses Skorohod and Young stochastic calculus for a multidimensional fractional Brownian motion as well as fractional calculus. It does not include the case $H = 1/2$ of the free Brownian motion.

More precisely, let us consider a family of independent fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$, $b = \{\{b_{ij}(t), t \geq 0\}, 1 \leq i \leq j \leq d\}$, and define the symmetric matrix fractional Brownian motion $B(t)$ by $B_{ij}(t) = b_{ij}(t)$ for $i < j$, and $B_{ii}(t) = \sqrt{2} b_{ii}(t)$.

As we are interested in functional limit theorems for the eigenvalues of the fractional Brownian motion, we will consider for $n \geq 1$ the following sequence of renormalized process $\{B^{(n)}(t)\}_{t \geq 0}$, given by

$$B^{(n)}(t) = \frac{1}{\sqrt{n}} B(t), \qquad \text{for } t > 0.$$

Following [12], it is possible to apply Itô's formula to the pathwise integral (in the sense of Young or Stratonovich) to obtain the following equation for the eigenvalues of the process $B^{(n)}$

$$\lambda_i^{(n)}(t) = \lambda_i^{(n)}(0) + \frac{1}{\sqrt{n}} \sum_{k \leq h} \int_0^t \frac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s)) \circ db_{kh}(s), \tag{1.2}$$

for any $t > 0$ and $i = 1, \ldots, d$. In equation (1.2) we have $\Phi_i^{(n)} = \lambda_i^{(n)}$ and,

$$\frac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}} = 2 u_{ik}^{(n)} u_{ih}^{(n)} 1_{\{k \neq h\}} + (u_{ik}^{(n)})^2 1_{\{k = h\}}.$$

In the last equation $u_{ik}^{(n)}$ denotes the k-th coordinate of the i-th eigenvector of the matrix $B^{(n)}$.

Now let us define the empirical measure-valued process which will be related to the functional limit theorems we are interested in

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \qquad t \geq 0. \tag{1.3}$$

As it is usual, for a probability measure μ and a μ-integrable function f we use the notation $\langle \mu_t, f \rangle = \int f(x) \mu(\mathrm{d}x)$. Hence, noting that the empirical measure is a point measure we have that for $f \in C_b^2$

$$\langle \mu_t^{(n)}, f \rangle = \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i^{(n)}(t)). \tag{1.4}$$

Therefore, if we apply Itô's formula (for the Stratonovich integral) to the last equation, we obtain

$$\langle \mu_t^{(n)}, f \rangle = \langle \mu_0^{(n)}, f \rangle + \frac{1}{n} \sum_{i=1}^{n} \int_0^t f_i^{\prime(n)}(\Phi^n(b(s)) \circ d\lambda_i^{(n)}(s). \tag{1.5}$$

The main result of this paper is the following functional limit for the empirical spectral measure-valued processes $\{\{\mu_t^{(n)}\}_{t\geq 0} : n \geq 1\}$ converging to the ncfBm. Let $\Pr(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ endowed with the topology of weak convergence and let $C(\mathbb{R}_+, \Pr(\mathbb{R}))$ be the space of continuous functions from $\mathbb{R}_+$ into $\Pr(\mathbb{R})$, endowed with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$.

Theorem 1. *Assume that $\mu_0^{(n)}$ converges weakly to δ_0. Then the family of measure-valued processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges weakly in $C(\mathbb{R}_+, \Pr(\mathbb{R}))$ to the unique continuous probability-measure valued function satisfying, for each $t \geq 0$ $f \in C_b^2(\mathbb{R})$,*

$$\langle \mu_t, f\rangle = f(0) + H \int_0^t ds \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} s^{2H-1} \mu_s(dx)\mu_s(dy). \tag{1.6}$$

Moreover, the family $(\mu_t)_{t\geq 0}$ corresponds to the law of a non-commutative fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$ and covariance (1.1) .

The case $H = 1/2$ of the free Brownian motion is known, see for example [3], [13]. The proof of Theorem 1 is for $H \in (1/2, 1)$ and it is done using Young stochastic integral results as well as fine estimations based on fractional calculus.

The paper is organized as follows. In Section 2, we prove that the family $\{(\mu_t^n)_{t\geq 0} : n \geq 1\}$ is tight in $C(\mathbb{R}_+, \Pr(\mathbb{R}))$. Estimations of the Young integral that appear in (1.5) by means of fractional calculus are first obtained. In Section 3, we prove that the weak limit of the sequence of measure-valued processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ satisfies the measure-valued equation (1.6). It is then proved, in Section 4, that the deterministic process $\{(\mu_t)_{t\geq 0}\}$ satisfying (1.6) corresponds to the law of a non-commutative fractional Brownian motion. For this we first show the stationary increments property by estimating moments of the type $\mathbb{E}(\mathrm{tr}(f(B_{t_2}^{(n_k)} - B_{t_1}^{(n_k)}, \ldots, B_{t_m}^{(n_k)} - B_{t_{m-1}}^{(n_k)}))$, for any function $f : \mathcal{H}_d^m \to \mathcal{H}_d$, where $\mathcal{H}_d$ denotes the space of d-symmetric matrices. The semicircular finite-dimensional distributions are obtained using results in [16].

