Continuous state branching processes in random environment: The Brownian case.

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Dedicated to the memory of Marc Yor.

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Abstract

Motivated by the works of Böinghoff and Huzenthaler [6] and Bansaye et al. [1], we introduce continuous state branching processes in a Brownian random environment. Roughly speaking, a process in this class behaves as a continuous state branching process but its dynamics are perturbed by an independent Brownian motion with drift. More precisely, we define a continuous state branching process in a Brownian random environment as the unique strong solution of a stochastic differential equation that satisfies a quenched branching property.

In this paper, we are interested in the long-term behaviour of such class of processes. In particular, we provide a necessary condition under which the process is conservative, i.e. that it does not explode a.s. at a finite time, and we also study the event of extinction conditionally on the environment. In the particular case where the branching mechanism is stable, we compute explicitly extinction and explosion probabilities. We show that three regimes arise for the speed of explosion in the case when the scaling index is negative. In the case when the scaling index is positive, the process is conservative and we prove that five regimes arise for the speed of extinction, as for discrete (time and space) branching processes in random environment and branching diffusions in random environment (see [6]). The precise asymptotics for the speed of extinction allow us to introduce the so called process conditioned to be never extinct or Q-process. The proofs of the speed of explosion and extinction are based on the precise asymptotic behaviour of exponential functionals of Brownian motion.

At the end of this paper, we discuss the immigration case which in particular provides an extension of the Cox-Ingersoll-Ross model in a random environment.

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1 Introduction

A continuous-state branching process (or CB-process for short) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\, x \ge 0)$ such that for any $x, y \ge 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$. CB-processes may be thought of as the continuous (in time and space) analogues of classical Bienaymé-Galton-Watson processes. CB-processes have been introduced by Jirina [13] and studied by many authors including Bingham [5], Grey [9], Lamperti [18], to name but a few. More precisely, a CB-process $Y = (Y_t, t \ge 0)$ is a Markov process satisfying the branching property such that 0 and ∞ are two absorbing states. In other words, let

$$T_0 = \inf\{t \ge 0 : Y_t = 0\} \qquad \text{and} \qquad T_\infty = \inf\{t \ge 0 : Y_t = \infty\}$$

denote the extinction and explosion times, respectively, then $Y_t = 0$ for every $t \ge T_0$ and $Y_t = \infty$ for every $t \ge T_\infty$.

The branching property implies that the Laplace transform of Y_t satisfies

$$\mathbb{E}_x\Big[e^{-\lambda Y_t}\Big] = \exp\{-x u_t(\lambda)\}, \qquad \text{for } \lambda \ge 0, \tag{1}$$

for some function $u_t(\lambda)$. According to Silverstein [21], the function $u_t(\lambda) : [0, \infty) \to [0, \infty]$ solves the integral equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\mathrm{d}s = \lambda, \tag{2}$$

where ψ satisfies the celebrated Lévy-Khincthine formula, i.e.

$$\psi(\lambda) = -q - a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\big)\mu(\mathrm{d}x), \tag{3}$$

where $q \ge 0$, $a \in \mathbb{R}$, $\gamma \ge 0$ and μ is a σ-finite measure such that $\int_{(0,\infty)} \big(1 \wedge x^2\big)\mu(\mathrm{d}x)$ is finite. The function $\psi : [0, \infty) \to (-\infty, \infty)$ is convex and is known as the branching mechanism of Y.

Plainly, equation (2) can be solved in terms of ψ, and this readily yields the law of the explosion and extinction times (see Grey [9]). More precisely, let η be the largest root of the branching mechanism ψ, i.e. $\eta = \sup\{\theta \ge 0 : \psi(\theta) = 0\}$. Then for every $x > 0$:

i) if $\eta = \infty$ or if $\int^\infty \mathrm{d}\theta/\psi(\theta) = \infty$, we have $\mathbb{P}_x(T_0 < \infty) = 0$,

ii) if $\eta < \infty$ and $\int^\infty \mathrm{d}\theta/\psi(\theta) < \infty$, we define

$$\phi(t) = \int_t^\infty \frac{\mathrm{d}\theta}{\psi(\theta)}, \qquad t \in (\eta, \infty).$$

The mapping $\phi : (\eta, \infty) \to (0, \infty)$ is bijective, and we write $\varphi : (0, \infty) \to (\eta, \infty)$ for its right-continuous inverse. Thus

$$\mathbb{P}_x(T_0 < t) = \exp\{-x\varphi(t)\},$$

iii) if $\eta = 0$ or if $-\int_{0+} \mathrm{d}\theta/\psi(\theta) = \infty$, we have $\mathbb{P}_x(T_\infty < \infty) = 0$,

iv) if $\eta > 0$ and $-\int_{0+} \mathrm{d}\theta/\psi(\theta) < \infty$, we define

$$g(t) = -\int_0^t \frac{\mathrm{d}\theta}{\psi(\theta)}, \qquad t \in (0, \eta).$$

The mapping $g : (0, \eta) \to (0, \infty)$ is bijective, we write $\gamma : (0, \infty) \to (0, \eta)$ for its right-continuous inverse. Thus

$$\mathbb{P}_x(T_\infty > t) = \exp\{-x\gamma(t)\}.$$

From (ii), we deduce that $\mathbb{P}_x(T_0 < \infty) = \exp\{-x\eta\}$. Hence, the latter identity and (i) imply that a CB-process has a finite time extinction a.s. if and only if

$$\int^\infty \frac{\mathrm{d}u}{\psi(u)} < \infty \qquad \text{and} \qquad \psi'(0+) \geq 0.$$

Similarly from (iv), we get that $\mathbb{P}_x(T_\infty < \infty) = 1 - \exp\{-x\eta\}$. Hence from the latter and (iii), we deduce that a CB-process has a finite time explosion with positive probability if and only if

$$-\int_{0+} \frac{\mathrm{d}u}{\psi(u)} < \infty \qquad \text{and} \qquad \psi'(0+) < 0.$$

The value of $\psi'(0+)$ also determines whether its associated CB-process will, on average, decrease, remain constant or increase. More precisely, under the assumption that $q = 0$, we observe that the first moment of a CB-process can be obtained by differentiating (1) with respect to λ. In particular, we may deduce

$$\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}, \qquad \text{for} \qquad t \geq 0.$$

Hence using the same terminology as for Bienaymé-Galton-Watson processes, in respective order, a CB-process is called *supercritical, critical* or *subcritical* depending on the behaviour of its mean, in other words on whether $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$.

The following two examples are of special interest in this paper since the Laplace exponent, i.e. the solution of (2), can be computed explicitly in a closed form. The first example that we present is the so-called Neveu branching process whose branching mechanism is given by $\psi(\lambda) = \lambda \ln \lambda$, for $\lambda \geq 0$. In this case $\psi'(0+) = -\infty$, $\eta = 1$, the process is supercritical and satisfies the integral conditions of (i) and (iii). Thus, the Neveu branching process does not explode neither become extinct at a finite time a.s. According to Theorem 12.7 in [15], we have that the Neveu branching process becomes extinct at infinity with positive probability. More precisely, if $(Y_t, t \geq 0)$ denotes the Neveu branching process, then

$$\mathbb{P}_x\left(\lim_{t\to\infty} Y_t = 0\right) = e^{-x}, \qquad x > 0.$$

The second example is the stable case with drift, in other words the branching mechanism satisfies $\psi(\lambda) = -a\lambda + c_\beta \lambda^{1+\beta}$, with β in $(-1,0)\cup(0,1]$ and c_β is a non-zero constant with the same sign as β. It is known that its associated CB-process can be obtained by scaling limits of Bienaymé-Galton-Watson processes with a fixed reproduction law. Moreover, the case $\beta = 1$ corresponds to the so-called Feller diffusion branching process.

The case $\beta \in (-1,0)$ has a particular behaviour. Its corresponding CB-process is supercritical, since $\psi'(0+) = -\infty$. We observe that η is infinite or finite according to whether $a \geq 0$ or $a < 0$. The process satisfied the integral conditions of (i) and (iv). Thus the stable CB process with $\beta \in (-1,0)$ does not become extinct at a finite time a.s., and it has a finite time explosion with positive probability. Moreover, the asymptotic behaviour of $\mathbb{P}_x(T_\infty > t)$ can be computed explicitly.

The case $\beta \in (0,1]$ has a completely different behaviour. Its associated CB-process is sub-critical, critical or supercritical depending on the value of a, since $\psi'(0+) = -a$, and satisfies the integral conditions in (ii) and (iii). We also have that $\eta = 0$ or $\eta > 0$ according as $a \leq 0$ or $a > 0$. In other words, the stable CB-process with $\beta \in (0,1]$ does not explode at a finite time a.s., and it becomes extinct at a finite time with positive probability. Moreover the asymptotic behaviour of $\mathbb{P}_x(T_0 < t)$ can be computed explicitly.

For our purposes, we recall that CB-processes can also be defined as the unique non-negative strong solution of the following stochastic differential equation (SDE for short)

$$\begin{aligned} Y_t =& Y_0 + a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s \\ &+ \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty]} \int_0^{Y_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{4}$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure independent of B, with intensity $\mathrm{d}s\Lambda(\mathrm{d}z)\mathrm{d}u$ where $\Lambda(\mathrm{d}z) = \mu(\mathrm{d}z) + q\delta_\infty(\mathrm{d}z)$, and $\widetilde{N}$ is the compensated measure of N, see for instance [8].

In this work, we are interested in studying a particular class of CB-processes in a random environment. More precisely, we are interested in the case when the random environment is driven by a Brownian motion with drift which is independent of the dynamics of the original process. A process in this class is defined as the unique strong solution of a stochastic differential equation that conditioned on the environment, satisfies the branching property. We will refer to such class of processes as *CB-processes in a Brownian random environment*.

Our motivation comes from the work of Böinghoff and Hutzenthaler [6], where they consider the case of branching diffusions in a Brownian random environment i.e. when the branching mechanism has no jump structure. The authors in [6] introduced this type of processes using a result of Kurtz [14], where a diffusion approximation of Bienaymé-Galton-Watson in random environment is studied. The scaling limit obtained in [14] turns out to be the strong solution of an SDE, that conditioned on the environment, satisfies the branching property. Böinghoff and Hutzenthaler computed the exact asymptotic behaviour of the survival probability using a time change method and in consequence, they also described the so called Q-process. This is the process conditioned to be never extinct. Similarly to the discrete case, the authors in [6] found a phase transition in the subcritical regime that depends on the parameters of the random environment.

Another class of CB-processes in random environment has been studied recently by Bansaye et al. [1]. The authors in [1] studied the particular case where the random environment is driven by a Lévy process with paths of bounded variation. Such type of processes are called CB-processes with catastrophes, motivated by the fact that the presence of a negative jump in the random environment represents that a proportion of a population, following the dynamics of the CB-process, is killed. Similarly to the diffusion case, CB-processes with catastrophes were also introduced as the strong solution of an SDE and conditioned on the environment, they satisfy the branching property. It is also important to note that CB-processes with catastrophes can also be obtained as the scaling limit of Bienaymé-Galton-Watson processes in random environment (see for instance [2]). Bansaye et al. also studied the survival probability but unlike the case studied in [6], they used a martingale technique since the time change technique does not hold in general. In the particular case where the branching mechanism is stable, the authors in [1] computed the exact asymptotic behaviour of the survival probability and obtained similar results to those found in [6].

One of our aims is to study explosion and extinction probabilities for CB-processes in a Brownian random environment. Up to our knowledge, the explosion case has never been studied before even in the discrete setting. In order to study explosion and extinction probabilities for a process in this class, we follow the martingale technique used in [1] to compute the Laplace exponent via a backward differential equation. With the Laplace exponent in hand, we are able to determine whether a process is conservative, i.e. that does not explode a.s. at a fixed time, or become extinct with positive probability. Nonetheless, it seems very difficult to deduce necessary and sufficient conditions for explosion and extinction probabilities. This is due to the fluctuations of the random environment. However, the stable and the Neveu cases will help us to understand the different situations that our main results cannot cover.

We give special attention to the case when the branching mechanism is stable i.e. $\psi(\lambda) = -a\lambda + c_\beta \lambda^{1+\beta}$, for $\beta \in (-1,0) \cup (0,1)$. Here, the Laplace exponent can be computed explicitly and we will show that it depends on the exponential functional of the random environment. Whenever $\beta \in (-1,0)$, we can compute the explosion probability at a fixed time and establish the asymptotic behaviour of the probability of no-extinction where we find three different regimes. Up to our knowledge, this behaviour was never observed. In the case when $\beta \in (0,1)$, we study the extinction probability and also establish the asymptotic behaviour of the survival probability where five different regimes appear. The asymptotic behaviour depends on the study of exponential functionals of Brownian motion.

From the speed of survival in the stable case, we can deduce the process conditioned to be never extinct or Q-process using a Doob h-transform technique.

We finish this paper studying the immigration case, which represents an example of affine processes in a random environment. More precisely, this family of processes is an extension of the so-called Cox-Ingersoll-Ross model, which is largely used in the financial literature, under the fluctuations of a random environment.

The remainder of the paper is structured as follows. In Section 2, we define and study CB-processes in a Brownian random environment. Section 3 is devoted to the long-term behaviour of this family of processes. In particular, we study explosion and extinction probabilities. In Section 4, we analyse the stable case. Here, we study the asymptotic behaviour of the no-explosion and survival probabilities as well as the process conditioned to be never extinct and the process conditioned on eventual extinction. Finally in Section 5, we study the immigration case.

2 CB-processes in a Brownian random environment

Motivated by the definition of branching diffusions in random environment (see Böinghoff and Huzenthaler [6]) and CB-processes with catastrophes (see Bansaye et al. [1]) we introduce, using the same notation as in the SDE (4), continuous state branching processes in a Brownian random environment (in short a CBBRE) as the unique non-negative strong solution of the following stochastic differential equation

$$\begin{aligned} Z_t =& Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s + \int_0^t Z_s \mathrm{d}S_s \\ &+ \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty]} \int_0^{Z_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{5}$$

where $S_t = \alpha t + \sigma B_t^{(e)}$, with $\alpha \in \mathbb{R}$, $\sigma \in (0,\infty)$ and $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion independent of B and the Poisson random measure N. The process S represents the *random environment*.

