CHAPTER 1

Excursion theory for Markov processes.

The aim of this chapter is to describe the evolution of a strong Markov process in terms of its behaviour between visits to a particular point in the state space. We first study the right-continuous inverse of the local time, which is a subordinator (possibly killed) whose jumps corresponds to the lengths of the excursion intervals. The study culminates with the description of the process in terms of a Poisson point process.

1. The inverse local time.

The local time L of a Markov processes, introduced in the previous chapter, is described most conveniently in terms of its right-continuous inverse:

$$L^{-1}(t) = \inf\Big\{s \geq 0 : L(s) > t\Big\}, \qquad t \geq 0.$$

The following notation will be also useful,

$$L^{-1}(t^-) = \inf\Big\{s \geq 0 : L(s) \geq t\Big\} = \lim_{s \to t^-} L(s), \qquad t \geq 0.$$

We start the study of L^{-1} with the following elementary properties.

PROPOSITION 1. *i) For every $t \geq 0$, $L^{-1}(t)$ and $L^{-1}(t^-)$ are stopping times.*
ii) The process L^{-1} is increasing, right-continuous and adapted to the filtration $(\mathcal{F}_{L^{-1}(t)})$.
iii) We have a.s. for all $t > 0$,

$$L^{-1}(L(t)) = \inf\Big\{L^{-1}(u) \geq 0 : L^{-1}(u) > t\Big\} = \inf\Big\{s > t : X_s = 0\Big\},$$

and

$$L^{-1}(L(t)^-) = \sup\Big\{L^{-1}(u) \geq 0 : L^{-1}(u) < t\Big\} = \sup\Big\{s < t : X_s = 0\Big\}.$$

In particular $L^{-1}(t) \in \mathcal{L}$ on $\{L^{-1}(t) < \infty\}$.

Proof: (i) For every, $s, t > 0$, we have

$$\Big\{L^{-1}(t) < s\Big\} = \Big\{L(s) > t\Big\},$$

since L is continuous. Hence $L^{-1}(t)$ is a stopping time because L is adapted to the right-continuous filtration $(\mathcal{F}_t)$. Since $L^{-1}(t^-)$ is a limit of stopping times and $(\mathcal{F}_t)$ is right-continuous, we deduce that it is also a stopping time.
(ii) From the definition of L^{-1}, it is increasing and right-continuous. The fact that it is adapted to the filtration $(\mathcal{F}_{L^{-1}(t)})$ follows from (i).
(iii) From the definition of L^{-1}, it is clear that

$$\begin{aligned} L^{-1}(L(t)) &= \inf\Big\{s \geq 0 : L(s) > L(t)\Big\} = \inf\Big\{L^{-1}(u) \geq 0 : u > L(t)\Big\} \\ &= \inf\Big\{L^{-1}(u) \geq 0 : L^{-1}(u) > t\Big\}. \end{aligned}$$

Now, we define $D_t = \inf\{s > t : X_s = 0\}$ and suppose that $D_t > t$. By Theorem **??**, L is constant in the interval $[t, D_t)$, hence $D_t \le L^{-1}(L(t))$. We may now assume that $D_t < \infty$, since otherwise there is nothing to prove. Then, D_t belongs to the support of the measure $\mathrm{d}L$ and is isolated on the left. On the other hand, since L is continuous, D_t cannot be isolated on the right, that is to say that $L(s) > L(D_t) = L(t)$ for all $s > D_t$; and hence $D_t \ge L^{-1}(L(t))$.
Next, we suppose that $D_t = t$, so that t belongs to the support of $\mathrm{d}L$ and is not isolated on the right. Therefore $L(s) > L(t)$, for all $s > t$, and hence $t \ge L^{-1}(L(t))$. The converse inequality is obvious.
The second identity in (iii) follows from similar arguments. Finally, on $\{L^{-1}(t) < \infty\}$, there exist s such that $L(s) = t$ and the first identity shows that $L^{-1}(t)$ is a zero of X. ■

Before we characterize the law of L^{-1}, we note that our previous result shows that $\mathrm{cl}(\mathcal{L})$ has no isolated points as we remarked in Proposition **??**. This is because L is continuous and the support of the measure $\mathrm{d}L$ coincides with $\mathrm{cl}(\mathcal{L})$. Another remarkable consequence of Proposition 1 is that the excursion intervals are the open intervals of the type $(L^{-1}(t^-), L^{-1}(t))$ whenever $L^{-1}(t^-) < L^{-1}(t)$.