2 Tightness

In this section, we will prove that the family $\{(\mu_t^n)_{n\geq 0} : n \geq 1\}$ is tight. Using (1.2) we obtain

$$\langle \mu_t^{(n)}, f\rangle = \langle \mu_0^{(n)}, f\rangle + \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{k\leq h} \int_0^t f_i'^{(n)}(\Phi^n(b(s))) \frac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s)) \circ db_{kh}(s).$$

Now using the relation between the Skorohod and Stratonovich integrals (see Theorem 2.3.8 in [5]), we obtain the following equation

$$\langle \mu_t^{(n)}, f\rangle = \langle \mu_0^{(n)}, f\rangle + \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{k\leq h} \int_0^t f_i'^{(n)}(\Phi^n(b(s))) \frac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s)) \delta b_{kh}(s)$$

$$
\begin{aligned}
&+\frac{\alpha_H}{n^2}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t\int_0^t D_r\left(f_i'^{(n)}(\Phi^n(b(s)))\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\right)|s-r|^{2H-2}drds\\
&=\langle\mu_0^{(n)},f\rangle+\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t f_i'^{(n)}(\Phi^n(b(s)))\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\delta b_{kh}(s)\\
&+\frac{H}{n^2}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t f_i''^{(n)}(\Phi^n(b(s)))\left(\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\right)^2 s^{2H-1}ds\\
&+\frac{H}{n^2}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t f_i'^{(n)}(\Phi^n(b(s)))\frac{\partial^2\Phi_i^{(n)}}{\partial (b_{kh}^{(n)})^2}(b(s))s^{2H-1}ds\\
&=\langle\mu_0^{(n)},f\rangle+\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t f_i'^{(n)}(\Phi^n(b(s)))\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\delta b_{kh}(s)\\
&+\frac{H}{n^2}\sum_{i=1}^{n}\int_0^t f_i''^{(n)}(\Phi^n(b(s)))s^{2H-1}ds+\frac{2H}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}\int_0^t\frac{f_i'^{(n)}(\Phi^n(b(s)))}{\lambda_i^{(n)}(s)-\lambda_j^{(n)}(s)}s^{2H-1}ds\\
&=\langle\mu_0^{(n)},f\rangle+\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t f_i'^{(n)}(\Phi^n(b(s)))\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\delta b_{kh}(s)\\
&+H\int_0^t\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}s^{2H-1}\mu_s^{(n)}(dx)\mu_s^{(n)}(dy)ds. \qquad (2.7)
\end{aligned}
$$

Where in the third equality we used the identity,

$$
\sum_{k\leq h}\frac{\partial^2\Phi_i^{(n)}}{\partial(b_{kh}^{(n)})^2}=2\sum_{j\neq i}\frac{1}{\lambda_i^{(n)}(s)-\lambda_j^{(n)}(s)}.
$$

see for instance page 15 in [12], for a proper reference.

We first make some estimations of the Young integral that appears in (1.5) by means of fractional calculus.

Lemma 1. *For any $T>0$ and any $f:\mathbb{R}\to\mathbb{R}$ such that f' and f'' are bounded, we have that there exists $\alpha<1$ and $\beta\in(1/2,H)$, with $\beta-\alpha+1>0$ such that*

$$
\begin{aligned}
&\left|\frac{1}{n}\sum_{j=1}^{n}\int_0^t f'(\lambda_i^{(n)}(u))\circ d\lambda_i^{(n)}(u)-\frac{1}{n}\sum_{j=1}^{n}\int_0^s f'(\lambda_i^{(n)}(u))\circ d\lambda_i^{(n)}(u)\right|\\
&\quad\leq\frac{G\|f''\|_\infty}{\Gamma(\alpha)}C_{\alpha,\beta}(2T)^{\beta+\alpha-1}\left(\|f'\|_\infty+\frac{G\|f''\|_\infty}{(\beta-\alpha)(\beta-\alpha+1)}\right)(t-s)^{\beta-\alpha+1},\quad 0<s\leq t\leq T,
\end{aligned}
\qquad (2.8)
$$

where G is a random variable with moments of any order, and $C_{\alpha,\beta}:=\left(1+\frac{(1-\alpha)}{(\beta+1)}\right)$.