The following theorem provides the existence of the CBBRE as a strong solution of (5) and, in some sense, characterizes its law given the environment. In order to introduce our main result, we define an auxiliary process that depends on S, as follows

$$K_t = S_t - \frac{\sigma^2}{2}t, \qquad \text{for} \qquad t \geq 0.$$

Theorem 1. *The stochastic differential equation (5) has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ satisfies the Markov property and its infinitesimal generator $\mathcal{L}$ satisfies, for every $f \in C^2(\mathbb{R}_+)$,*

$$\begin{aligned} \mathcal{L}f(x) =& x(a+\alpha)f'(x) + \left(\frac{1}{2}\sigma^2 x^2 + x\gamma^2\right) f''(x) \\ &+ x\int_{(0,\infty]} \left(f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{z<1\}}\right) \Lambda(\mathrm{d}z). \end{aligned} \tag{6}$$

Furthermore, the process Z conditioned on K, satisfies the branching property and for every $t > 0$

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = \exp\Big\{-z v_t(0,\lambda,K)\Big\} \qquad a.s., \tag{7}$$

where for every $(\lambda,\delta) \in (\mathbb{R}_+, C(\mathbb{R}_+))$, $v_t : s \in [0,t] \mapsto v_t(s,\lambda,\delta)$ is the unique solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = e^{\delta_s}\psi(v_t(s,\lambda,\delta)e^{-\delta_s}), \qquad v_t(t,\lambda,\delta) = \lambda, \tag{8}$$

and ψ is the branching mechanism of the underlying CB-process.

Proof. First, we prove the existence of a unique strong solution of the stochastic differential equation

$$\begin{aligned} Z_t^{(n)} &= Z_0^{(n)} + a\int_0^t (Z_s^{(n)} \wedge n)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2(Z_s^{(n)} \wedge n)}\mathrm{d}B_s + \int_0^t (Z_s^{(n)} \wedge n)\mathrm{d}S_s \\ &\quad + \int_0^t \int_{(0,1)} \int_0^{(Z_s^{(n)}\wedge n)-} (z \wedge n)\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t \int_{[1,\infty]} \int_0^{(Z_s^{(n)}\wedge n)-} (z\wedge n)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned} \tag{9}$$

In order to do so, we define the local martingale

$$\begin{aligned} M_t^{(n)} &= \int_0^t \mathbf{1}_{\{Z_s^{(n)}=0\}}\mathrm{d}B_s + \int_0^t \mathbf{1}_{\{Z_s^{(n)}>0\}} \frac{\sqrt{2\gamma^2(Z_s^{(n)}\wedge n)}}{\sqrt{2\gamma^2(Z_s^{(n)}\wedge n) + \sigma^2(Z_s^{(n)}\wedge n)^2}}\mathrm{d}B_s \\ &\quad + \int_0^t \mathbf{1}_{\{Z_s^{(n)}=0\}}\mathrm{d}B_s^{(e)} + \int_0^t \mathbf{1}_{\{Z_s^{(n)}>0\}} \frac{\sigma(Z_s^{(n)}\wedge n)}{\sqrt{2\gamma^2(Z_s^{(n)}\wedge n) + \sigma^2(Z_s^{(n)}\wedge n)^2}}\mathrm{d}B_s^{(e)}, \qquad t \geq 0. \end{aligned}$$

Observe that $M^{(n)}$ posses continuous paths and that $\langle M^{(n)}\rangle_t = t$, for $t \geq 0$. By Lévy's characterization theorem, the local martingale $M^{(n)} = (M_t^{(n)}, t \geq 0)$ is a standard Brownian motion which is independent of the Poisson random measure N thanks to Theorem 6.3 in Ikeda and Watanabe [11]. Then, the SDE (9) can be written as follows

$$\begin{aligned} Z_t^{(n)} &= Z_0^{(n)} + (a+\alpha)\int_0^t (Z_s^{(n)} \wedge n)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2(Z_s^{(n)}\wedge n) + \sigma^2(Z_s^{(n)}\wedge n)^2}\mathrm{d}M_s^{(n)} \\ &\quad + \int_0^t \int_{(0,1)} \int_0^{(Z_s^{(n)}\wedge n)-} (z\wedge n)\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{(Z_s^{(n)}\wedge n)-} (z\wedge n)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned} \tag{10}$$

Following the notation in [8], the conditions from Theorem 5.1 in Fu and Li [8] are satisfied by taking $U_0 = U_1 = (0,\infty] \times (0,\infty)$, $V_n = (1/n,\infty) \times (0,n)$, $\mu_0(\mathrm{d}z,\mathrm{d}u) = \mathbf{1}_{\{z<1\}}\mu(\mathrm{d}z)\mathrm{d}u$, $\mu_1(\mathrm{d}z,\mathrm{d}u) = \mathbf{1}_{\{z\geq 1\}}\mu(\mathrm{d}z)\mathrm{d}u + q\delta_\infty(\mathrm{d}z)\mathrm{d}u$, and the functions

$$\begin{aligned} b(x) &= (a+\alpha)(x\wedge n), & \sigma(x) &= \big(\sigma^2(x\wedge n)^2 + 2\gamma^2(x\wedge n)\big)^{1/2}, \\ g_0(x,z,u) &= (z\wedge n)\mathbf{1}_{\{u\leq x\wedge n\}}, & g_1(x,z,u) &= (z\wedge n)\mathbf{1}_{\{u\leq x\wedge n\}}. \end{aligned}$$

Then, there exists a unique non-negative strong solution to (10) and therefore there exists a unique non-negative weak solution to (9). By Lemma 3 (see the Appendix) the pathwise uniqueness of (9) holds. This guarantees that there is an unique non-negative strong solution to (9).

For $m \geq 1$ let $\tau_m = \inf\{t \geq 0 : Z_t^{(m)} \geq m\}$. Since $0 \leq Z_t^{(m)} \leq m$ for $0 \leq t < \tau_m$, the trajectory $t \mapsto Z_t^{(m)}$ has no jumps larger than m on the interval $[0, \tau_m)$. Then, we have that $Z_t^{(m)}$ satisfies (5) for $0 \leq t < \tau_m$. For $n \geq m \geq 1$, let $(Y_t, t \geq 0)$ be the strong solution to

$$\begin{aligned} Y_t =& Z^{(m)}_{\tau_m-} + (a+\alpha)\int_0^t (Y_s \wedge n)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2(Y_s \wedge n)}\mathrm{d}B_{\tau_m+s} \\ &+ \int_0^t (Y_s \wedge n)\mathrm{d}B^{(e)}_{\tau_m+s} + \int_0^t \int_{(0,1)} \int_0^{(Y_s\wedge n)-} (z\wedge n)\widetilde{N}(\tau_m + \mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &\qquad + \int_0^t \int_{[1,\infty]} \int_0^{(Y_s\wedge n)-} (z\wedge n)N(\tau_m + ds, dz, du). \end{aligned}$$

Next, we define $Y_t^{(n)} = Z_t^{(m)}$ for $0 \leq t < \tau_m$ and $Y_t^{(n)} = Y_{t-\tau_m}$ for $t \geq \tau_m$. It is not difficult to see that $(Y_t^{(n)})$ is a solution to (9). By the strong uniqueness, we get that $Z_t^{(n)} = Y_t^{(n)}$ for all $t \geq 0$. In particular, $Z_t^{(n)} = Z_t^{(m)} < m$, for $0 \leq t < \tau_m$. Consequently, the sequence $\{\tau_m\}_{m\geq 1}$ is non-decreasing. We define the process $(Z_t, t \geq 0)$ as follows

$$Z_t = \begin{cases} Z_t^{(m)} & \text{if} \quad t < \tau_m, \\ \infty & \text{if} \quad t \geq \lim\limits_{m\to\infty} \tau_m. \end{cases}$$

Therefore, the process $(Z_t, t \geq 0)$ is a weak solution to (5).

Now, we consider Z' and Z'', two solutions of (5) and define

$$\tau'_m = \inf\{t \geq 0 : Z'_t \geq m\}, \qquad \tau''_m = \inf\{t \geq 0 : Z''_t \geq m\},$$

and $\sigma_m = \tau'_m \wedge \tau''_m$. Thus, Z' and Z'' satisfy (9) on $[0, \sigma_m)$, implying that both processes are indistinguishable on $[0, \sigma_m)$. If $\sigma_\infty = \lim\limits_{m\to\infty} \sigma_m < \infty$, we have two situations to consider, either Z' or Z'' explodes at σ_∞ continuously or by a jump of infinite size. If Z' or Z'' explodes continuously then both processes explodes at the same time since they are indistinguishable on $[0, \sigma_m)$, for all $m \geq 1$. If Z' or Z'' explodes by a jump of infinite size, then this jump comes from an atom of the Poisson random measure N, so that both processes have it. Since after this time both processes are equal to ∞, and since the integral with respect to the Poisson process diverges, we get that Z' and Z'' are indistinguishable. Then, there is a unique strong solution to (5) that we denote by $(Z_t, t \geq 0)$. The strong uniqueness implies the strong Markov property and by Itô's formula it is easy to show that the infinitesimal generator of $(Z_t, t \geq 0)$ is given by (6).

The branching property of Z_t conditioned on K, follows from the branching property of the underlying CB-process. In order to deduce (7), we follow similar arguments as used in Bansaye [1]. To this purpose, we introduce $\widetilde{Z}_t = Z_t e^{-K_t}$ and take $F \in C^{1,2}(\mathbb{R}_+, \mathbb{R}_+ \cup \{\infty\})$. An application of Itô's formula guarantees that $F(t, \widetilde{Z}_t)$ conditioned on K is a local martingale if and only if for every $t \geq 0$,

$$
\begin{aligned}
0 = &\int_0^t \left(\frac{\partial}{\partial t} F(s, \widetilde{Z}_s) + a\widetilde{Z}_s \frac{\partial}{\partial x} F(s, \widetilde{Z}_s) + \gamma^2 e^{-K_s} \widetilde{Z}_s \frac{\partial^2}{\partial x^2} F(s, \widetilde{Z}_s) \right) \mathrm{d}s \\
&+ \int_0^t \int_0^\infty Z_s \left(F(s, \widetilde{Z}_s + z e^{-K_s}) - F(s, \widetilde{Z}_s) - \frac{\partial}{\partial x} F(s, \widetilde{Z}_s) z e^{-K_s} \mathbf{1}_{\{z<1\}} \right) \Lambda(\mathrm{d}z)\mathrm{d}s.
\end{aligned}
$$

By choosing $F(s, x) = \exp\{-x v_t(s, \lambda, K)\}$, where $v_t(s, \lambda, K)$ is differentiable with respect to the variable s, non-negative and such that $v_t(t, \lambda, K) = \lambda$ for all $\lambda \geq 0$, we observe that $(F(s, \widetilde{Z}_s), s \leq t)$ conditioned on K is a martingale if and only if

$$
\begin{aligned}
\frac{\partial}{\partial s} v_t(s, \lambda, K) = &-q e^{K_s} - a v_t(s, \lambda, K) + \gamma^2 (v_t(s, \lambda, K))^2 e^{-K_s} \\
&+ e^{K_s} \int_0^\infty \left(e^{-e^{-K_s} v_t(s,\lambda,K) z} - 1 + e^{-K_s} v_t(s, \lambda, K) z \mathbf{1}_{\{z<1\}} \right) \mu(\mathrm{d}z),
\end{aligned}
$$

which is equivalent to $v_t(s, \lambda, K)$ satisfying (8). Since ψ is locally Lipschitz on $(0, \infty)$, the existence and uniqueness of v_t follows from the Picard-Lindelöf theorem.

The above implies that the process $(\exp\{-\widetilde{Z}_s v_t(s, \lambda, K)\}, 0 \leq s \leq t)$ conditioned on K is a martingale, and hence

$$
\mathbb{E}_z \left[\exp\left\{ -\lambda \widetilde{Z}_t \right\} \big| K \right] = \mathbb{E}_z \left[\exp\left\{ -\widetilde{Z}_0 v_t(0, \lambda, K) \right\} \big| K \right] = \exp\{-z v_t(0, \lambda, K)\},
$$

which completes the proof. □

Remark 1. *Observe that in the case when $|\psi'(0+)| < \infty$, the auxiliary process can be taken as follows*

$$
K_t^{(0)} = \sigma B_t^{(e)} + \mathbf{m}t, \qquad \text{for} \quad t \geq 0,
$$

where

$$
\mathbf{m} = \alpha - \psi'(0+) - \frac{\sigma^2}{2}.
$$

Following the same arguments as in the last part of the proof of the previous Theorem and replacing K with $K^{(0)}$, one can deduce that $v_t(s, \lambda, K^{(0)})$ is the unique solution to the backward differential equation

$$
\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) = e^{K_s^{(0)}} \psi_0(v_t(s, \lambda, K^{(0)}) e^{-K_s^{(0)}}), \tag{11}
$$

where

$$
\psi_0(\lambda) = -q + \gamma^2 \lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \right) \mu(\mathrm{d}x). \tag{12}
$$

In this case, the process Z conditioned on $K^{(0)}$, satisfies, for every $t > 0$,

$$
\mathbb{E}_z \Big[\exp\Big\{ -\lambda Z_t e^{-K_t^{(0)}} \Big\} \Big| K^{(0)} \Big] = \exp\Big\{ -z v_t(0, \lambda, K^{(0)}) \Big\} \qquad a.s. \tag{13}
$$

Before we continue with the exposition of this manuscript, we would like to provide some examples where we can compute explicitly the Laplace exponent of the CB-process in a Brownian random environment.

Example 1. (Neveu case): The Neveu branching process in a Brownian random environment has branching mechanism given by $\psi(u) = u\log(u)$, for $u \geq 0$. In this particular case the backward differential equation (8) satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = v_t(s,\lambda,\delta)\log(e^{-\delta_s} v_t(s,\lambda,\delta)).$$

Providing that $v_t(t,\lambda,\delta) = \lambda$, one can solve the above equation and after some straightforward computation we deduce

$$v_t(s,\lambda,\delta) = \exp\left\{ e^s \left(\int_s^t e^{-u}\delta_u \mathrm{d}u + \log(\lambda)e^{-t} \right) \right\}.$$

Hence, from identity (7) we get

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = \exp\left\{ -z\lambda^{e^{-t}} \exp\left\{ \int_0^t e^{-s}K_s \mathrm{d}s \right\}\right\} \qquad \text{a.s.} \tag{14}$$

Observe that the r.v. $\int_0^t e^{-s}K_s\mathrm{d}s$, is normal distributed with mean $(\alpha - \frac{\sigma^2}{2})(1-e^{-t}-te^{-t})$ and variance $\frac{\sigma^2}{2}(1+4e^{-t}-3e^{-2t})$, for $t \geq 0$. In other words, the Laplace transform of $Z_t e^{-K_t}$ can be determined by the Laplace transform of a log-normal distribution which we know exists but there is not a explicit form of it.