THEOREM 1. *The inverse local time L^{-1} is a subordinator with Lévy measure Π, drift coefficient $d \ge 0$ and killed at rate $\overline{\Pi}(\infty)$. One has for all $t, \lambda > 0$*

$$\mathbb{E}\Big(\exp\big\{-\lambda L^{-1}(t)\big\}\Big) = \exp\left\{-t\lambda\left(d + \int_0^\infty e^{-\lambda r}\overline{\Pi}(r)\mathrm{d}r\right)\right\}.$$

Proof: We first suppose that $\overline{\Pi}(\infty) = 0$, in this case we know that there is no infinite excursion a.s. (see Lemma 5). In particular $d_1(c) < \infty$ and by iteration of the strong Markov property, we deduce that $d_n(c) < \infty$ a.s. for every $n \ge 1$. From Proposition **??**, the strong Markov property and the additivity of the local time, $L(d_n(c))$ can be expressed as the sum of n independent exponential random variables with parameter 1. In particular, $L(\infty) = \lim_n L(d_n(c)) = \infty$, a.s.
From Proposition 1 part (i), we may apply the strong Markov property at $L^{-1}(t)$ and then the process $X \circ \theta_{L^{-1}(t)}$ has the same law as X and is independent of $\mathcal{F}_{L^{-1}(t)}$. From the additivity of L, we have that the local time $\tilde{L}$ of $X \circ \theta_{L^{-1}(t)}$ is defined by $\tilde{L}(s) = L(L^{-1}(t) + s) - t$, and the inverse local time of $X \circ \theta_{L^{-1}(t)}$ is

$$\begin{aligned}\tilde{L}^{-1}(s) &= \inf\Big\{u \ge 0 : \tilde{L}(u) > s\Big\} = \inf\Big\{u \ge 0 : L(L^{-1}(t) + u) > s + t\Big\}\\ &= \inf\Big\{u \ge L^{-1}(t) : L(u) > s + t\Big\} = L^{-1}(s + t) - L^{-1}(t),\end{aligned}$$

which proves that L^{-1} has homogeneous independent increments, and since its simple paths are increasing and right-continuous, L^{-1} is a subordinator.
The Lévy measure Π of L^{-1} is the characteristic measure of the Poisson point process of its jumps, ΔL^{-1}. For each $a > 0$, we define $T_a = \inf\{t \ge 0 : \Delta L^{-1}_t > a\}$, the instant of the first jump with length $\ell > a$. From Proposition 1 part (iii), we see that T_a coincides with the local time evaluated on the right-end point of the first excursion interval with $\ell > a$. Then from Proposition **??**, T_a has an exponential distribution with parameter $\overline{\Pi}(a)$. Now, from Lemma 3, we deduce that $\overline{\Pi}(a) = \Pi(a, \infty)$.
Recall that the excursion intervals are the open intervals which appear in the canonical decomposition of the open set $\mathrm{cl}(\mathcal{L})^c$. Using again the correspondence between the jumps

of L^{-1} and the length of the excursion intervals of X, we have

$$L^{-1}(t) = \int_0^{L^{-1}(t)} \mathbb{1}_{\mathrm{cl}(\mathcal{L})}(s)\mathrm{d}s + \sum_{s\leq t} \Delta L^{-1}(s).$$

By Corollary 2, $\mathfrak{d}L(L^{-1}(t)) = \mathfrak{d}t$, which coincides with the integral of the right. That is

$$L^{-1}(t) = \mathfrak{d}t + \sum_{s\leq t} \Delta L^{-1}(s),$$

which shows that the drift coefficient of L^{-1} is $\mathfrak{d}$.
Now, we suppose that $\overline{\Pi}(\infty) > 0$. Using similar arguments as above, we see that for every $0 < t < t'$, the law of the inverse local time up to time t, $(L^{-1}(s), 0 \leq s \leq t)$, is the same conditionally on $\{L^{-1}(t) < \infty\}$ as conditionally on $\{L^{-1}(t') < \infty\}$ and coincides with the law of a subordinator σ restricted to the time interval $[0, t]$. Since the events $\{L^{-1}(t) < \infty\}$ and $\{L(\infty) < t\}$ are the same, we may rephrase the preceding assertion by claiming that $(L^{-1}(s), 0 \leq s < L(\infty))$ has the same law as the killed subrodinator $(\sigma_t, t < \tau)$, where τ is independent of σ and has an exponential distribution with parameter $\overline{\Pi}(\infty)$.
We denote the Lévy measure of σ by Π. Let T_a be as before and note that $T_a = L(d_1(a))$. Therefore we have for every $a > 0$,

$$\begin{aligned} 1 - \exp\{-t\Pi(a,\infty)\} &= \mathbb{P}(\exists s < t \,:\, \Delta\sigma_s > a) = \mathbb{P}(T_a < t | L(\infty) > t) \\ &= \exp\{t\overline{\Pi}(\infty)\}\mathbb{P}(L(d_1(a)) < t, L(\infty) > t). \end{aligned}$$