Proof. Let us consider a function $f:\mathbb{R}\to\mathbb{R}$ such that f',f'' are bounded. In this case, by using inequality (4.11) in [9], we have, for $s < t$,

$$
\begin{aligned}
&\left|\sum_{j=1}^{n}\int_0^t f'(\lambda_i^{(n)}(u))\circ d\lambda_i^{(n)}(u) - \sum_{j=1}^{n}\int_0^s f'(\lambda_i^{(n)}(u))\circ d\lambda_i^{(n)}(u)\right| \\
&\leq \sum_{j=1}^{n}\left|\int_0^t f'(\lambda_i^{(n)}(u))\circ d\lambda_i^{(n)}(u) - \int_0^s f'(\lambda_i^{(n)}(u))\circ d\lambda_i^{(n)}(u)\right| \\
&\leq \sum_{j=1}^{n}\Lambda_\alpha(\lambda_i^{(n)})\int_s^t \left(\frac{|f'(\lambda_i^{(n)}(r))|}{(r-s)^\alpha}dr + \alpha\int_s^r \frac{|f'(\lambda_i^{(n)}(r)) - f'(\lambda_i^{(n)}(y))|}{(r-y)^{\alpha+1}}dy\right)dr, \qquad (2.9)
\end{aligned}
$$

where $\Lambda_\alpha(\lambda_i^{(n)}) := \dfrac{1}{\Gamma(1-\alpha)}\sup_{0<s<t<T}|(D_{t-}^{1-\alpha}\lambda_i^{(n)}(t-))(s)|$.

Now let us work with each of the integral terms in expression (2.9). Using the fact that f' is bounded then we have, for all $i = 1, \ldots, n$ and $\alpha < 1$,

$$
\int_s^t \frac{|f'(\lambda_i^{(n)}(r))|}{(r-s)^\alpha}dr \leq \|f'\|_\infty \int_s^t \frac{1}{(r-s)^\alpha}dr = \|f'\|_\infty (r-s)^{1-\alpha}\Big|_s^t = \|f'\|_\infty (t-s)^{1-\alpha}. \qquad (2.10)
$$

On the other hand by identity (5.1) in [12], we know that $\lambda_i^{(n)}$ is Hölder continuous of order $\beta < H$, again using that the function f'' is bounded and applying the Mean Value Theorem, we deduce

$$
|f'(\lambda_i^{(n)}(r)) - f'(\lambda_i^{(n)}(s))| \leq \|f''\|_\infty |\lambda_i^{(n)}(r) - \lambda_i^{(n)}(s)| \leq \|f''\|_\infty G|r-s|^\beta,
$$

where G is a random variable with moments of all orders. Hence by the previous inequality, for $\beta > \alpha$, we have

$$
\begin{aligned}
\int_s^t\int_s^r \frac{|f'(\lambda_i^{(n)}(r)) - f'(\lambda_i^{(n)}(y))|}{(r-y)^{\alpha+1}}dydr &\leq G\|f''\|_\infty \int_s^t\int_s^r |r-y|^{\beta-\alpha-1}dydr \\
&= \frac{G\|f''\|_\infty}{\beta-\alpha}\int_s^t (r-s)^{\beta-\alpha}dr \\
&= \frac{G\|f''\|_\infty}{(\beta-\alpha)(\beta-\alpha+1)}(t-s)^{\beta-\alpha+1}. \qquad (2.11)
\end{aligned}
$$

Finally for $s < t$ and $\beta > 1-\alpha$, we obtain the following inequality

$$
\begin{aligned}
|(D_{t-}^{1-\alpha}\lambda_i^{(n)}(t-))(s)| &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{|\lambda_i^{(n)}(s) - \lambda_i^{(n)}(t)|}{(t-s)^{1-\alpha}} + (1-\alpha)\int_s^t \frac{|\lambda_i^{(n)}(s) - \lambda_i^{(n)}(y)|}{(t-s)^{2-\alpha}}dy\right) \\
&\leq \frac{G\|f''\|_\infty}{\Gamma(\alpha)}\left(|t-s|^{\beta+\alpha-1} + \frac{(1-\alpha)}{(\beta+1)}|t-s|^{\beta+\alpha-1}\right).
\end{aligned}
$$

Therefore, we obtain

$$
\Lambda_\alpha(\lambda_i^{(n)}) = \sup_{0<s<t<T}|(D_{t-}^{1-\alpha}\lambda_i^{(n)}(t-))(s)| \leq \frac{G\|f''\|_\infty}{\Gamma(\alpha)}\left(1 + \frac{(1-\alpha)}{(\beta+1)}\right)(2T)^{\beta+\alpha-1}. \qquad (2.12)
$$

Hence using that $H > 1/2$, we can find $\alpha < 1$ and $\beta > 1/2$ such that $\beta > \alpha$ and $\beta > 1-\alpha$. Finally, putting the inequalities (2.10), (2.11), and (2.12) together, we deduce the result. $\square$

With the estimates given in the previous lemma we are ready to prove tightness, and hence we have the following result

Theorem 2. *The family of measures* $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ *is tight.*