Example 2. (Feller case): Assume that $a = \mu(0,\infty) = 0$, thus the CB-process in a Brownian random environment (5) is reduced to the following SDE

$$Z_t = Z_0 + \alpha\int_0^t Z_s \mathrm{d}s + \sigma\int_0^t Z_s \mathrm{d}B_s^{(e)} + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s,$$

where the random environment is given by $S_t = \alpha t + \sigma B_t^{(e)}$. This SDE is equivalent to the strong solution of the SDE

$$\begin{aligned} dZ_t =& \frac{\sigma^2}{2} Z_t \mathrm{d}t + Z_t\mathrm{d}K_t + \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s, \\ dK_t =& a_0\mathrm{d}t + \sigma\mathrm{d}B_t^{(e)}, \end{aligned}$$

where $a_0 = \alpha - \sigma^2/2$, which is the branching diffusion in random environment studied by Böinghoff and Hutzenthaler [6].

Observe that in this case $\psi'(0+) = 0$, implying that the auxiliary processes K and $K^{(0)}$ coincide. Hence the backward differential equation (11) satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = \gamma^2 v_t^2(s,\lambda,\delta)e^{-\delta_s}.$$

If $v_t(t,\lambda,\delta)=\lambda$, the above equation can be solved and after some computations one can deduce

$$v_t(s,\lambda,\delta)=\left(\lambda^{-1}+\gamma^2\int_s^t e^{-\delta_u}\mathrm{d}u\right)^{-1}.$$

Hence, from identity (7) we get

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big]=\exp\left\{-z\left(\lambda^{-1}+\gamma^2\int_0^t e^{-K_u}\mathrm{d}u\right)^{-1}\right\}\qquad \text{a.s.}\tag{15}$$

The r.v. $\int_0^t e^{-K_u}\mathrm{d}u$, is known as the exponential functional of the process K, which is a Brownian motion with drift, and it has been deeply studied by many authors, see for instance [4, 7, 19].

Example 3. (Stable case): For our last example, we assume that the branching mechanism is of the form

$$\psi(\lambda)=-a\lambda+c_\beta\lambda^{\beta+1},\qquad \lambda\geq 0,$$

for some $\beta\in(-1,0)\cup(0,1)$, $a\in\mathbb{R}$, and c_β is such that

$$\begin{cases} c_\beta<0 & \text{if } \beta\in(-1,0),\\ c_\beta>0 & \text{if } \beta\in(0,1).\end{cases}$$

Under this assumption, the process Z satisfies the following stochastic differential equation

$$Z_t=Z_0+a\int_0^t Z_s\mathrm{d}s+\int_0^t Z_s\mathrm{d}S_s+\int_0^t\int_0^\infty\int_0^{Z_{s-}} z\widehat{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\tag{16}$$

where $S_t=\alpha t+\sigma B_t^{(e)}$, for $t\geq 0$, is a Brownian motion with drift and

$$\widehat{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)=\begin{cases} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) & \text{if } \beta\in(-1,0),\\ \widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) & \text{if } \beta\in(0,1),\end{cases}$$

where N is an independent Poisson random measure with intensity

$$\frac{c_\beta\beta(\beta+1)}{\Gamma(1-\beta)}\frac{1}{z^{2+\beta}}\mathrm{d}s\mathrm{d}z\mathrm{d}u,$$

and $\widetilde{N}$ is its compensated version.

In this case, we note

$$\psi'(0+)=\begin{cases} -\infty & \text{if } \beta\in(-1,0),\\ -a & \text{if } \beta\in(0,1).\end{cases}$$

Hence, when $\beta\in(0,1)$, we have $K_t^{(0)}=K_t+at$, for $t\geq 0$. In both cases, we use the backward differential equation (8) and observe that it satisfies

$$\frac{\partial}{\partial s}v_t(s,\lambda,\delta)=-av_t(s,\lambda,\delta)+c_\beta v_t^{\beta+1}(s,\lambda,\delta)e^{-\beta\delta_s}.$$

Similarly to the Feller case, assuming that $v_t(t,\lambda,\delta)=\lambda$, we can solve the above equation and after some straightforward computations, we get

$$v_t(s,\lambda,\delta)=e^{as}\left((\lambda e^{at})^{-\beta}+\beta c_\beta\int_s^t e^{-\beta(\delta_u+au)}\mathrm{d}u\right)^{-1/\beta}.$$

Hence, from (7) we get the following a.s. identity

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big]=\exp\left\{-z\left((\lambda e^{at})^{-\beta}+\beta c_\beta\int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\},$$

which clearly implies the following a.s. identity

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-(K_t+at)}\Big\}\Big|K\Big]=\exp\left\{-z\left(\lambda^{-\beta}+\beta c_\beta\int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}. \quad (17)$$

We finish this example by observing that the r.v. $\int_0^t e^{-\beta(K_u+au)}\mathrm{d}u$ is the exponential functional of the Brownian motion with drift $(\beta(K_u+au),u\geq 0)$.

3 Long-term behaviour

Similarly to the CB-processes case, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*. Recall that the event of explosion at fixed time t, is given by $\{Z_t=\infty\}$. When $\mathbb{P}_z(Z_t<\infty)=1$, for all $t>0$ and $z>0$, we say the process is *conservative*. In the second event, we observe from the definition of Z that if $Z_t=0$ for some $t>0$, then $Z_{t+s}=0$ for all $s\geq 0$, which makes 0 an absorbing state. As Z_t is to be thought of as the size of a given population at time t, the event $\{\lim_{t\to\infty}Z_t=0\}$ is referred as *extinction*.

Up to our knowledge, *explosion* has never been studied before for branching processes in random environment even in the discrete setting. Most of the results that appear in the literature are related to *extinction*. In this section, we first provide a sufficient condition under which the process Z is conservative and an example where we can determine explicitly the probability of explosion. Under the condition that the process is conservative, we study the probability of extinction under the influence of the random environment.

In our particular case, the events of explosion and absorption are not so easy to deduce in full generality. In the next section, we provide an example under which both events can be computed explicitly, as well as their asymptotic behaviour when time increases.

3.1 Explosion and conservative processes

Recall that $\psi'(0+)\in[-\infty,\infty)$, and that whenever $|\psi'(0+)|<\infty$, we write

$$\mathbf{m}=\alpha+\psi'(0+)-\frac{\sigma^2}{2}.$$

The following proposition provides necessary conditions under which the process Z is conservative.

Proposition 1. *Assume that $q = 0$ and $|\psi'(0+)| < \infty$, then a CBBRE with branching mechanism ψ is conservative.*

Proof. Recall that under our assumption, the auxiliary process takes the form

$$K_t^{(0)} = \sigma B_t^{(e)} + \mathbf{m}t, \qquad \text{for} \quad t \geq 0,$$

and that $v_t(s, \lambda, K^{(0)})$ is the unique solution to the backward differential equation (11) with $q = 0$. From identity (13), we know that

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t^{(0)}}\Big\}\Big] = \mathbb{E}\Big[\exp\Big\{-z v_t(0, \lambda, K^{(0)})\Big\}\Big].$$

Thus if we take λ goes to 0, we deduce

$$\mathbb{P}_z\big(Z_t < \infty\big) = \lim_{\lambda \downarrow 0} \mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t^{(0)}}\Big\}\Big] = \mathbb{E}\left[\exp\left\{-z \lim_{\lambda \downarrow 0} v_t(0, \lambda, K^{(0)})\right\}\right],$$

where the limits are justified by monotonicity and dominated convergence. This implies that a CBBRE is conservative if and only if

$$\lim_{\lambda \downarrow 0} v_t(0, \lambda, K^{(0)}) = 0.$$

Let us introduce the function $\Phi(\lambda) = \lambda^{-1}\psi_0(\lambda)$ and observe that $\Phi(0) = \psi_0'(0+) = 0$. Since ψ_0 is convex, we deduce that Φ is increasing. Finally, if we solve equation (11) with $\psi_0(\lambda) = \lambda\Phi(\lambda)$, we get

$$v_t(s, \lambda, K) = \lambda \exp\left\{-\int_s^t \Phi(e^{-K_r} v_t(r, \lambda, K))\mathrm{d}r\right\}.$$

Therefore, since Φ is increasing and $\Phi(0) = 0$, we have

$$0 \leq \lim_{\lambda \to 0} v_t(0, \lambda, K) = \lim_{\lambda \to 0} \lambda \exp\left\{-\int_0^t \Phi(e^{-K_r} v_t(r, \lambda, K))\mathrm{d}r\right\} \leq \lim_{\lambda \to 0} \lambda = 0,$$

implying that Z is conservative.

□

Recall that in the case when there is no random environment, i.e. $\sigma = 0$, we know that a CB-process with branching mechanism ψ is conservative if and only if

$$\int_{0+} \frac{\mathrm{d}u}{|\psi(u)|} = \infty.$$

In the case when the random environment is present, it is not so clear how to get a necessary and sufficient condition in terms of the branching mechanism since the random environment is affecting the monotonicity of ψ in the backward differential equation (8).

We now provide two interesting examples in the case when $\psi'(0+) = -\infty$, that behave completely differently.

1. Stable case with $\beta \in (-1, 0)$. Recall that in this case $\psi(u) = -au + c_\beta u^{\beta+1}$, where $a \in \mathbb{R}$ and c_β is a negative constant. From straightforward computations, we get

$$\psi'(0+) = -\infty, \qquad \text{and} \qquad \int_{0+} \frac{\mathrm{d}u}{|\psi(u)|} < \infty,$$

On the other hand, from identity (17) and taking λ goes to 0, we deduce

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K\Big) = \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)}\mathrm{d}u\right)^{-1/\beta}\right\} \qquad \text{a.s.,} \tag{18}$$

implying

$$\mathbb{P}_z\Big(Z_t = \infty \Big| K\Big) = 1 - \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)}\mathrm{d}u\right)^{-1/\beta}\right\} > 0.$$

In other words the stable CBBRE with $\beta \in (-1, 0)$ explodes with positive probability for any $t > 0$. Moreover, if the process $(K_u + au, u \geq 0)$ does not drift to $-\infty$, i.e. $\alpha + a - \sigma^2/2 \geq 0$, we deduce from Theorem 1 in Bertoin and Yor [4],

$$\int_0^t e^{-\beta(K_u + au)}\mathrm{d}u \to \infty, \qquad \text{as} \quad t \to \infty,$$

implying

$$\lim_{t\to\infty} Z_t = \infty, \qquad \text{a.s.}$$

On the other hand, if the process $(K_u + au, u \geq 0)$ drifts to $-\infty$, i.e. $\alpha + a - \sigma^2/2 < 0$, we have an interesting long-term behaviour of the process Z. In fact, we deduce from the Dominated Convergence Theorem

$$\mathbb{P}_z\Big(Z_\infty = \infty\Big) = 1 - \mathbb{E}\left[\exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta(K_u + au)}\mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

The above probability is positive since

$$\int_0^\infty e^{-\beta(K_u + au)}\mathrm{d}u < \infty \qquad \text{a.s.,}$$

according to Theorem 1 in Bertoin and Yor [4].

In this particular case, we will discuss the asymptotic behaviour of the probability of explosion in Section 4.

2. Neveu case. In this case, recall that $\psi(u) = u \log u$. In particular

$$\psi'(0+) = -\infty \qquad \text{and} \qquad \int_{0+} \frac{\mathrm{d}u}{|u \log u|} = \infty.$$

By taking λ goes to 0 in (14), one can see that the process is conservative conditionally on the environment, i.e.

$$\mathbb{P}_z\left(Z_t < \infty \middle| K\right) = 1,$$

for all $t \in (0, \infty)$ and $z \in [0, \infty)$.

3.2 Extinction probabilities

Here, we consider CBBREs that are conservative. The following result provides a criteria that depends on the behaviour of the auxiliary process $K^{(0)}$, which allows us to compute the probability of extinction of a CBBRE. Recall that the event of extinction is equal to $\{\lim_{t\to\infty} Z_t = 0\}$.

Proposition 2. *Assume that $q = 0$ and $|\psi'(0+)| < \infty$. Let $(Z_t, t \geq 0)$ be a CBBRE with branching mechanism given by ψ.*

i) If $\mathbf{m} < 0$, then $\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t = 0\Big| K^{(0)}\Big) = 1$, a.s.

ii) If $\mathbf{m} = 0$, then $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t = 0\middle| K^{(0)}\right) = 1$, a.s. Moreover if $\gamma > 0$ then

$$\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t = 0\Big| K^{(0)}\Big) = 1, \text{ a.s.}$$

iii) If $\mathbf{m} > 0$ and

$$\int^{\infty} x \ln x \, \mu(\mathrm{d}x) < \infty,$$

then $\mathbb{P}_z\Big(\liminf_{t\to\infty} Z_t > 0\Big| K^{(0)}\Big) > 0$ a.s., and there exists a non-negative finite r.v. W such that

$$Z_t e^{-K_t^{(0)}} \underset{t\to\infty}{\longrightarrow} W, \ \text{a.s} \qquad \text{and} \qquad \big\{W = 0\big\} = \Big\{\lim_{t\to\infty} Z_t = 0\Big\}.$$

Moreover if $\gamma > 0$, we have

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\middle| K^{(0)}\right) > 0, \quad \text{a.s.}$$

and in particular,

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \geq \left(1 + \frac{z\sigma^2}{\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

Proof. Recall that under our assumption, i.e. $|\psi'(0+)| < \infty$, the auxiliary process can be written as

$$K_t^{(0)} = \sigma B_t^{(e)} + \mathbf{m}t, \qquad \text{for} \quad t \geq 0,$$

and the function $v_t(s, \lambda, K^{(0)})$ satisfies the backward differential equation (11).