On the one hand, by Proposition **??**, the law of $L(d_1(a))$ conditionally on $\{d_1(a) < \infty\}$ is the exponential distribution with parameter $\overline{\Pi}(a)$. On the other hand, Lemma 7 implies that

$$\mathbb{P}(d_1(a) < \infty) = \mathbb{P}(\ell_1(a) < \infty) = 1 - \frac{\overline{\Pi}(\infty)}{\overline{\Pi}(a)}.$$

Now applying the Markov property at $d_1(a)$ and Proposition **??**, we see

$$\begin{aligned} \mathbb{P}(L(d_1(a)) < t, L(\infty) > t) &= \mathbb{P}(L(d_1(a)) < t, L(\infty) > t, d_1(a) < \infty) \\ &= \mathbb{P}\Big(\mathbb{P}\big(L(\infty)\circ\theta_{d_1(a)} > t - L(d_1(a))\big); L(d_1(a)) < t, d_1(a) < \infty\Big) \\ &= \mathbb{P}\Big(\exp\{-(t - L(d_1(a)))\overline{\Pi}(\infty)\}; L(d_1(a)) < t, d_1(a) < \infty\Big) \\ &= \left(1 - \frac{\overline{\Pi}(\infty)}{\overline{\Pi}(a)}\right)\int_0^t \overline{\Pi}(a)\exp\{-s\overline{\Pi}(a)\}\exp\{-(t-s)\overline{\Pi}(\infty)\}\mathrm{d}s \\ &= \big(\overline{\Pi}(a) - \overline{\Pi}(\infty)\big)\exp\{-t\overline{\Pi}(\infty)\}\int_0^t \exp\{-s(\overline{\Pi}(a) - \overline{\Pi}(\infty))\}\mathrm{d}s \\ &= \exp\{-t\overline{\Pi}(\infty)\}\Big(1 - \exp\{-t(\overline{\Pi}(a) - \overline{\Pi}(\infty))\}\Big). \end{aligned}$$

Hence $\Pi(a,\infty) = \overline{\Pi}(a) - \overline{\Pi}(\infty)$, finally we check as in the case $\overline{\Pi}(\infty) = 0$ that the drift of σ is $\mathfrak{d}$.
The identity for the Laplace exponent follows from the Lévy-Kintchine formula (Theorem 3) by integration by parts. ■

2. Excursion processes.

In what follows, we will deal with the Skorokhod space of càdlàg paths. Specifically, take an isolated point ∂ which will serve as cemetery point. Consider

$$\Omega' = \mathcal{D}\Big([0,\infty), S \cup \{\partial\}\Big),$$

the set of paths $\omega : [0,\infty) \to S \cup \{\partial\}$ with lifetime

$$\zeta = \inf\Big\{t \geq 0 : \omega(t) = \partial\Big\}$$

which are right-continuous on $[0,\infty)$, have a left limit on $(0,\infty)$ and stay at the cemetery point ∂ after the lifetime ζ. This space is endowed with the Skorokhod's topology under which the space Ω' is a Pollish space.

In order to start with the description of the excursion process, we first introduce the space of excursions. Let $\delta > 0$ and denote by U^δ the set of excursion with lifetime (or length) $\zeta > \delta$, that is

$$U^\delta = \Big\{\omega \in \Omega' : \zeta > \delta \text{ and } \omega(t) \neq 0 \text{ for all } 0 < t < \zeta\Big\},$$

and by $U = \cup_{\delta > 0} U^\delta$ the space of excursions. These sets are endowed with the topology induced by the Skorokhod's topology.
For each $a > 0$ with $\overline{\Pi}(a) > 0$, denote by $n(\cdot|\zeta > a)$ the probability measure on U^a corresponding to the law of the process $(X_{g_1(a)+t}, 0 \leq t \leq \ell_1(a))$ under $\mathbb{P}$. This probability is called the law of the excursions of X with lifetime bigger than a.