Proof. Following the ideas in [14], it is easily seen using (2.8) in Lemma 1 that for every $0 \leq t_1 \leq t_2 \leq T$, $n \geq 1$ and $f \in \mathcal{C}_b^2$,

$$\begin{aligned}
\mathbb{E}&\left(\left|\langle \mu_{t_1}^{(n)}, f\rangle - \langle \mu_{t_2}^{(n)}, f\rangle\right|^4\right) \\
&= \mathbb{E}\left(\left|\sum_{j=1}^{n}\int_0^t f'(\lambda_i^{(n)}(u)) \circ d\lambda_i^{(n)}(u) - \sum_{j=1}^{n}\int_0^s f'(\lambda_i^{(n)}(u)) \circ d\lambda_i^{(n)}(u)\right|^4\right) \\
&\leq \left(K_1(T,f)\mathbb{E}(G^4) + K_2(T,f)\mathbb{E}(G^8)\right)|t_2 - t_1|^{4(\beta-\alpha+1)},
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
K_1(T,f) &= \left(\frac{\|f''\|_\infty \|f'\|_\infty}{\Gamma(\alpha)}\right)^4 C_{\alpha,\beta}(2T)^{4(\beta+\alpha-1)} \quad \text{and,} \\
K_2(T,f) &= \left(\frac{\|f''\|_\infty^2}{\Gamma(\alpha)(\beta-\alpha)(\beta-\alpha+1)}\right)^4 C_{\alpha,\beta}(2T)^{4(\beta+\alpha-1)}.
\end{aligned} \tag{2.14}$$

Therefore, by the well known criterion that appears in [4] (see pp. 107), we have that the sequence of continuous real processes $\{(\langle \mu_t^{(n)}, f\rangle)_{t\geq 0} : n \geq 1\}$ is tight and consequently the sequence of processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ is tight in the space $\mathcal{C}(\mathbb{R}_+, \mathrm{Pr}(\mathbb{R}))$. □

3 Weak convergence of the empirical measure of eigenvalues

In the previous section, we proved that the family of measures $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ is tight. Now we will proceed to identify the limit of any subsequence of the family, to this end we first need to prove that

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{j=1}^{d}\sum_{k\leq h}\int_0^t f'(\Phi_i(b(s)))\frac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\delta b_{kh}(s) = 0,$$

almost surely.

Following the ideas of Nualart et al. [10], the following result provides a $\mathbb{L}_p$ estimate for the supremum of the multidimensional Skorohod integral, although the steps follow verbatim from Theorem 4 in [10], for the sake of completeness we provide a proof.

Lemma 2. *For any* $T > 0$*, any* $f : \mathbb{R} \to \mathbb{R}$ *such that* f' *and* f'' *are bounded, and* $p \in (1/H, 2)$ *we have that*

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \sup_{0\leq t\leq T}\left|\sum_{j=1}^{d}\sum_{k\leq h}\int_0^t f'(\Phi_i(b(s)))\frac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\delta b_{kh}(s)\right|^p = 0, \qquad \mathbb{P}\text{-}a.s. \tag{3.15}$$

Proof. Let us use the following notation for the Skorohod integral with respect to the multi-dimensional fractional Brownian motion $(b_t)_{t\geq 0}$:

$$\int_0^t g^i(b_s)\delta b_s = \sum_{k\leq h}\int_0^t g^i_{kh}(b_s)\delta b_{kh}(s),$$

where $g^i_{kh}(b_s) := f'(\Phi_i(b(s)))\dfrac{\partial \Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))$, for each $i = 1,\dots,n$.

The following $\mathbb{L}_p$ estimates will be very useful in what follows, so let $u \in \mathbb{L}_n^{2,1}$ then for any $p \geq 1/H$, we have

$$\left|\int_0^T u_s\delta b_s\right|^p \leq c_{p,T,H}\left(\|\mathbb{E}(u)\|^p_{\mathbb{L}^{1/H}([0,T])} + \mathbb{E}\|Du\|^p_{\mathbb{L}^{1/H}([0,T])}\right). \tag{3.16}$$

This result is a consequence of Meyer's inequalities and appears for the one dimensional case in identity (5.40) of [8] and for the multidimensional case when $H = 1/2$ in Proposition 3.5 of [11].

Thus let $p \in (1/H, 2)$, and set $\alpha > 0$ such that $1 - H < \alpha < 1 - 1/p$, then using the equality $c_\alpha = \int_r^t (t-\theta)^{-\alpha}(\theta - r)^{\alpha-1}d\theta$, we can write

$$\int_0^t g^i(b_s)\delta b_s = c_\alpha^{-1}\int_0^t g^i(b_s)\left(\int_s^t (t-r)^{-\alpha}(r-s)^{\alpha-1}dr\right)\delta b_s.$$

Using Fubini's stochastic theorem (see for instance [8]), we have

$$\int_0^t g^i(b_s)\delta b_s = c_\alpha^{-1}\int_0^t (t-r)^{-\alpha}\left(\int_0^r g^i(b_s)(r-s)^{\alpha-1}\delta b_s\right)dr.$$