Similarly to the last part of the proof of Theorem 1, one can prove that $Z_t e^{-K_t^{(0)}}$ is a non-negative local martingale. Therefore $Z_t e^{-K_t^{(0)}}$ is a non-negative supermartingale and it converges a.s. to a non-negative finite random variable, here denoted by W. This implies the statement of part (i) and the first statement of part (ii).

In order to prove the second statement of part (ii), we observe that if $\gamma > 0$, then the backward differential equation (11) satisfies

$$\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) \geq \gamma^2 v_t(s, \lambda, K^{(0)})^2 e^{-K_s^{(0)}}.$$

Therefore

$$v_t(s,\lambda,K^{(0)}) \leq \left(\frac{1}{\lambda}+\gamma^2\int_s^t e^{-K_s^{(0)}}\mathrm{d}s\right)^{-1},$$

which implies the following inequality,

$$\mathbb{P}_z(Z_t=0|K^{(0)}) \geq \exp\left\{-z\left(\gamma^2\int_0^t e^{-K_s^{(0)}}\mathrm{d}s\right)^{-1}\right\}. \tag{19}$$

Since $\mathbf{m} \leq 0$, we deduce from Theorem 1 in Bertoin and Yor [4]

$$\int_0^t e^{-K_s^{(0)}}\mathrm{d}s \to \infty, \qquad \text{as} \quad t\to\infty. \tag{20}$$

In consequence, we have

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\Big|K^{(0)}\right) = 1 \quad \text{a.s.}$$

Now, we prove part (iii). We first note that $v_t(\cdot,\lambda,K^{(0)})$, the solution to the backward differential equation (11), is non-decreasing on $[0,t]$ (since ψ_0 is positive). Thus for all $s\in[0,t]$, $v_t(s,\lambda,K^{(0)}) \leq \lambda$. From the proof of part (ii) in Proposition 1, we know that the function $\Phi(\lambda)=\lambda^{-1}\psi_0(\lambda)$, is increasing. Hence

$$\frac{\partial}{\partial s}v_t(s,\lambda,K^{(0)}) = v_t(s,\lambda,K^{(0)})\Phi(v_t(s,\lambda,K^{(0)})e^{-K_s^{(0)}}) \leq v_t(s,\lambda,K^{(0)})\Phi(\lambda e^{-K_s^{(0)}}).$$

Therefore, for every $s\leq t$, we have

$$v_t(s,\lambda,K^{(0)}) \geq \lambda\exp\left\{-\int_s^t \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}.$$

In particular,

$$\lim_{t\to\infty} v_t(0,\lambda,K^{(0)}) \geq \lambda\exp\left\{-\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}.$$

According to Proposition 3.3 in Salminen and Yor [20], we have

$$\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s<\infty \quad \text{a.s.,} \qquad \text{if and only if} \qquad \int_0^\infty \Phi(\lambda e^{-y})\mathrm{d}y<\infty.$$

Observe

$$\begin{aligned}
\int_0^\infty \Phi(\lambda e^{-y})\mathrm{d}y &= \int_0^\lambda \frac{\Phi(\theta)}{\theta}\mathrm{d}\theta \\
&= \gamma^2\lambda + \int_0^\lambda \frac{\mathrm{d}\theta}{\theta^2}\int_{(0,\infty)}(e^{-\theta x}-1+\theta x)\mu(\mathrm{d}x) \\
&= \gamma^2\lambda + \int_{(0,\infty)}\mu(\mathrm{d}x)\int_0^\lambda (e^{-\theta x}-1+\theta x)\frac{\mathrm{d}\theta}{\theta^2} \\
&= \gamma^2\lambda + \int_{(0,\infty)} x\left(\int_0^{\lambda x}(e^{-y}-1+y)\frac{\mathrm{d}y}{y^2}\right)\mu(\mathrm{d}x).
\end{aligned}$$

Since the function

$$g_\lambda(x)=\int_0^{\lambda x}(e^{-y}-1+y)\frac{\mathrm{d}y}{y^2},$$

is equivalent to $\lambda x/2$ as $x\to 0$ and equivalent to $\ln x$ as $x\to\infty$, we deduce that

$$\int_0^\infty \Phi(\lambda e^{-y})\mathrm{d}y<\infty \qquad \text{if and only if} \qquad \int^\infty x\ln x\mu(\mathrm{d}x)<\infty.$$

In other words,

$$\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s<\infty \quad \text{a.s.,} \qquad \text{if and only if} \qquad \int^\infty x\ln x\mu(\mathrm{d}x)<\infty.$$

If the integral condition from above is satisfied, then

$$\lim_{t\to\infty} v_t(0,\lambda,K^{(0)})\geq \lambda\exp\left\{-\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}>0,$$

implying

$$\mathbb{E}_z\left[e^{-\lambda W}\Big|K^{(0)}\right]\leq \exp\left\{-z\;\lambda\exp\left\{-\int_0^\infty \Phi(\lambda e^{-K_s^{(0)}})\mathrm{d}s\right\}\right\}<1,$$

and in particular $\mathbb{P}_z\Big(\liminf_{t\to\infty} Z_t>0\Big|K^{(0)}\Big)>0$ a.s. Next, we use Lemma 20 in [1] and the branching property of Z, to deduce

$$\{W=0\}=\Big\{\lim_{t\to\infty}Z_t=0\Big\}.$$

Now assume $\gamma>0$. From inequality (19) and the Dominate Convergence Theorem, we deduce

$$\mathbb{P}_z\left(\lim_{t\to\infty}Z_t=0\Big|K^{(0)}\right)\geq \exp\left\{-z\left(\gamma^2\int_0^\infty e^{-K_s^{(0)}}\mathrm{d}s\right)^{-1}\right\}\quad \text{a.s.}$$

On the other hand, since $\mathbf{m}>0$, we deduce from Theorem 1 in Bertoin and Yor [4] (or Proposition 3.3 in Salminen and Yor [20])

$$\int_0^\infty e^{-K_s^{(0)}}\mathrm{d}s<\infty, \qquad \text{a.s.,} \tag{21}$$

implying that $\mathbb{P}_z\big(\lim_{t\to\infty}Z_t=0\big|K^{(0)}\big)>0$, a.s. Finally, according to Dufresne [7],

$$\int_0^\infty e^{-K_s^{(0)}}\mathrm{d}s \quad \text{has the same law as} \quad \Big(2\Gamma_{\frac{2\mathbf{m}}{\sigma^2}}\Big)^{-1}, \tag{22}$$

where Γ_v is Gamma r.v. with parameter v. After straightforward computations, we deduce

$$\mathbb{P}_z\left(\lim_{t\to\infty}Z_t=0\right)\geq\left(1+\frac{z\sigma^2}{\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

The proof of Proposition 2 is now complete. □

Remark 2. *When $\mathbf{m} > 0$, we also have an upper bound for the probability of extinction under the assumption*

$$\gamma > 0 \qquad \text{and} \qquad \kappa := \int_1^\infty x^2 \mu(\mathrm{d}x) < \infty.$$

More precisely, under the above assumption we have

$$\left(1 + \frac{z\sigma^2}{2\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}} \leq \mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \leq \left(1 + \frac{1}{2}\frac{z\sigma^2}{\gamma^2 + \kappa}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

The upper bound follows from the a.s. inequality

$$\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) \leq (\gamma^2 + \kappa) v_t(s, \lambda, K^{(0)})^2 e^{-K_s^{(0)}}.$$

It is important to note that when the branching mechanism is of the form $\psi(u) = -au + c_\beta u^{\beta+1}$ for $\beta \in (0, 1]$, one can deduce directly from (15) and (17) by taking λ and t go to ∞, and (20) that

$$\lim_{t\to\infty} Z_t = 0, \quad \text{a.s.,} \qquad \text{for} \quad \mathbf{m} \leq 0.$$

Similarly, using (21), in the case when $\mathbf{m} > 0$ we have

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \big| K\right) = \exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta(K_u + au)}\mathrm{d}u\right)^{-1/\beta}\right\}, \qquad \text{a.s.,}$$

and in particular

$$\mathbb{P}\Big(W = 0\Big) = \mathbb{P}_z\Big(\lim_{t\to\infty} Z_t = 0\Big) = \mathbb{E}_z\left[\exp\left\{-z\left(\beta c_\beta \int_0^\infty e^{-\beta(K_u + au)}\mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

The latter probability can be computed explicitly using (22).

We finish this section with a remark on the Neveu case. If we take λ goes to ∞, in (14), we obtain that the Neveu CBBRE survives conditionally on the environment, in other words

$$\mathbb{P}_z\left(Z_t > 0 \big| K\right) = 1,$$

for all $t \in (0, \infty)$ and $z \in [0, \infty)$. Moreover since the process has càdlàg paths, we deduce the Neveu CBBRE survives a.s., i.e.

$$\mathbb{P}_z\Big(Z_t > 0, \text{ for all } t \geq 0\Big) = 1.$$

On the one hand, using integration by parts we obtain

$$\int_0^t e^{-s} K_s \mathrm{d}s = \sigma \int_0^t e^{-s} \mathrm{d}B_s^{(e)} + (1 - e^{-t})\left(\alpha - \frac{\sigma^2}{2}\right) - e^{-t} K_t.$$

Since

$$\left\langle \sigma \int_0^{\cdot} e^{-s} \mathrm{d}B_s^{(e)} \right\rangle_t = \frac{\sigma^2}{2}(1 - e^{-2t}) < \infty \qquad \text{and} \qquad \mathbb{E}\left[\left\langle \sigma \int_0^{\cdot} e^{-s} \mathrm{d}B_s^{(e)} \right\rangle_\infty \right] = \frac{\sigma^2}{2},$$

we have that

$$\lim_{t\to\infty} \int_0^t e^{-s} B_s^{(e)} ds = \sigma \int_0^\infty e^{-s} dB_s^{(e)} + \alpha - \frac{\sigma^2}{2},$$

exists and its law is Gaussian with mean $\alpha - \frac{\sigma^2}{2}$ and variance $\frac{\sigma^2}{2}$. Hence, if we take take t goes to ∞ in (14), we observe

$$\mathbb{E}_z\Big[\exp\Big\{ -\lambda \lim_{t\to\infty} Z_t e^{-K_t} \Big\} \Big| K \Big] = \exp\left\{ -z \exp\left\{ \int_0^\infty e^{-s} K_s \mathrm{d}s \right\} \right\}.$$

Since the right-hand side of the above identity does not depend on λ, this implies that

$$\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t e^{-K_t} = 0 \Big| K \Big) = \exp\left\{ -z \exp\left\{ \int_0^\infty e^{-s} K_s \mathrm{d}s \right\} \right\}$$

and taking expectations in the above identity, we deduce

$$\mathbb{P}_z\Big(\lim_{t\to\infty} Z_t e^{-K_t} = 0 \Big) = \mathbb{E}\left[\exp\left\{ -z \exp\left\{ \int_0^\infty e^{-s} K_s \mathrm{d}s \right\} \right\} \right].$$

In conclusion, the Neveu process is conservative and survives a.s., but when $\alpha < \frac{\sigma^2}{2}$, the extinction probability is given by the Laplace transform of a log-normal distribution.

Finally, if we multiply (14) by e^{K_t}, differentiate with respect to λ and then we take expectations on both sides, we deduce

$$\begin{aligned} &\mathbb{E}_z\Big[Z_t \exp\Big\{ -\lambda Z_t e^{-K_t} \Big\} \Big] \\ &\qquad = z e^{-t} \lambda^{e^{-t}-1} \mathbb{E}\left[\exp\left\{ K_t + \int_0^t e^{-s} K_s \mathrm{d}s - z \lambda^{e^{-t}} \exp\left\{ \int_0^t e^{-s} K_s \mathrm{d}s \right\} \right\} \right]. \end{aligned}$$

Taking limits as λ goes to 0, it is clear

$$\mathbb{E}\left[Z_t \right] = \infty, \qquad \text{for} \quad t > 0.$$

4 Stable case

The stable case is perhaps one of the most interesting examples of CB-processes. One of the advantages of this class of CB-processes is that we can perform explicit computations of many functionals, see for instance [3, 16, 17], and that they appear in many other areas of probability such as coalescent theory, fragmentation theory, Lévy trees and self-similar Markov process to name but a few. As we will see below, we can also perform a lot of explicit computations when the stable CB-process is affected by a Brownian random environment.

In the sequel, we shall assume that the branching mechanism satisfies $\psi(u) = -au + c_\beta u^{\beta+1}$, for $u \geq 0$, and $\beta \in (-1,0) \cup (0,1)$. Recall that $a \in \mathbb{R}$, and

$$\begin{cases} c_\beta < 0 & \text{if } \beta \in (-1,0), \\ c_\beta > 0 & \text{if } \beta \in (0,1). \end{cases}$$

Böinghoff and Hutzenthaler [6] studied the particular case when $\beta = 1$, also known as the Feller diffusion case. The authors in [6] gave a precise asymptotic behaviour for the survival probability and also studied the so-called Q-process. In this section, we prove similar results for the case when $\beta \in (0,1)$ and we obtain new results on the asymptotic behaviour of non-explosion for the case when $\beta \in (-1,0)$.

Recall from identity (17) that the stable CBBRE $Z = (Z_t, t \geq 0)$, satisfies

$$\mathbb{E}_z\Big[\exp\Big\{ -\lambda Z_t e^{-(K_t+at)}\Big\}\Big|K\Big] = \exp\left\{-z\left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}.$$

If we take λ goes to ∞, in the above identity we obtain for all $z, t > 0$,

$$\mathbb{P}_z\left(Z_t > 0\middle|K\right) = 1 - \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}\mathbf{1}_{\{\beta>0\}}, \qquad \text{a.s.,} \qquad (23)$$

where the same holds true for the Feller case by taking $\beta = 1$ and $c_\beta = \gamma^2$.

On the other hand, if we take λ goes to 0, we deduce

$$\mathbb{P}_z\Big(Z_t < \infty\Big|K\Big) = \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}\mathbf{1}_{\{\beta<0\}} + \mathbf{1}_{\{\beta>0\}} \qquad \text{a.s.}$$

It is then clear that if $\beta \in (-1,0)$, then the survival probability equals 1, for all $t \geq 0$. If $\beta \in (0,1]$ then the process is conservative.