PROPOSITION 2. *Let $a > 0$ such that $\overline{\Pi}(a) > 0$. For any $b \in (0,a)$ and measurable event $\Lambda \in U^a$, we have*

$$\overline{\Pi}(a) n(\Lambda|\zeta > a) = \overline{\Pi}(b) n(\Lambda|\zeta > b).$$

Proof: From Proposition **??**, we know

$$\frac{\overline{\Pi}(a)}{\overline{\Pi}(b)} = \mathbb{P}\Big(N_b(g_1(a)) = 0\Big),$$

which is in fact the probability that the first excursion with $\ell > b$ has $\ell > a$. Hence, the law of the first excursion with $\ell > a$ conditioned on $\{N_b(g_1(a)) = 0\}$ is

$$\frac{\overline{\Pi}(b)}{\overline{\Pi}(a)} n(\cdot, \zeta > a|\zeta > b).$$

According to Proposition **??**, the first excursion with $\ell > a$ is independent of $N_b(g_1(a))$ which implies that the previous probability is equal to $n(\cdot|\zeta > a)$, which proves the result.
■

A natural consequence of the above result is the existence of a unique measure n on U, called the excursion measure of X, such that

$$n(\Lambda) = \overline{\Pi}(a) n(\Lambda|\zeta > a) \qquad \text{for every measurable} \quad \Lambda \subset U^a.$$

In particular, $n(\zeta > a) = \overline{\Pi}(a)$. Another important consequence is that the excursion measure n has the simple Markov property. More precisely, take $a > 0$ and note that

$$g_1(a) + a = \inf\{t \geq a : X_s \neq 0 \text{ for all } s \in [t-a, t]\}$$

is a stopping time. The strong Markov property of X and the definition of the excursion measure imply that under n, conditionally on $\{\varepsilon(a) = x, a < \zeta\}$ (where ε denotes the

generic excursion and ζ its lifetime), the shifted process $(\varepsilon(t+a), 0 \leq t < \zeta - a)$ is independent of $(\varepsilon(t), 0 \leq t \leq a)$ and is distributed as $(X_t, 0 \leq t < R_0)$ under $\mathbb{P}_x$.
Again our definition of the excursion measure depends on the constant $c > 0$ but changing such a constant would only affect the excursion measure by a constant multiplicative factor.
Now, we introduce the excursion process of X denoted by $e = (e_t, t \geq 0)$. The excursion process e take values in $U \cup \{\delta\}$ and is given by

$$(1.1) \qquad e_t = \Big(X_{s+L^{-1}(t^-)}, 0 \leq s < L^{-1}(t) - L^{-1}(t^-)\Big) \quad \text{if} \quad L^{-1}(t^-) < L^{-1}(t),$$

and $e_t = \delta$ otherwise.
Before we establish our next result, we recall the definition of a stopped Poisson point process. Let $\Delta = (\Delta_t, t \geq 0)$ be a Poisson point process and define the random time $T_B = \inf\{t \geq 0 : \Delta_t \in B\}$ for B measurable. The process $(\Delta_t, 0 \leq t \leq T_B)$ is called the Poisson point process stopped at the first point in B.

THEOREM 2. *(Itô, 1970) i) If* 0 *is recurrent, then* e *is a Poisson point process with characteristic measure* n.
ii) If 0 *is transient, then* $e = (e_t, 0 \leq t \leq L(\infty))$ *is a Poisson point process with characteristic measure* n*, stopped at the first point in* U^∞*, the space of excursions with infinite lifetime.*

Proof: We first prove the case when 0 is recurrent. From Proposition 1, we have that for every $t \geq 0$, $L^{-1}(t)$ is a stopping time and hence $(\mathcal{H}_t, t \geq 0)$, where $H_t = \mathcal{F}_{L^{-1}(t)}$, is a filtration. We may verify that for every $\epsilon > 0$ and measurable $B \subset U^\epsilon$ the counting process