Hölder's inequality and condition $\alpha < 1 - 1/p$ yields

$$\left|\int_0^t g^i(b_s)\delta b_s\right|^p \leq c_{\alpha,p}\int_0^T\left|\int_0^r g^i(b_s)(r-s)^{\alpha-1}\delta b_s\right|^p dr,$$

where it follows that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t g^i(b_s)\delta b_s\right|^p\right) \leq c_{\alpha,p}\mathbb{E}\left(\int_0^T\left|\int_0^r g^i(b_s)(r-s)^{\alpha-1}\delta b_s\right|^p dr\right),$$

Now using inequality (3.16), we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t g^i(b_s)\delta b_s\right|^p\right) \leq C_{\alpha,p}\Bigg\{&\int_0^T\left(\int_0^r |\mathbb{E}(g^i(b_s))|^{1/H}(r-s)^{\frac{\alpha-1}{H}}ds\right)^{pH}dr\\ &+\mathbb{E}\left(\int_0^T\left(\int_0^r\int_0^T (r-s)^{\frac{\alpha-1}{H}}|D_\theta g^i(b_s)|^{1/H}d\theta ds\right)^{pH}dr\right)\Bigg\}.\end{aligned}$$

So using again Hölder's inequality and taking into account that $\alpha > 1 - H$, we deduce

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t g^i(b_s)\delta b_s\right|^p\right) \leq C\left[\int_0^T |\mathbb{E}(g^i(b_s))|^p ds + \mathbb{E}\left(\int_0^T\left(\int_0^T |D_s g^i(b_r)|^{\frac{1}{H}}ds\right)^{pH}dr\right)\right], \tag{3.17}$$

where the constant $C > 0$ depends on p, H, and T.

Now, we proceed to estimate each of the two integrals in the right hand side of (3.17), so recalling the definition of g, it is clear

$$|g^i|^2 = \sum_{k\leq h}\left(f'(\Phi_i(b(s)))\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\right)^2 \leq \|f'\|_\infty^2\sum_{k\leq h}\left(\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\right)^2 \leq 4\|f'\|_\infty^2.$$

Therefore, by Jensen's inequality we get

$$\int_0^T |\mathbb{E}g^i(b_s)|^p ds \leq 2^p\|f'\|_\infty^p T. \tag{3.18}$$

For the second integral in the right hand side of (3.17), we first compute an upper bound for the norm of the Malliavin derivative of g

$$\begin{aligned}
|D_s g^i(b_s)|^2 &= \sum_{k\leq h}(D_s^{kh}g_{kh}^i(b_s))^2 = \sum_{k\leq h}\left(f''(\Phi(b_s))\left(\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\right)^2 + f'(\Phi(b_s))\frac{\partial^2\Phi_i^{(n)}}{\left(\partial b_{kh}^{(n)}\right)^2}(b(s))\right)^2\\
&\leq 4\sum_{k\leq h}\left(f''(\Phi(b_s))^2\left(\frac{\partial\Phi_i^{(n)}}{\partial b_{kh}^{(n)}}(b(s))\right)^4 + f'(\Phi(b_s))^2\left(\frac{\partial^2\Phi_i^{(n)}}{\left(\partial b_{kh}^{(n)}\right)^2}(b(s))\right)^2\right)\\
&\leq 4\left\{16\|f''\|_\infty^2 + \|f'\|_\infty^2\sum_{k\leq h}\left(\frac{\partial^2\Phi_i^{(n)}}{\left(\partial b_{kh}^{(n)}\right)^2}(b(s))\right)^2\right\}.
\end{aligned}\tag{3.19}$$

On the other hand from page 9 in [12] and Jensen's inequality, it follows

$$\begin{aligned}
\sum_{k\leq h}\left(\frac{\partial^2\Phi_i^{(n)}}{\left(\partial b_{kh}^{(n)}\right)^2}(b(s))\right)^2 &= \sum_{k\leq h}\left(2\sum_{j\neq i}\frac{|u_{ik}u_{jh}+u_{jk}u_{ih}|^2}{\lambda_i^{(n)}-\lambda_j^{(n)}}\right)^2\\
&\leq 2^5 n\sum_{k\leq h}\sum_{i\neq j}\frac{|u_{ik}u_{jh}+u_{jk}u_{ih}|^2}{(\lambda_i^{(n)}-\lambda_j^{(n)})^2}\\
&\leq 2^7 n\sum_{i\neq j}(\lambda_i^{(n)}-\lambda_j^{(n)})^{-2}.
\end{aligned}\tag{3.20}$$

Since $pH > 1$ and using Jensen'e inequality, we deduce

$$\mathbb{E}\left(\int_0^T\left(\int_0^T |D_s g^i(b_r)|^{\frac{1}{H}}ds\right)^{pH}dr\right) = T^{pH-1}\int_0^T\int_0^T \mathbb{E}|D_s g(b_r)|^p ds dr.$$