From the above identities, a natural question arises: *can we determine the asymptotic behaviour, when t goes to ∞, of $\mathbb{P}_z(Z_t > 0)$ for $\beta \in (0,1]$ and $\mathbb{P}_z(Z_t < \infty)$ for $\beta \in (-1,0)$?* We observe below that the answer of this question depends on a fine study of the asymptotic behaviour of

$$\mathbb{E}\left[\exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u+au)}\mathrm{d}u\right)^{-1/\beta}\right\}\right].$$

For this purpose, let us recall some interesting facts of the exponential functional of a Brownian motion with drift.

4.1 Exponential functional of a Brownian motion with drift

In what follows, the following functional will be of particular interest. Let $I_t^{(\eta)}$ be the exponential functional of a Brownian motion with drift $\eta \in \mathbb{R}$, in other words

$$I_t^{(\eta)} := \int_0^t \exp\Big\{2(\eta s + B_s)\Big\}\mathrm{d}s, \qquad t \in [0,\infty).$$

The law of such random variable have been deeply studied by many authors. Up to our knowledge this is the unique example for which there exist an explicit formula for the joint distribution of $(I_t^{(\eta)}, B_t + \eta t)$, see for instance Proposition 2 in Matsumoto and Yor [19]. In particular, for all $t,\ u \in (0, \infty)$ and $x \in \mathbb{R}$, we have

$$\mathbb{P}\left(I_t^{(\eta)} \in \mathrm{d}u \middle| B_t + \eta t = x\right) = \frac{\sqrt{2\pi t}}{u} \exp\left\{\frac{x^2}{2t}\right\} \exp\left\{-\frac{1}{2u}\left(1 + e^{2x}\right)\right\} \theta_{e^x/u}(t)\mathrm{d}u, \qquad (24)$$

where

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\frac{\pi^2}{2t}} \int_0^\infty e^{-\frac{y^2}{2t} - r\cosh(y)} \sinh(y) \sin\left(\frac{\pi y}{t}\right) \mathrm{d}y, \qquad r > 0.$$

Lemma 1. *Let $\eta \in \mathbb{R}$ and $p \geq 0$. Then for every $t > 0$, we have*

i) $$\mathbb{E}\left[\left(I_t^{(\eta)}\right)^{-p}\right] = e^{(2p^2 - 2p\eta)t} \mathbb{E}\left[\left(I_t^{(-(\eta-2p))}\right)^{-p}\right],$$

ii) $$\mathbb{E}\left[\left(I_t^{(\eta)}\right)^{-2p}\right] \leq e^{(2p^2 - 2p\eta)t} \mathbb{E}\left[\left(I_{t/2}^{(-(\eta-2p))}\right)^{-p}\right] \mathbb{E}\left[\left(I_{t/2}^{((\eta-2p))}\right)^{-p}\right].$$

Proof. Using the time reversal property for Brownian motion, we observe that the process $(\eta t + B_t - \eta(t-s) - B_{t-s}, 0 \leq s \leq t)$ has the same law as $(\eta s + B_s, 0 \leq s \leq t)$. Then, we deduce that

$$\int_0^t e^{2(\eta s + B_s)} \mathrm{d}s \quad \text{has the same law as} \quad e^{2(\eta t + B_t)} \int_0^t e^{-2(\eta s + B_s)} \mathrm{d}s.$$

We now introduce the exponential change of measure known as the Esscher transform or Girsanov's formula

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\lambda)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda B_t - \frac{\lambda^2}{2} t}, \qquad \text{for} \quad \lambda \in \mathbb{R},$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by the Brownian motion B which is naturally completed. Observe that under $\mathbb{P}^{(\lambda)}$, the process B is a Brownian motion with drift λ. Hence, taking $\lambda = -2p$, we deduce

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t e^{2(\eta s + B_s)} \mathrm{d}s\right)^{-p}\right] &= \mathbb{E}\left[e^{-2p(\eta t + B_t)} \left(\int_0^t e^{-2(\eta s + B_s)} \mathrm{d}s\right)^{-p}\right] \\
&= e^{-2p\eta t} e^{2p^2 t} \mathbb{E}^{(-2p)} \left[\left(\int_0^t e^{-2(\eta s + B_s)} \mathrm{d}s\right)^{-p}\right] \\
&= e^{-2p\eta t} e^{2p^2 t} \mathbb{E}\left[\left(\int_0^t e^{-2((\eta - 2p)s + B_s)} \mathrm{d}s\right)^{-p}\right],
\end{aligned}$$

which implies the first identity, thanks to the symmetry property of Brownian motion.

In order to get the second identity, we observe

$$\int_0^t e^{2(\eta s + B_s)} \mathrm{d}s = \int_0^{t/2} e^{2(\eta s + B_s)} \mathrm{d}s + e^{\eta t + 2B_{t/2}} \int_0^{t/2} e^{2(\eta s + \tilde{B}_s)} \mathrm{d}s,$$

where $\tilde{B}_s = B_{s+t/2} - B_{t/2}$, $s \geq 0$, is a Brownian motion which is independent of $(B_u, 0 \leq u \leq t/2)$. Therefore, using part (i), we deduce

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-2p}\right]\\
&\qquad\qquad \leq \mathbb{E}\left[\left(e^{\eta t+2B_{t/2}}\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\mathbb{E}\left[\left(\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\\
&\qquad\qquad \leq e^{(p^2-\eta p)t}\mathbb{E}\left[\left(e^{\eta t+2B_{t/2}}\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\mathbb{E}\left[\left(I_{t/2}^{(-(\eta-2p))}\right)^{-p}\right].
\end{aligned}$$

On the other hand from the Esscher transform with $\lambda = -2p$, we get

$$\begin{aligned}
\mathbb{E}\left[\left(e^{\eta t+2B_{t/2}}\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right] &= e^{-p\eta t}e^{p^2 t}\mathbb{E}^{(-2p)}\left[\left(\int_0^{t/2} e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-p}\right]\\
&= e^{-p\eta t}e^{p^2 t}\mathbb{E}\left[\left(\int_0^{t/2} e^{2((\eta-2p)s+B_s)}\mathrm{ds}\right)^{-p}\right].
\end{aligned}$$

Putting all the pieces together, we deduce

$$\mathbb{E}\left[\left(\int_0^t e^{2(\eta s+B_s)}\mathrm{ds}\right)^{-2p}\right] \leq e^{(2p^2-\eta 2p)t}\mathbb{E}\left[\left(I_{t/2}^{((\eta-2p))}\right)^{-p}\right]\mathbb{E}\left[\left(I_{t/2}^{(-(\eta-2p))}\right)^{-p}\right].$$

The proof of the Lemma is now complete. □

We are also interested in

$$I_\infty^{(\eta)} := \int_0^\infty \exp\Big\{2(\eta s + B_s)\Big\}\mathrm{d}s,$$

which is finite a.s. whenever $\eta < 0$. We recall that according to Dufresne [7],

$$I_\infty^{(\eta)} \quad \text{has the same law as} \quad \Big(2\Gamma_{-\eta}\Big)^{-1}, \tag{25}$$

where Γ_v is Gamma r.v. with parameter v.

4.2 Explosion probability

Throughout this section, we assume that $\beta \in (-1, 0)$. As we will see in the main result of this section, the asymptotic behaviour of the probability of explosion depends on the value of

$$\mathbf{m} = \alpha + a - \frac{\sigma^2}{2}.$$

We recall that when $\sigma = 0$, i.e there is no random environment, the stable CB-processes explodes with positive probability. In fact, when $a = 0$, we can compute explicitly the asymptotic behaviour of the probability of explosion.

When a Brownian random environment affects the stable CB-process, the process behaves completely different. In fact, it also explodes with positive probability but we have three different regimes of the asymptotic behaviour of the non-explosion probability that depends on the parameters of the random environment. Up to our knowledge, this behaviour was never observed or studied before. We call these regimes *subcritical-explosion*, *critical-explosion* or *supercritical-explosion* depending on whether this probability stays positive, converge to zero polynomially fast or converges to zero exponentially fast.

Let

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c_\beta}\right)^{1/\beta},$$

and define

$$g(x) := \exp\left\{-\mathbf{k}x^{1/\beta}\right\}, \qquad \text{for} \qquad x \geq 0.$$

From identity (18) and the scaling property, we deduce for $\beta \in (-1,0)$ and $\eta > -1$,

$$\mathbb{P}_z\left(Z_t < \infty\right) = \mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right] = \int_0^\infty g(z^\beta v)p_{t\sigma_e^2/4,\eta}(v)\mathrm{d}v, \tag{26}$$

where $p_{\nu,\eta}$ denotes the density function of $1/2I^{(\eta)}_\nu$ which according to Matsumoto and Yor [19], satisfies

$$p_{\nu,\eta}(x) = \frac{e^{-\eta^2\nu/2}e^{\pi^2/2\nu}}{\sqrt{2}\pi^2\sqrt{\nu}}\Gamma\left(\frac{\eta+2}{2}\right)e^{-x}x^{-(\eta+1)/2}\int_0^\infty\int_0^\infty e^{\xi^2/2\nu}s^{(\eta-1)/2}e^{-xs} \times \frac{\sinh(\xi)\cosh(\xi)\sin(\pi\xi/\nu)}{(s+\cosh(\xi)^2)^{\frac{\eta+2}{2}}}\mathrm{d}\xi\mathrm{d}s. \tag{27}$$

We also denote

$$\mathcal{L}_{\eta,\beta}(\theta) = \mathbb{E}\Big[e^{-\theta\Gamma^{1/\beta}_{-\eta}}\Big], \qquad \text{for} \quad \theta \geq 0.$$

Theorem 2. *Let $(Z_t, t \geq 0)$ be the stable CBBRE with index $\beta \in (-1,0)$ defined by the SDE (16) with $Z_0 = z > 0$.*

i) Subcritical-explosion. If $\mathbf{m} < 0$, then

$$\lim_{t\to\infty}\mathbb{P}_z\left(Z_t < \infty\right) = \mathcal{L}_{\eta,\beta}\left(z\mathbf{k}\right). \tag{28}$$

ii) Critical-explosion. If $\mathbf{m} = 0$, then

$$\lim_{t\to\infty}\sqrt{t}\ \mathbb{P}_z\left(Z_t < \infty\right) = -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\int_0^\infty e^{-z\mathbf{k}x^{1/\beta}-x}\frac{\mathrm{d}x}{x}. \tag{29}$$

iii) Supercritical-explosion. If $\mathbf{m} > 0$, then

$$\lim_{t\to\infty}t^{\frac{3}{2}}e^{\frac{\mathbf{m}^2 t}{2\sigma^2}}\mathbb{P}_z\left(Z_t < \infty\right) = -\frac{8}{\beta^3\sigma^3}\int_0^\infty g(z^\beta v)\phi_\eta(v)\mathrm{d}v, \tag{30}$$

where

$$\phi_\eta(v) = \int_0^\infty\int_0^\infty \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{\eta+2}{2}\right)e^{-v}v^{-\eta/2}u^{(\eta-1)/2}e^{-u}\frac{\sinh(\xi)\cosh(\xi)\xi}{(u+v\cosh(\xi)^2)^{\frac{\eta+2}{2}}}\mathrm{d}\xi\mathrm{d}u.$$

Proof. Our arguments follows from similar reasoning as in the proof of Theorem 5 in Böinghoff and Hutzenthaler [6]. For this reason, following the same notation as in [6], we just provide the fundamental ideas of the proof.

The subcritical-explosion case (i) follows from the identity in law by Dufresne (25). More precisely, from (25), (26) and the Dominated Convergence Theorem, we deduce

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}_z\left(Z_t<\infty\right)&=\mathbb{E}\left[\exp\left\{-z\mathbf{k}\Gamma_{-\eta}^{1/\beta}\right\}\right]\\&=\mathcal{L}_{\eta,\beta}\left(z\mathbf{k}\right).\end{aligned}$$

In order to prove the critical-explosion case (ii), we use Lemma 12 in [6]. From identity (26) and applying Lemma 12 in [6] to

$$g(z^\beta x)=\exp\left\{-z\mathbf{k}x^{1/\beta}\right\}\leq\frac{x^{-1/\beta}}{z\mathbf{k}},\qquad\text{for}\quad x\geq 0,\tag{31}$$

we get

$$\begin{aligned}\lim_{t\to\infty}\sqrt{t}\ \mathbb{P}_z\left(Z_t<\infty\right)&=-\frac{2}{\beta\sigma}\lim_{t\to\infty}\sqrt{\frac{t\sigma^2\beta^2}{4}}\ \mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2t/4}}\right)\right]\\&=-\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\int_0^\infty e^{-z\mathbf{k}x^{1/\beta}-x}\frac{\mathrm{d}x}{x},\end{aligned}$$

which is finite since the inequality (31) holds.

We now consider the supercritical-explosion case (iii). Observe that for all $n\geq 0$,

$$g(z^\beta x)=\exp\left\{-z\mathbf{k}x^{1/\beta}\right\}\leq\frac{x^{-n/\beta}}{n!(z\mathbf{k})^n},\qquad\text{for}\quad x\geq 0.$$

Therefore using the above inequality for a fixed n, Lemma 13 in [6] and identity (26), we obtain that for $0<\mathbf{m}<n\sigma^2/2$, the following limit holds

$$\begin{aligned}\lim_{t\to\infty}t^{3/2}\ e^{\mathbf{m}^2t/2\sigma^2}\mathbb{P}_z\left(Z_t>0\right)&=\lim_{t\to\infty}t^{3/2}\ e^{\eta^2\beta^2\sigma^2t/8}\mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2t/4}}\right)\right]\\&=-\frac{8}{\beta^3\sigma^3}\int_0^\infty e^{-z\mathbf{k}v^{1/\beta}}\phi_\eta(v)\mathrm{d}v,\end{aligned}$$

where ϕ_η is defined as in the statement of the Theorem. Since this limit holds for any $n\geq 1$, we deduce that it must hold for $\mathbf{m}>0$. This completes the proof. □

4.3 Survival probability

Throughout this section, we assume that $\beta\in(0,1)$. One of the aims of this section is to compute the asymptotic behaviour of the survival probability and we will see that it depends on the value of $\mathbf{m}$. We find five different regimes as in the Feller case (see for instance Theorem 5 in [6]) and CB processes with catastrophes (see for instance Proposition

5 in [1]). Recall that in the classical theory of branching processes, the survival probability stays positive, converges to zero polynomially fast or converges to zero exponentially fast, depending of whether the process is *supercritical* ($\mathbf{m} > 0$), *critical* ($\mathbf{m} = 0$) or *subcritical* ($\mathbf{m} < 0$), respectively. In the stable CBBRE there is another phase transition in the subcritical regime. This phase transition occurs when $\mathbf{m} = -\sigma^2$. We say that the stable CBBRE is *weakly subcritical* if $-\sigma^2 < \mathbf{m} < 0$, *intermediately subcritical* if $\mathbf{m} = -\sigma^2$ and *strongly subcritical* if $\mathbf{m} < -\sigma^2$.