Therefore using (3.19), (3.20), the fact that $p/2 < 1$ and using again Jensen's inequality, we obtain

$$\begin{aligned}
\mathbb{E}|D_s g^i(b_r)|^p &\leq 2^{3/2p}\left\{\|f''\|_\infty^p + \|f'\|_\infty^p 2^{7/2p}n^{p/2}\mathbb{E}\left(\sum_{i\neq j}(\lambda_i^{(n)}-\lambda_j^{(n)})^{-2}\right)^{p/2}\right\}\\
&\leq 2^{3/2p}\left\{\|f''\|_\infty^p + \|f'\|_\infty^p 2^{7/2p}n^{p-1}\left(\sum_{i\neq j}\mathbb{E}(\lambda_i^{(n)}-\lambda_j^{(n)})^{-2}\right)^{p/2}\right\}.
\end{aligned}\tag{3.21}$$

In page 10 of [12], we can find the following estimate for the sum of the $-p$-moments of the differences between the eigenvalues

$$\sum_{i\neq j}\mathbb{E}(\lambda_i^{(n)}-\lambda_j^{(n)})^{-2}\leq s^{-2H}.$$

So using the above estimate in (3.21), it is clear

$$\begin{aligned}\mathbb{E}|D_s g^i(b_r)|^p &\leq 2^{3/2p}\left\{\|f''\|_\infty^p+\|f'\|_\infty^p 2^{7/2p}n^{p-1}s^{-pH}\right\}\\ &:= C_1+C_2n^{p-1}s^{-pH}.\end{aligned}$$

This implies,

$$\begin{aligned}T^{pH-1}\int_0^T\int_0^T\mathbb{E}|D_s g^i(b_r)|^p ds dr &\leq T^{pH-1}\int_0^T s(C_1+C_2n^{p-1}s^{-pH})ds\\ &= C_1(T,p,f)+C_2(T,p,f)n^{p-1}.\end{aligned}$$

Therefore putting all the pieces together, we have

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t g^i(b_s)\delta b_s\right|^p\right)\leq D_1(T,p,f)+C_1(T,p,f)+C_2(T,p,f)n^{p-1},$$

where $D_1(T,p,f),C_1(T,p,f)$, and $C_2(T,p,f)$ are constants that only depend on T,f, and p.

In order to complete proof, we observe by using Jensen's inequality and the fact that $p>1$, the following

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\frac{1}{n^{3/2}}\sum_{i=1}^n\int_0^t g^i(b_s)\delta b_s\right|^p\right) &\leq n^{p-1}n^{-3/2p}\sum_{i=1}^n\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t g^i(b_s)\delta b_s\right|^p\right)\\ &= n^{p-1}n^{-3/2p}(D_1(T,p,f)+C_1(T,p,f)+C_2(T,p,f)n^{p-1}).\end{aligned}$$

Hence for $\varepsilon,T>0$, taking into account that $p\in(1/H,2)$, we obtain

$$\begin{aligned}&\sum_n\mathbb{P}\left(\sup_{0\leq t\leq T}\left|\frac{1}{n^{3/2}}\sum_{i=1}^n\int_0^t g^i(b_s)\delta b_s\right|^p>\varepsilon\right)\\ &\quad\leq\frac{1}{\varepsilon^p}\sum_n\left(n^{p-1}n^{-3/2p}(D_1(T,p,f)+C_1(T,p,f)+C_2(T,p,f)n^{p-1})\right)\\ &\quad\leq K\sum_n(n^{-1/2p-1}+n^{1/2p-2})<\infty,\end{aligned}$$

and therefore,

$$\sup_{0\leq t\leq T}\left|\frac{1}{n^{3/2}}\sum_{i=1}^n\int_0^t g^i(b_s)\delta b_s\right|^p\to 0,\qquad\mathbb{P}\text{-a.s.},$$

as n goes to $+\infty$. □

With the previous results we are ready to show that the weak limit of the sequence of measure-valued processes $\{(\mu_t^{(n)})_{t\geq 0}:n\geq 1\}$ satisfies the measure valued equation (1.6).

Proof. From Theorem 2, we know that the family $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ is relative compact, and therefore by (2.7) and Lemma 2 it is clear that any weak limit $(\mu_t)_{t\geq 0}$ of a subsequence $(\mu_t^{(n_k)})_{t\geq 0}$ should satisfy (1.6). Applying (2.7) to the deterministic sequence of functions

$$f_j(x) = \frac{1}{x - z_j}, \qquad z_j \in (\mathbb{Q} \times \mathbb{Q}) \cap \mathbb{C}^+,$$

and using a continuity argument, we get that the Cauchy-Stieltjes transform $(G_t)_{t\geq 0}$ of $(\mu_t)_{t\geq 0}$ satisfies the integral equation

$$G_t(z) = -\frac{1}{z} + H \int_0^t s^{2H-1} ds \int_{\mathbb{R}^2} \frac{\mu_s(dx)\mu_s(dy)}{(x-z)(y-z)^2}, \qquad t \geq 0, z \in \mathbb{C}^+. \tag{3.22}$$