Recall that

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c_\beta}\right)^{1/\beta},$$

and define

$$f(x) := 1 - \exp\left\{-\mathbf{k}x^{1/\beta}\right\}, \qquad \text{for} \qquad x \geq 0.$$

From identity (23) and the scaling property, we deduce for $\beta > 0$ and $\eta > -1$,

$$\mathbb{P}_z\left(Z_t > 0\right) = \mathbb{E}\left[f\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right] = \int_0^\infty f(z^\beta v)p_{\beta^2\sigma^2 t/4,\eta}(v)\mathrm{d}v, \tag{32}$$

where $p_{\nu,\eta}$ denotes the density function of $1/2I_\nu^{(\eta)}$ and is given in (27).

Theorem 3. *Let $(Z_t, t \geq 0)$ be the stable CBBRE with index $\beta \in (0,1)$ defined by the SDE (16) with $Z_0 = z > 0$.*

i) Supercritical. If $\mathbf{m} > 0$, then

$$\lim_{t\to\infty}\mathbb{P}_z\left(Z_t > 0\right) = 1 - \sum_{n=0}^\infty \frac{(-z\mathbf{k})^n}{n!}\frac{\Gamma(\frac{n}{\beta} - \eta)}{\Gamma(-\eta)}. \tag{33}$$

ii) Critical. If $\mathbf{m} = 0$, then

$$\lim_{t\to\infty}\sqrt{t}\ \mathbb{P}_z\left(Z_t > 0\right) = -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\sum_{n=1}^\infty\frac{(-z\mathbf{k})^n}{n!}\Gamma\left(\frac{n}{\beta}\right). \tag{34}$$

iii) Weakly subcritical. If $\mathbf{m} \in (-\sigma^2, 0)$, then

$$\lim_{t\to\infty} t^{\frac{3}{2}}e^{\frac{\mathbf{m}^2 t}{2\sigma^2}}\mathbb{P}_z\left(Z_t > 0\right) = \frac{8}{\beta^3\sigma^3}\int_0^\infty f(z^\beta v)\phi_\eta(v)\mathrm{d}v, \tag{35}$$

where

$$\phi_\eta(v) = \int_0^\infty\int_0^\infty \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{\eta+2}{2}\right)e^{-v}v^{-\eta/2}u^{(\eta-1)/2}e^{-u}\frac{\sinh(\xi)\cosh(\xi)\xi}{(u + v\cosh(\xi)^2)^{\frac{\eta+2}{2}}}\mathrm{d}\xi\mathrm{d}u.$$

iv) Intermediately subcritical. If $\mathbf{m} = -\sigma^2$, then

$$\lim_{t\to\infty}\sqrt{t}e^{\sigma^2 t/2}\mathbb{P}_z\left(Z_t > 0\right) = z\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\frac{1}{\beta}\right). \tag{36}$$

v) Strongly subcritical. If $\mathbf{m} < -\sigma^2$*, then*

$$\lim_{t\to\infty} e^{-\frac{1}{2}(2\mathbf{m}+\sigma^2)t}\mathbb{P}_z\left(Z_t > 0\right) = z\mathbf{k}\frac{\Gamma(\eta - 1/\beta)}{\Gamma(\eta - 2/\beta)}. \tag{37}$$

Proof. As in the proof of Theorem 2, and following the same notation as in [6], we just provide the fundamental ideas of the proof.

The supercritical case (i) follows from the identity in law by Dufresne (25). More precisely, from (25), (32) and the Dominated Convergence Theorem, we deduce

$$\lim_{t\to\infty}\mathbb{P}_z\left(Z_t > 0\right) = \mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\Gamma_{-\eta}^{1/\beta}\right\}\right] = 1 - \sum_{n=0}^{\infty}(-z\mathbf{k})^n\frac{\Gamma(\frac{n}{\beta}-\eta)}{n!\Gamma(-\eta)}.$$

In order to prove the critical case (ii), we use Lemma 12 in [6]. From identity (32) and applying Lemma 12 in [6] to

$$g(x) := 1 - \exp\left\{-z\mathbf{k}x^{1/\beta}\right\} \le z\mathbf{k}x^{1/\beta}, \quad x \ge 0, \tag{38}$$

we get

$$\begin{aligned}\lim_{t\to\infty}\sqrt{t}\ \mathbb{P}_z\left(Z_t > 0\right) &= \frac{2}{\beta\sigma}\lim_{t\to\infty}\sqrt{\frac{t\sigma^2\beta^2}{4}}\ \mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2 t/4}\right)^{-1/\beta}\right\}\right]\\ &= \frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\int_0^\infty\left(1 - \exp\left\{-z\mathbf{k}x^{1/\beta}\right\}\right)\frac{e^{-x}}{x}\mathrm{d}x.\end{aligned}$$

By Fubini's theorem, it is easy to show that, for all $q \ge 0$

$$\int_0^\infty\left(1 - e^{-qx^{1/\beta}}\right)\frac{e^{-x}}{x}\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\Gamma\left(\frac{n}{\beta}\right)q^n,$$

which implies (34).

We now consider the weakly subcritical case (iii). Recall that inequality (38) still holds, then using Lemma 13 in [6] and identity (32), we obtain

$$\begin{aligned}\lim_{t\to\infty}t^{3/2}\ e^{\mathbf{m}^2 t/2\sigma^2}\mathbb{P}_z\left(Z_t > 0\right) &= \lim_{t\to\infty}t^{3/2}\ e^{\eta^2\beta^2\sigma^2 t/8}\mathbb{E}\left[g\left(\frac{z^\beta}{2I^{(\eta)}_{\beta^2\sigma^2 t/4}}\right)\right]\\ &= \frac{8}{\beta^3\sigma^3}\int_0^\infty\left(1 - \exp\left\{-z\mathbf{k}v^{1/\beta}\right\}\right)\phi_\eta(v)\mathrm{d}v,\end{aligned}$$

where ϕ_η is defined as in the statement of the Theorem.

In the remaining two cases we will use Lemma 9 in [6] and Lemma 1. For the intermediately subcritical case (iv), we observe that $\eta = 2/\beta$. Hence, applying Lemma 1 with $p = 1/\beta$, we get

$$\mathbb{E}\left[\left(I^{(\eta)}_t\right)^{-1/\beta}\right] = e^{-\frac{2}{\beta^2}t}\mathbb{E}\left[\left(I^{(0)}_t\right)^{-1/\beta}\right] \quad\text{and}\quad \mathbb{E}\left[\left(I^{(\eta)}_t\right)^{-2/\beta}\right] \le e^{-\frac{2}{\beta^2}t}\mathbb{E}\left[\left(I^{(0)}_{t/2}\right)^{-1/\beta}\right]^2.$$

Now, applying Lemma 12 in [6], we deduce

$$\lim_{t\to\infty}\sqrt{t}\ \mathbb{E}\left[\left(2I_t^{(\eta)}\right)^{-1/\beta}\right] = \int_0^\infty \frac{1}{\sqrt{2\pi}}\frac{e^{-a}}{a}a^{1/\beta}da = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{\beta}\right),$$

and

$$\lim_{t\to\infty}\sqrt{t}\ \mathbb{E}\left[\left(2I_t^{(\eta)}\right)^{-2/\beta}\right] = 0.$$

Therefore, we can apply Lemma 9 in [6] with $c_t = \sqrt{t}\, e^{2t}$ and $Y_t = \left(2I_t^{(\eta)}\right)^{-1/\beta}$, and obtain

$$\begin{aligned}
\lim_{t\to\infty}\sqrt{t}e^{\sigma^2 t/2}\mathbb{P}_z\left(Z_t>0\right) &= \lim_{t\to\infty}\sqrt{t}e^{\sigma^2 t/2}\mathbb{E}\left[1-\exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2 t/4}\right)^{-1/\beta}\right\}\right]\\
&= \lim_{t\to\infty}\sqrt{t}e^{\sigma^2 t/2}z\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\beta^2\sigma^2 t/4}\right)^{-1/\beta}\right]\\
&= z\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\frac{1}{\beta}\right).
\end{aligned}$$

Finally for the strongly subcritical case, we use again Lemma 1 with $p = 1/\beta$. First observe that $\eta - 2/\beta > 0$. Thus, the Monotone Convergence Theorem and the identity of Dufresne (25) yield

$$\lim_{t\to\infty}\mathbb{E}\left[\left(2I_{t/2}^{(-(\eta-2/\beta))}\right)^{-1/\beta}\right] = \mathbb{E}\left[\left(2I_\infty^{(-(\eta-2/\beta))}\right)^{-1/\beta}\right] = \mathbb{E}\left[\left(\Gamma_{\eta-2/\beta}\right)^{1/\beta}\right] = \frac{\Gamma(\eta-1/\beta)}{\Gamma(\eta-2/\beta)}.$$

Since $I_t^{(\eta-2/\beta)}$ goes to ∞ as t increases, from the Monotone Convergence Theorem, we get

$$\lim_{t\to\infty}\mathbb{E}\left[\left(2I_{t/2}^{(\eta-2/\beta)}\right)^{-1/\beta}\right] = 0.$$

Hence by applying Lemma 9 in [6] with $c_t = e^{-(2/\beta^2-2\eta/\beta)t}$ and $Y_t = \left(2I_t^{(\eta)}\right)^{-1/\beta}$ we obtain that

$$\begin{aligned}
\lim_{t\to\infty}e^{-\frac{1}{2}(2\mathbf{m}+\sigma^2)t}\mathbb{P}_z\left(Z_t>0\right) &= \lim_{t\to\infty}c_{\beta^2\sigma^2 t/4}\mathbb{E}\left[1-\exp\left\{-\mathbf{k}\left(2I^{(\eta)}_{\beta^2\sigma^2 t/4}\right)^{-1/\beta}\right\}\right]\\
&= z\mathbf{k}\frac{\Gamma(\eta-1/\beta)}{\Gamma(\eta-2/\beta)}.
\end{aligned}$$

This completes the proof. □

4.4 Conditioned stable CBBRE

Here, we are interested in studying two conditioned versions of the stable CBBRE: the process conditioned to be never extinct (or Q-process) and the process conditioned on eventual extinction. Our methodology follows similar arguments as those used in Lambert [17] and extend the results obtained by Böinghoff and Hutzenthaler [6] and Hutzenthaler [10] in the stable case with $\beta \in (0,1)$.

4.4.1 The process conditioned to be never extinct

In order to study the stable CBBRE conditioned to be never extinct, we need the following Lemma.

Lemma 2. *For every $t \geq 0$, Z_t is integrable.*

Proof. Differentiating the Laplace transform (17) of Z_t in λ and taking limits as $\lambda \downarrow 0$, on both sides, we deduce

$$\mathbb{E}_z\left[Z_t \middle| K\right] = z e^{\mathbf{m}t + \sigma B_t},$$

which is an integrable random variable. □

Recall that

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c_\beta}\right)^{1/\beta}.$$

We now define the function $U : [0, \infty) \to (0, \infty)$ as follows

$$U(z) = \begin{cases} -\frac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\displaystyle\sum_{n=1}^{\infty}\frac{(-z\mathbf{k})^n}{n!}\Gamma\left(\frac{n}{\beta}\right) & \text{if } \mathbf{m} = 0, \\ \frac{8}{\beta^3\sigma^3}\displaystyle\int_0^\infty \left(1 - e^{-z\mathbf{k}v^{1/\beta}}\right)\phi_\eta(v)\mathrm{d}v & \text{if } \mathbf{m} \in (-\sigma^2, 0), \\ z\dfrac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\dfrac{1}{\beta}\right) & \text{if } \mathbf{m} = -\sigma^2, \\ z\mathbf{k}\dfrac{\Gamma(\eta - 1/\beta)}{\Gamma(\eta - 2/\beta)} & \text{if } \mathbf{m} < -\sigma^2, \end{cases}$$

where the function ϕ_η is given as in Theorem 3. We also introduce

$$\theta := \theta(\mathbf{m}, \sigma) = \begin{cases} 0 & \text{if } \mathbf{m} = 0, \\ \dfrac{\mathbf{m}^2}{2\sigma^2} & \text{if } \mathbf{m} \in (-\sigma^2, 0), \\ -\dfrac{2\mathbf{m} + \sigma^2}{2} & \text{if } \mathbf{m} \leq -\sigma^2. \end{cases}$$

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by Z and $T_0 = \inf\{t \geq 0 : Z_t = 0\}$ be the extinction time of the process Z. The next Proposition states, in the critical and subcritical cases, the existence of the Q-process.