From (3.22) it is easily seen that G_t is the unique solution to the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t} G_t(z) = H s^{2H-1} G_t(z) \frac{\partial}{\partial z} G_t(z), & t > 0, \\ G_0(z) = -\frac{1}{z}, & z \in \mathbb{C}^+, \end{cases}$$

and consequently is the family of fractional semicircle laws $(\mu_t^{fsc})_{t\geq 0}$. Therefore all limits of subsequences of $(\mu_t^{(n)})_{t\geq 0}$ coincide with $(\mu_t^{fsc})_{t\geq 0}$ and thus the sequence $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges weakly to $(\mu_t^{fsc})_{t\geq 0}$. □

4 Convergence to the non-commutative fBm

In this section, we prove that the deterministic process $(\mu_t)_{t\geq 0}$ corresponds to the law of a non commutative fractional Brownian motion. The intuitive idea is as follows: By tightness of the sequence of processes $\{(\mu_t^{(n)})_{t\geq 0}; n \geq 1\}$ in the space $C(\mathbb{R}_+, \Pr(\mathbb{R}))$, the weak limit $(\mu_t)_{t\geq 0}$ of any subsequence $\{(\mu_t^{(n_k)})_{t\geq 0}; k \geq 1\}$ should satisfy (1.6), for any $t \geq 0$ and $f \in C_b^2$. Therefore by the uniqueness of solutions to equation (3.22), it is easy to check that

$$G_t(z) = \frac{1}{2t^{2H}} \left(\sqrt{z^2 - 4t^{2H}} - z \right), \qquad t \geq 0,\ z \in \mathbb{C}^+.$$

Which is the Cauchy-Stieljes transform of a semi-circle law with variance at time $t > 0$ given by t^{2H}, and hence the law of a non-commutative fractional Brownian motion at time t.

In the following subsections we will prove some results that are needed to formally prove the convergence to a non-commutative fractional Brownian motion.

4.1 Stationarity

Let us recall that the sequence of processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ is tight in $C(\mathbb{R}_+, \Pr(\mathbb{R}))$, this implies that the sequence is relatively compact, in other words, there exists a subsequence $\{(\mu_t^{(n_k)})_{t\geq 0} : k \geq 1\}$ that converges weakly to a process that we denote by $\{(\mu_t)_{t\geq 0}\}$ in $C(\mathbb{R}_+, \Pr(\mathbb{R}))$.

Given the fact that the weak convergence of processes in $C(\mathbb{R}_+, \Pr(\mathbb{R}))$ implies the convergence of the finite-dimensional distributions, then for each bounded and continuous function $g : \mathbb{R}^m \to \mathbb{R}$, and for each sequence of times $0 \leq t_1 < \cdots < t_m$ it follows,

$$\langle \mu_{t_1,\ldots,t_m}^{(n_k)}, g \rangle \xrightarrow{\mathcal{L}} \langle \mu_{t_1,\ldots,t_m}, g \rangle, \qquad \text{as } k \to \infty. \tag{4.23}$$

Let us now consider $(B_t^{n_k})_{t\geq 0}$ the symmetric fractional Brownian matrix, such that the empirical measure of it eigenvalues (see (1.3)) is given by $(\mu_t^{(n_k)})_{t\geq 0}$. Let $\mathcal{H}_d$ denote the space of d-symmetric matrices, then for any function $f:\mathcal{H}_d^m\to\mathcal{H}_d$, we have

$$\begin{aligned}\mathbb{E}\Big(&\mathbf{tr}(f(B_{t_2}^{n_k}-B_{t_1}^{n_k},\ldots,B_{t_m}^{n_k}-B_{t_{m-1}}^{n_k})\Big) \\ &=\mathbb{E}\left(\int_{\mathbb{R}^m} f(x_2-x_1,\ldots,x_m-x_{m-1})\mu_{t_1,t_2,\ldots,t_m}^{(n_k)}(dx_1,dx_2,\ldots,dx_m)\right),\end{aligned}\tag{4.24}$$

where $\mathbf{tr}$ denotes the trace of the matrix $f(B_{t_2}^{n_k}-B_{t_1}^{n_k},\ldots,B_{t_m}^{n_k}-B_{t_{m-1}}^{n_k})$. In a similar way we have

$$\begin{aligned}\mathbb{E}\Big(&\mathbf{tr}(f(B_{t_2-t_1}^{n_k},\ldots,B_{t_m-t_{m-1}}^{n_k})\Big) \\ &=\mathbb{E}\left(\int_{\mathbb{R}^m} f(x_1,\ldots,x_m)\mu_{t_1-t_2,\ldots,t_m-t_{m-1}}^{(n_k)}(dx_1,dx_2,\ldots,dx_m)\right).\end{aligned}\tag{4.25}$$

So using identities (4.24) and (4.25) and the stationarity of the increments of the fractional Brownian motion, we obtain

$$\begin{aligned}\mathbb{E}\Bigg(&\int_{\mathbb{R}^m} f(x_2-x_1,\ldots,x_m-x_{m-1})\mu_{t_1,t_2,\ldots,t_m}^{(n_k)}(dx_1,dx_2,\ldots,dx_m)\Bigg) \\ &=\mathbb{E}\left(\int_{\mathbb{R}^m} f(x_1,\ldots,x_m)\mu_{t_1-t_2,\ldots,t_m-t_{m-1}}^{(n_k)}(dx_1,dx_2,\ldots,dx_m)\right).\end{aligned}\tag{4.26}$$

Therefore, from (4.23), we obtain

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}\Bigg(&\int_{\mathbb{R}^m} f(x_2-x_1,\ldots,x_m-x_{m-1})\mu_{t_1,t_2,\ldots,t_m}^{(n_k)}(dx_1,dx_2,\ldots,dx_m)\Bigg) \\ &=\int_{\mathbb{R}^m} f(x_2-x_1,\ldots,x_m-x_{m-1})\mu_{t_1,t_2,\ldots,t_m}(dx_1,dx_2,\ldots,dx_m),\end{aligned}$$

and also

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}\Bigg(&\int_{\mathbb{R}^m} f(x_1,\ldots,x_m)\mu_{t_1-t_2,\ldots,t_m-t_{m-1}}^{(n_k)}(dx_1,dx_2,\ldots,dx_m)\Bigg) \\ &=\int_{\mathbb{R}^m} f(x_1,\ldots,x_m)\mu_{t_1-t_2,\ldots,t_m-t_{m-1}}(dx_1,dx_2,\ldots,dx_m).\end{aligned}$$

The fact that we do not take expectation in the right hand side of each of the last two terms is due to the fact that the limit is deterministic (since from Lemma 2 the stochastic term given in the Skorohod integral vanishes in the limit). This implies

$$\begin{aligned}&\int_{\mathbb{R}^m} f(x_2-x_1,\ldots,x_m-x_{m-1})\mu_{t_1,t_2,\ldots,t_m}(dx_1,dx_2,\ldots,dx_m) \\ &\qquad=\int_{\mathbb{R}^m} f(x_1,\ldots,x_m)\mu_{t_1-t_2,\ldots,t_m-t_{m-1}}(dx_1,dx_2,\ldots,dx_m).\end{aligned}$$

Hence, we can conclude that the deterministic limit $(\mu_t)_{t\geq 0}$ is stationary.

4.2 Semicircular finite-dimensional distributions

Let us take a set of times $0 \le t_1 < \cdots < t_m$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, then for any measurable and continuous function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}\Big(\mathbf{tr}(f(\lambda_1 B^{n_k}_{t_1} + \ldots + \lambda_m B^{n_k}_{t_m})\Big) \\ = \mathbb{E}\left(\int_{\mathbb{R}^m} f(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_m x_m)\mu^{(n_k)}_{t_1,t_2,\dots,t_m}(dx_1, dx_2, \dots, dx_m)\right).$$

From Theorem 2.2 in [16], we know

$$\lim_{k\to\infty} \mathbb{E}\Big(\mathbf{tr}(f(\lambda_1 B^{n_k}_{t_1} + \ldots + \lambda_m B^{n_k}_{t_m})\Big) \\ = \int_{\mathbb{R}^m} f(\lambda_1 x_1 + \cdots + \lambda_m x_m)\mu_{t_1,t_2,\dots,t_m}(dx_1, dx_2, \dots, dx_m),$$

has a semicircle distribution. Meaning that if we denote by $(X_1, \dots, X_m)$ the random vector with distribution $\mu_{t_1,t_2,\dots,t_m}$ then the random variable $\lambda_1 X_1 + \cdots + \lambda_m X_m$ has a semicircle distribution. Therefore the process $(\mu_t)_{t\ge 0}$ is the law of a semicircular process.

4.3 Non commutative fractional Brownian motion

Now we know that the limit $(\mu_t)_{t\ge 0}$ is the law of a semicircular stationary process, so in order to identify the limit as the law of a non commutative fractional Brownian motion we compute its covariance, therefore for $t \ge s \ge 0$ we obtain using (1.6) (with $f(x) = x^2$) and the stationarity of the laws $(\mu_t)_{t\ge 0}$ that

$$\int_{\mathbb{R}^2} (xy)\mu_{s,t}(dxdy) = \frac{1}{2}\left(\int_{\mathbb{R}^2} x^2 \mu_{s,t}(dxdy) - \int_{\mathbb{R}^2} y^2 \mu_{s,t}(dxdy) - \int_{\mathbb{R}^2} (x-y)^2 \mu_{s,t}(dxdy)\right) \\ = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Therefore, we conclude that $(\mu_t)_{t\ge 0}$ is the law of a non-commutative fractional Brownian motion.

Finally noting that this holds for any subsequence, we can conclude that the whole sequence $\{(\mu^{(n)}_t)_{t\ge 0} : n \ge 1\}$ converges in law to the deterministic process $(\mu_t)_{t\ge 0}$ which is characterized by being the law of a non-commutative fractional Brownian motion.

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