Proposition 3. *Let $(Z_t, t \geq 0)$ be the stable CBBRE with index $\beta \in (0, 1)$ defined by the SDE (16) with $Z_0 = z > 0$. Then for $\mathbf{m} \leq 0$:*

i) The conditional laws $\mathbb{P}_z(\cdot \mid T_0 > t + s)$ converge as $s \to \infty$ to a limit denoted by $\mathbb{P}_z^\natural$, in the sense that for any $t \geq 0$ and $\Lambda \in \mathcal{F}_t$,

$$\lim_{s\to\infty}\mathbb{P}_z(\Lambda \mid T_0 > t + s) = \mathbb{P}_z^\natural(\Lambda).$$

ii) The probability measure $\mathbb{P}^\natural$ can be expressed as an h-transform of $\mathbb{P}$ based on the martingale

$$D_t = e^{\theta t} U(Z_t),$$

in the sense that

$$\mathrm{d}\mathbb{P}^\natural_z\big|_{\mathcal{F}_t} = \frac{D_t}{U(z)} \mathrm{d}\mathbb{P}_z\big|_{\mathcal{F}_t}.$$

Proof. We first prove part (i). Let $z, s, t > 0$, and $\Lambda \in \mathcal{F}_t$. From the Markov property, we observe

$$\mathbb{P}_z\left(\Lambda \mid T_0 > t+s\right) = \frac{\mathbb{P}_z\Big(\Lambda; T_0 > t+s\Big)}{\mathbb{P}_z\left(T_0 > t+s\right)} = \mathbb{E}_z\left[\frac{\mathbb{P}_{Z_t}\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)}; \Lambda, T_0 > t\right]. \tag{39}$$

On the other hand, since the mapping $t \mapsto I_t^{(\eta)}$ is increasing and the function $f(x) = 1 - \exp\left\{-\mathbf{k}x^{1/\beta}\right\}$ is decreasing, we deduce from (32) and the Markov property that for any $z, y > 0$,

$$\begin{aligned}
0 \le \frac{\mathbb{P}_y\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)} &= \frac{\mathbb{E}\left[1 - \exp\left\{-y\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right\}\right]}{\mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 (t+s)/4}\right)^{-1/\beta}\right\}\right]} \\
&\le \frac{y\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right]}{\mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right\}\right]}.
\end{aligned}$$

Moreover, since $I_t^{(\eta)}$ diverge as t goes to ∞, we have

$$\frac{z\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right]}{2} \le \mathbb{E}\left[1 - \exp\left\{-z\mathbf{k}\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right\}\right] \le z\mathbf{k}\mathbb{E}\left[\left(2I^{(\eta)}_{\sigma^2\beta^2 s/4}\right)^{-1/\beta}\right],$$

for s sufficiently large. Then for any s greater than some bound chosen independently of $Z_t(\omega)$, we necessarily have

$$0 \le \frac{\mathbb{P}_{Z_t}\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)} \le \frac{2}{z} Z_t.$$

Now, from the asymptotic behaviour (34), (35), (36) and (37), we get

$$\lim_{s\to\infty} \frac{\mathbb{P}_{Z_t}\left(Z_s > 0\right)}{\mathbb{P}_z\left(Z_{t+s} > 0\right)} = \frac{e^{\theta t} U(Z_t)}{U(z)}.$$

Hence, Dominated Convergence and identity (39) imply

$$\lim_{s\to\infty} \mathbb{P}_z\left(\Lambda \mid T_0 > t+s\right) = \mathbb{E}_z\left[\frac{e^{\theta t} U(Z_t)}{U(z)}, \Lambda\right]. \tag{40}$$

Next, we prove part (ii). In order to do so, we use (40) with $\Lambda = \Omega$ to deduce

$$\mathbb{E}_z\left[e^{\theta t} U(Z_t)\right] = U(z).$$

Therefore from the Markov property, we obtain

$$\mathbb{E}_z\left[e^{\theta(t+s)} U(Z_{t+s}) \middle| \mathcal{F}_s\right] = e^{\theta s}\mathbb{E}_{Z_s}\left[e^{\theta t} U(Z_t)\right] = e^{\theta s} U(Z_s),$$

implying that D is a martingale. □

4.4.2 The process conditioned on eventual extinction

Here, we assume that $\mathbf{m} > 0$ and define for $z > 0$.

$$U_*(z) := \sum_{n=0}^{\infty} \frac{(-z\mathbf{k})^n}{n!} \frac{\Gamma(n/\beta - \eta)}{\Gamma(-\eta)}.$$

In the supercritical case we are interested in the process conditioned on eventual extinction.

Proposition 4. *Let $(Z_t, t \geq 0)$ be the stable CBBRE with index $\beta \in (0,1)$ defined by the SDE (16) with $Z_0 = z > 0$. Then for $\mathbf{m} > 0$, the conditional law*

$$\mathbb{P}^*_z(\cdot) = \mathbb{P}_z\left(\cdot \mid T_0 < \infty\right),$$

satisfies for any $t \geq 0$,

$$\mathrm{d}\mathbb{P}^*_z\big|_{\mathcal{F}_t} = \frac{U_*(Z_t)}{U_*(z)} \mathrm{d}\mathbb{P}_z\big|_{\mathcal{F}_t}.$$

Moreover, $(U_*(Z_t), t \geq 0)$ is a martingale.

Proof. Let z, $t \geq 0$ and $\Lambda \in \mathcal{F}_t$, then

$$\mathbb{P}_z\left(\Lambda \middle| T_0 < \infty\right) = \frac{\mathbb{P}_z\left(\Lambda, T_0 < \infty\right)}{\mathbb{P}_z\left(T_0 < \infty\right)} = \lim_{s\to\infty} \frac{\mathbb{P}_z\left(\Lambda, Z_{t+s} = 0\right)}{U_*(z)}.$$

On the other hand, the Markov property implies

$$\mathbb{P}_z\left(\Lambda, Z_{t+s} = 0\right) = \mathbb{E}_z\left[\mathbb{P}_z\left(Z_{t+s} = 0 \middle| \mathcal{F}_t\right), \Lambda\right] = \mathbb{E}_z\left[\mathbb{P}_{Z_t}\left(Z_s = 0\right), \Lambda\right].$$

Therefore using the Dominated Convergence Theorem, we deduce

$$\mathbb{P}_z\left(\Lambda \middle| T_0 < \infty\right) = \lim_{s\to\infty} \frac{\mathbb{E}_z\left[\mathbb{P}_{Z_t}\left(Z_s = 0\right), \Lambda\right]}{U_*(z)} = \frac{\mathbb{E}_z\left[U_*(Z_t), \Lambda\right]}{U_*(z)}.$$

The proof that $(U_*(Z_t), t \geq 0)$ is a martingale follows from the same argument as in the proof of part (ii) of Proposition 3. □

5 The immigration case.

In this section, we introduce continuous state branching processes with immigration in a Brownian random environment. In particular, this class of processes is an extension of the Cox-Ingersoll-Ross model in a random environment. For simplicity, we introduce such class of processes under the assumption that the branching mechanism posses finite mean.

Recall that a CB-process with immigration (or CBI-process) is a strong Markov process taking values in $[0,\infty]$, where 0 is no longer an absorbing state. It is characterized by its branching mechanism,

$$\psi(u) = -au + \gamma^2 u^2 + \int_{(0,\infty)} (e^{-ux} - 1 + ux)\mu(\mathrm{d}x),$$

and its immigration mechanism,

$$\phi(u) = \mathtt{d}u + \int_0^\infty (1 - e^{-ut})\nu(\mathrm{d}t),$$

where $a \in \mathbb{R}$, $\gamma, \mathtt{d} \geq 0$ and

$$\int_0^\infty (x \wedge x^2)\mu(\mathrm{d}x) + \int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < \infty. \tag{41}$$

It is well-known that if $(Y_t, t \geq 0)$ is a process in this class, then its semi-group is characterized by

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\left\{-xu_t(\lambda) - \int_0^t \phi(u_s)\mathrm{d}s\right\}, \qquad \text{for} \quad \lambda \geq 0,$$

where u_t solves

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)), \qquad u_0(\lambda) = \lambda.$$

According to Fu and Li [8], a CBI-process can be defined as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Y_t = Y_0 + &\int_0^t (\mathtt{d} + aY_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s \\ &+ \int_0^t \int_{(0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,\infty)} zM(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $M(\mathrm{d}s, \mathrm{d}z)$ are two independent Poisson random measures with intensities $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ and $\mathrm{d}s\nu(\mathrm{d}z)$, respectively, satisfying the integral condition (41) and $\widetilde{N}$ is the compensated measure of N. The processes B, N and M are mutually independent.

Motivated from the definition of CBBRE, we introduce a continuous state branching process with immigration in a Brownian random environment (in short a CBIBRE) as the unique non-negative strong solution of the stochastic differential equation

$$\begin{aligned} Z_t =& Z_0 + \int_0^t (\mathtt{d} + aZ_s)\,\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s + \int_0^t Z_s \mathrm{d}S_s \\ &+ \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,\infty)} zM(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{42}$$

where $S_t = \alpha t + \sigma B_t^{(e)}$, with $\alpha \in \mathbb{R}$, $\sigma > 0$ and $(B_t^{(e)}, t \geq 0)$ is a standard Brownian motion independent of B and the Poisson random measures N and M. Similarly as in the case with no immigration, we define the auxiliary process

$$K_t^{(0)} = \sigma B_t^{(e)} + \mathbf{m}t, \qquad \text{for} \qquad t \geq 0,$$

where

$$\mathbf{m} = \alpha + a - \frac{\sigma^2}{2}.$$

The following Theorem provides the existence of the CBIBRE as the unique strong solution of (42). We defer its proof to the Appendix since it follows from similar arguments as those used in Theorem 1.

Theorem 4. *The stochastic differential equation (42) has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and its infinitesimal generator $\mathcal{A}$ satisfies, for every $f \in C^2(\mathbb{R}_+)$,*

$$\begin{aligned} \mathcal{A}f(x) =& \frac{1}{2}\sigma^2 x^2 f''(x) + \Big(x(a+\alpha) + \mathtt{d}\Big) f'(x) + x\gamma^2 f''(x) \\ &+ \int_{(0,\infty)} (f(x+z) - f(x))\nu(\mathrm{d}z) + x\int_{(0,\infty)} \left(f(x+z) - f(x) - zf'(x)\right)\mu(\mathrm{d}z). \end{aligned} \tag{43}$$

Furthermore, the process Z, conditioned on $K^{(0)}$, satisfies the branching property and for every $t > 0$

$$\begin{aligned} &\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t^{(0)}}\right\} \big| K^{(0)}\right] \\ &\qquad = \exp\left\{-z v_t(0, \lambda, K^{(0)}) - \int_0^t \phi\Big(v_t(r, \lambda, K^{(0)}) e^{-K_r^{(0)}}\Big)\mathrm{d}r\right\} \quad a.s., \end{aligned} \tag{44}$$

where for every $(\lambda, \delta) \in (\mathbb{R}_+, C(\mathbb{R}_+))$, $v_t : s \in [0,t] \mapsto v_t(s, \lambda, \delta)$ is the unique solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = e^{\delta_s}\psi_0(v_t(s, \lambda, \delta)e^{-\delta_s}), \qquad v_t(t, \lambda, \delta) = \lambda, \tag{45}$$

and $\psi_0(\lambda) = \psi(\lambda) + a\lambda$, for $\lambda \geq 0$.

We finish this section with some examples of CBIBRE, all of them related to the stable CBBRE.

Example 4. (Stable CBIBRE) Here we assume that the branching and immigration mechanisms are of the form $\psi(\lambda) = -a\lambda + c\lambda^{\beta+1}$ and $\phi(\lambda) = \kappa\lambda^{\beta}$, where $\beta \in (0,1)$, $c, \kappa > 0$ and $a \in \mathbb{R}$. Hence, the stable CBIBRE-process is given as the unique non-negative strong solution of the stochastic differential equation

$$Z_t = Z_0 + \int_0^t aZ_s\mathrm{d}s + \int_0^t Z_s\mathrm{d}S_s + \int_0^t\int_0^\infty\int_0^{Z_{s-}} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t\int_0^\infty zM(\mathrm{d}s,\mathrm{d}z),$$

where $S_t = \alpha t + \sigma^2 B_t^{(e)}$ is a Brownian motion with drift, and N and M are two independent Poisson random measures with intensities

$$\frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\frac{1}{z^{2+\beta}}\mathrm{d}s\mathrm{d}z\mathrm{d}u \qquad \text{and} \qquad \frac{\kappa\beta}{\Gamma(1-\beta)}\frac{1}{z^{1+\beta}}\mathrm{d}s\mathrm{d}z.$$

Its associated infinitesimal generator $\mathcal{A}$ satisfies, for every $f \in C^2(\mathbb{R}_+)$,

$$\begin{aligned}\mathcal{A}f(x) = \frac{1}{2}\sigma^2x^2f''(x) + x(a+\alpha)f'(x) + \frac{\kappa\beta}{\Gamma(1-\beta)}\int_{(0,\infty)}(f(x+z)-f(x))\frac{\mathrm{d}z}{z^{1+\beta}}\\ + x\frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\int_{(0,\infty)}\left(f(x+z)-f(x)-zf'(x)\right)\frac{\mathrm{d}z}{z^{2+\beta}}.\end{aligned} \quad (46)$$

From (44) we get the following a.s. identity

$$\begin{aligned}\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t^{(0)}}\right\}\Big|K^{(0)}\right] = &\exp\left\{-z\left(\beta c\int_0^t e^{-\beta K_s^{(0)}}\mathrm{d}s + \lambda^{-\beta}\right)^{-1/\beta}\right\}\\ &\times\exp\left\{-\frac{\kappa}{\beta c}\ln\left(\beta c\lambda^\beta\int_0^t e^{-\beta K_s^{(0)}}\mathrm{d}s + 1\right)\right\}.\end{aligned} \quad (47)$$

If we take limits as z goes to 0, we deduce that the entrance law at 0 of the process $(Z_te^{-K_t^{(0)}}, t \geq 0)$ satisfies

$$\mathbb{E}_0\left[\exp\left\{-\lambda Z_t e^{-K_t^{(0)}}\right\}\Big|K^{(0)}\right] = \exp\left\{-\frac{\kappa}{\beta c}\ln\left(\beta c\lambda^\beta\int_0^t e^{-\beta K_s^{(0)}}\mathrm{d}s + 1\right)\right\}.$$

If we take limits as λ goes to ∞ in (47), we obtain

$$\mathbb{P}_z\Big(Z_t > 0\Big|K^{(0)}\Big) = 1, \qquad \text{for} \quad z \geq 0.$$

Similarly if we take limits as λ goes to 0 in (47), we deduce

$$\mathbb{P}_z\Big(Z_t < \infty\Big|K^{(0)}\Big) = 1, \qquad \text{for} \quad z \geq 0.$$

In other words, the stable CBIBRE is conservative and positive at finite time a.s.

An interesting question is to study the long-term behaviour of the stable CBIBRE. Now, if we take limits as t goes to ∞ in (47), we deduce that when $\mathbf{m} > 0$,

$$\begin{aligned}\mathbb{E}_z\left[\exp\left\{-\lambda\lim_{t\to\infty} Z_t e^{-K_t^{(0)}}\right\}\right] = &\mathbb{E}\Bigg[\exp\left\{-z\left(\beta c\int_0^\infty e^{-\beta K_s^{(0)}}\mathrm{d}s + \lambda^{-\beta}\right)^{-1/\beta}\right\}\\ &\times\exp\left\{-\frac{\kappa}{\beta c}\ln\left(\beta c\lambda^\beta\int_0^\infty e^{-\beta K_s^{(0)}}\mathrm{d}s + 1\right)\right\}\Bigg],\end{aligned}$$

where we recall that $\int_0^\infty e^{-\beta K_s^{(0)}}\mathrm{d}s$ has the same law as $\Big(2\Gamma_{-\eta}\Big)^{-1}$, with Γ_v is a Gamma r.v. with parameter v. In other words, $Z_t e^{-K_t^{(0)}}$ converges in distribution to a r.v. whose Laplace transform is given by the previous identity.

If $\mathbf{m} \le 0$, we deduce

$$\lim_{t\to\infty} Z_t e^{-K_t^{(0)}} = \infty, \qquad \mathbb{P}_z - \text{a.s.}$$

We observe that when $\mathbf{m} = 0$, the process $K^{(0)}$ oscillates implying that

$$\lim_{t\to\infty} Z_t = \infty, \qquad \mathbb{P}_z - \text{a.s.}$$

Example 5. (Stable CBBRE Q-process). Here, we assume $\mathbf{m} \le -\sigma^2$. Recall that

$$\eta := -\frac{2}{\beta\sigma^2}\mathbf{m} \qquad \text{and} \qquad \mathbf{k} = \left(\frac{\beta\sigma^2}{2c}\right)^{1/\beta}.$$

From Proposition 3, we deduce that the form of the infinitesimal generator of the stable CBBRE Q-process, here denoted by $\mathcal{L}^\natural$, satisfies for $f \in \mathrm{Dom}(\mathcal{L})$

$$\begin{aligned}\mathcal{L}^\natural f(z) &= \mathcal{L}f(z) + z^2\sigma^2 f'(z)\frac{U'(z)}{U(z)} \\ &\quad + \frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\frac{z}{U(z)}\int_0^\infty \Big(f(y+z) - f(z)\Big)\Big(U(z+y) - U(z)\Big)\frac{\mathrm{d}y}{y^{2+\beta}},\end{aligned}$$

with

$$U(z) = \begin{cases} z\dfrac{\sqrt{2}}{\sqrt{\pi}\beta\sigma}\mathbf{k}\Gamma\left(\dfrac{1}{\beta}\right) & \text{if } \mathbf{m} = -\sigma^2, \\ z\mathbf{k}\dfrac{\Gamma(\eta - 1/\beta)}{\Gamma(\eta - 2/\beta)} & \text{if } \mathbf{m} < -\sigma^2. \end{cases}$$

Replacing the form of U in the infinitesimal generator $\mathcal{L}^\natural$ in both cases, we get

$$\begin{aligned}\mathcal{L}^\natural f(z) = \frac{1}{2}\sigma^2 z^2 f''(z) + (a + \alpha + \sigma^2)zf'(z) + \frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\int_0^\infty \Big(f(z+x) - f(x)\Big)\frac{\mathrm{d}z}{z^{1+\beta}} \\ + z\frac{c\beta(\beta+1)}{\Gamma(1-\beta)}\int_{(0,\infty)} \left(f(x+z) - f(z) - xf'(z)\right)\frac{\mathrm{d}x}{x^{2+\beta}}.\end{aligned}$$

From the form of the infinitesimal generator of the stable CBIBRE (46), we deduce that the stable CBBRE Q-process is a stable CBIBRE with branching and immigration mechanisms given by

$$\psi(\lambda) = -a\lambda + c\lambda^{1+\beta} \qquad \text{and} \qquad \phi(\lambda) = c(\beta+1)\lambda^\beta,$$

and the random environment $(S_t, t \ge 0)$ satisfies

$$S_t = (\alpha + \sigma^2)t + \sigma B_t^{(e)}, \qquad t \ge 0.$$

6 Appendix

Lemma 3. *The pathwise uniqueness holds for the positive solutions of*

$$Z_t^{(n)} = Z_0^{(n)} + a\int_0^t (Z_s^{(n)} \wedge n)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 (Z_s^{(n)} \wedge n)}\mathrm{d}B_s + \int_0^t (Z_s^{(n)} \wedge n)\mathrm{d}S_s$$
$$+ \int_0^t \int_{(0,1)} \int_0^{(Z_s^{(n)} \wedge n)-} (z \wedge n)\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{(Z_s^{(n)} \wedge n)-} (z \wedge n)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \tag{48}$$

for every $n \in \mathbb{N}$.

Proof. For each integer $k \geq 0$, define $a_k = \exp\{-k(k+1)/2\}$. Let $x \mapsto g_k(x)$ be a positive continuous function supported on the interval (a_k, a_{k-1}) such that $g_k(x) \leq 2(kx)^{-1}$ for every $x > 0$, and

$$\int_{a_k}^{a_{k-1}} g_k(x)\mathrm{d}x = 1.$$

For $k \geq 0$, we also let

$$f_k(z) = \int_0^{|z|} \mathrm{d}y \int_0^y g_k(x)\mathrm{d}x, \qquad z \in \mathbb{R}.$$

Observe that $|f_k'(z)| \leq 1$ and

$$0 \leq |z| f_k''(z) = |z| g_k(|z|) \leq 2k^{-1}, \qquad \text{for} \quad z \in \mathbb{R}.$$

Furthermore, we have $f_k(z) \uparrow |z|$ as $k \to \infty$. Let Z_t and Z_t' be two solutions of (48), and let $Y_t = Z_t - Z_t'$, for $t \geq 0$. Thus, the process $Y = (Y_t, t \geq 0)$ satisfies the following SDE

$$\begin{aligned} Y_t =& Y_0 + (a+\alpha)\int_0^t \big(Z_s \wedge n - Z_s' \wedge n\big)\mathrm{d}s \\ &+ \sqrt{2\gamma^2}\int_0^t \Big(\sqrt{Z_s \wedge n} - \sqrt{Z_s' \wedge n}\Big)\mathrm{d}B_s + \sigma \int_0^t \big(Z_s \wedge n - Z_s' \wedge n\big)\mathrm{d}B_s \\ &+ \int_0^t \int_{(0,1)} \int_{(Z_s' \wedge n)-}^{(Z_s \wedge n)-} (z \wedge n)\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_{(Z_s' \wedge n)-}^{(Z_s \wedge n)-} (z \wedge n)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned}$$

Applying Ito's formula, we deduce

$$\begin{aligned} f_k(Y_t) = & f_k(Y_0) + (a+\alpha)\int_0^t f_k'(Y_s)\big(Z_s \wedge n - Z_s' \wedge n\big)\mathrm{d}s \\ &+ \int_0^t f_k''(Y_s)\left(\gamma^2\Big(\sqrt{Z_s \wedge n} - \sqrt{Z_s' \wedge n}\Big)^2 + \frac{\sigma^2}{2}\big(Z_s \wedge n - Z_s' \wedge n\big)^2\right)\mathrm{d}s \\ &+ \int_0^t \int_{(0,\infty)} \big(Z_s \wedge n - Z_s' \wedge n\big)\Big(f_k(Y_s + z \wedge n) - f_k(Y_s) - f_k'(Y_s)(z \wedge n)\Big)\mathrm{d}s\mu(\mathrm{d}z) \\ &+ \int_0^t \int_{(1,\infty)} \big(Z_s \wedge n - Z_s' \wedge n\big) f_k'(Y_s)(z \wedge n)\mathrm{d}s\mu(\mathrm{d}z) + \text{Martingale}. \end{aligned} \tag{49}$$

Also observe that $|f_k(a+x)-f_k(a)|\le |x|$ for $a,x\in\mathbb{R}$, and for $ax\ge 0$

$$|f_k(a+x)-f_k(a)-zf_k'(a)|\le (2|ax|)\wedge(k^{-1}|x|^2).$$

Now, using the fact that $|(z\wedge n)-(y\wedge n)|\le |z-y|\wedge n$ and taking expectations in both parts of equation (49), we obtain

$$\begin{aligned}\mathbb{E}\left[f_k(Y_t)\right]\le&\mathbb{E}\left[f_k(Y_0)\right]+\Big(|a+\alpha|+n\mu\big([1,\infty)\big)\Big)\int_0^t\mathbb{E}\left[|Y_s|\right]\mathrm{d}s+\left(\gamma^2+\frac{\sigma^2 n}{2}\right)\frac{2t}{k}\\&+\int_0^t\int_{(0,\infty)}n\Big(2(z\wedge n)\mathbb{E}\left[|Y_s|\right]\wedge k^{-1}(z\wedge n)^2\Big)\mathrm{d}s\mu(\mathrm{d}z).\end{aligned}$$

Therefore, taking limits as $k\to\infty$, we deduce

$$\mathbb{E}\left[|Y_t|\right]\le\mathbb{E}\left[|Y_0|\right]+\Big(|a+\alpha|+n\mu\big([1,\infty)\big)\Big)\int_0^t\mathbb{E}\left[|Y_s|\right]\mathrm{d}s.$$

By Gronwall's inequality, we observe that the pathwise uniqueness holds for the positive solutions of (48) for every $n\in\mathbb{N}$. □

Proof of Theorem 4. We just explain the main steps of the proof, since it follows from similar arguments as those used in Theorem 1. We first introduce the local martingale

$$\begin{aligned}C_t=&\int_0^t\mathbf{1}_{\{Z_s=0\}}\mathrm{d}B_s+\int_0^t\mathbf{1}_{\{Z_s=0\}}\mathrm{d}B_s^{(e)}\\&+\int_0^t\mathbf{1}_{\{Z_s>0\}}\frac{\sqrt{2\gamma Z_s}}{\sqrt{2\gamma Z_s+\sigma^2Z_s^2}}\mathrm{d}B_s+\int_0^t\mathbf{1}_{\{Z_s>0\}}\frac{\sigma Z_s}{\sqrt{2\gamma Z_s+\sigma^2Z_s^2}}\mathrm{d}B_s^{(e)}.\end{aligned}$$

By Lévy's characterization theorem, the local martingale $C=(C_t,t\ge 0)$ is a standard Brownian motion which is independent of the Poisson random measures N and M. Then, the SDE (42) can be written as follows

$$\begin{aligned}Z_t=&Z_0+\int_0^t\Big(\mathtt{d}+(a+\alpha)\Big)Z_s\mathrm{d}s+\int_0^t\sqrt{2\gamma^2Z_s+\sigma^2Z_s^2}\mathrm{d}C_s\\&+\int_0^t\int_{(0,\infty)}\int_0^{Z_{s-}}z\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)+\int_0^t\int_{(0,\infty)}zM(\mathrm{d}s,\mathrm{d}z).\end{aligned}\tag{50}$$

Then, from Theorem 5.1 in Fu and Li [8], there exists a unique non-negative strong solution to (50). This implies that there exists a unique non-negative weak solution to (42). By an analogous argument given in Lemma 3, the pathwise uniqueness of (42) holds and therefore there is a unique non-negative strong solution to (42). The uniqueness implies the strong Markov property and by Itô's formula it is easy to show that the infinitesimal generator of Z satisfies (43).

The branching property of Z_t conditioned on $K^{(0)}$, is due to the CBI-process. Let $\widetilde{Z}_t=Z_te^{-K_t^{(0)}}$ and $F\in C^{1,2}(\mathbb{R}_+,\mathbb{R})$, Itô's formula again guarantees that $F(t,\widetilde{Z}_t)$ conditioned on

$K^{(0)}$ is a local martingale if and only if for every $t \geq 0$,

$$0 = \int_0^t \left(\frac{\partial}{\partial t} F(s, \widetilde{Z}_s) + \mathtt{d} \frac{\partial}{\partial x} F(s, \widetilde{Z}_s) + \gamma^2 e^{-K_s^{(0)}} \widetilde{Z}_s \frac{\partial^2}{\partial x^2} F(s, \widetilde{Z}_s) \right) \mathrm{d}s$$
$$+ \int_0^t \int_0^\infty Z_s \left(F(s, \widetilde{Z}_s + z e^{-K_s^{(0)}}) - F(s, \widetilde{Z}_s) - \frac{\partial}{\partial x} F(s, \widetilde{Z}_s) z e^{-K_s^{(0)}} \right) \mu(\mathrm{d}z) \mathrm{d}s$$
$$+ \int_0^t \int_0^\infty \left(F(s, \widetilde{Z}_s + z e^{-K_s^{(0)}}) - F(s, \widetilde{Z}_s) \right) \nu(\mathrm{d}z) \mathrm{d}s.$$

We choose $F(s,x) = \exp\left\{ -x v_t(s, \lambda, K^{(0)}) - \int_s^t \phi(v_t(r, \lambda, K^{(0)}) e^{-K_r^{(0)}}) \mathrm{d}r \right\}$ where $v_t(s, \lambda, K^{(0)})$ is differentiable with respect to the variable s, non-negative and such that $v_t(t, \lambda, K^{(0)}) = \lambda$ for all $\lambda \geq 0$. We observe that F is bounded, therefore, $(F(s, \widetilde{Z}_s), s \leq t)$ conditioned on $K^{(0)}$ is a martingale if and only if

$$\frac{\partial}{\partial s} v_t(s, \lambda, K^{(0)}) = \gamma^2 (v_t(s, \lambda, K^{(0)}))^2 e^{-K_s^{(0)}}$$
$$+ e^{K_s^{(0)}} \int_0^\infty \left(e^{-e^{-K_s^{(0)}} v_t(s, \lambda, K^{(0)}) z} - 1 + e^{-K_s^{(0)}} v_t(s, \lambda, K^{(0)}) z \right) \mu(\mathrm{d}z),$$

which is equivalent to the backward differential equation given in (45) and (11). The existence and unicity of v_t follows from the Picard-Lindelof theorem.
Therefore, the process $\left(\exp\left\{ -\widetilde{Z}_s v_t(s, \lambda, K^{(0)}) \right\}, 0 \leq s \leq t \right)$ conditioned on $K^{(0)}$ is a martingale, and hence

$$\mathbb{E}_z \left[\exp\left\{ -\lambda \widetilde{Z}_t \right\} \middle| K^{(0)} \right] = \exp\left\{ -z v_t(0, \lambda, K^{(0)}) - \int_0^t \phi(v_t(r, \lambda, K^{(0)}) e^{-K_r^{(0)}}) \mathrm{d}r \right\}.$$

□

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