$$N_t^B = \text{card}\{0 < s \leq t; e_s \in B\} \qquad t \geq 0,$$

is an $(\mathcal{H}_t)$-Poisson process with intensity $n(B)$. Indeed, let $B_1, \ldots, B_k$ be pairwise disjoint measurable sets, then their respective counting processes never jump simultaneously and therefore will be independent. One then deduce that the associated random measure defined by $M = \sum_{t \geq 0} \delta_{(t, e_t)}$ is a Poisson measure with intensity $\lambda \otimes n$, where λ is the Lebesgue measure on $[0, \infty)$.
For every $s, t \geq 0$, $N_{t+s}^B - N_t^B$ is the number of excursions of X in B which were completed during the time interval $(L^{-1}(t), L^{-1}(t+s)]$. Now, we consider the process $X \circ \theta_{L^{-1}(t)}$ and note that it is independent of $\mathcal{H}_t$ and has the same law as X (from the strong Markov property and the fact that $L^{-1}(t)$ is a zero of X). Denote by $\tilde{L}$ and $\tilde{L}^{-1}$ for the local time and the inverse local time, respectively. The additivity of L implies that for every $u \geq 0$,

$$L^{-1}(t+u) = L^{-1}(t) + \inf\{s \geq 0 : \tilde{L}(s) \geq u\} = L^{-1}(t) + \tilde{L}^{-1}(t),$$

and therefore $N_{t+s}^B - N_t^B = \tilde{N}_s^B$ is the number of excursions of $\tilde{X}$ in B which were completed during the time interval $(0, \tilde{L}^{-1}(s)]$. As a consequence $N_{t+s}^B - N_t^B$ has the same law as N_s^B and is independent of $\mathcal{H}_t$. This shows that N_t^B is a subordinator adapted to $(\mathcal{H}_t)$ which increases only by jumps a.s. equal to 1, hence a $(\mathcal{H}_t)$-Poisson process and hence e is a Poisson point process.
Let ν be the characteristic measure of e. We see from Lemma 3 that for every $u > 0$, the conditional law $\nu(\cdot|U^a)$ is the law of the excursions with lifetime $\zeta > a$, that is

$$\nu(\cdot, \zeta > a)/\nu(\zeta > a) = n(\cdot|\zeta > a) = n(\cdot, \zeta > a)/n(\zeta > a).$$

On the other hand, the local time evaluated on the right-end point of the first excursion interval with length $\ell > a$, $L(d_1(a))$, is the instant of the first point of e in U^a, and we know from Proposition **??** that $L(d_1(a))$ has an exponential distribution with parameter $\overline{\Pi}(a)$. Hence again from Lemma 3, we deduce that $\nu(\zeta > a) = \overline{\Pi}(a)$ and the measures ν

and n coincide on U^a. Since $U = \cup_{a>0} U^a$, the proof of part (i) is complete.
Part (ii) follows from similar arguments as those used above. Actually, we just need to prove that the point process defined by

$$e'_t = \begin{cases} \delta & \text{if } e_t \in U^\infty, \\ e_t & \text{otherwise}, \end{cases}$$

is a Poisson point process with characteristic measure $n(\cdot, (U^\infty)^c)$ and independent of $(T_{U^\infty}, e_{T_{U^\infty}})$, where T_{U^∞} is the intant of the first point in U^∞. We leave the details to the reader. ■

We finish this chapter with some comments on the cases of holding points and of irregular points. Assume that 0 is a holding point and consider the sequence of successive exits from 0 and returns to 0, $R_0 < F_1 < R_1, \ldots$, where $R_0 = 0$, $R_n = \inf\{t > F_n : X_t = 0\}$, $F_{n+1} = \inf\{t > R_n : X_t \neq 0\}$. Note that the Markov property implies that $X_{F_1} \neq 0$ and that F_1 has an exponential distribution and is independent of the first excursion $(X_{F_1+t}, 0 \leq t < R_1 - S_1)$. Also note that on the event $\{R_1 < \infty\}$, we have $X_{R_1} = 0$ so iterating the strong Markov property we see that

$$\mathcal{L} = \bigcup_{n \geq 1} [R_{n-1}, F_n),$$

and there exist an obvious continuous additive functional which increases exactly on $\mathrm{cl}(\mathcal{L})$,

$$A_t = \int_0^t \mathbb{I}_{\{X_s = 0\}} u ds, \qquad t \geq 0.$$

We may define the local time at 0 as any process $L = (L(t), t \geq 0)$ such that $\mathfrak{d} L(t) = A_t$, for all $t \geq 0$. It is not difficult to check that the right-continuous inverse L^{-1} is continous except at $L(F_i)$, $i \geq 1$ and also that it is a subordinator possibly killed with drift coefficient $\mathfrak{d}$. Moreover, the excursion process is a Poisson point process possibly stopped at the instant of the first point in U^∞ which characteristic measure is proportional to the law of the first excursion $(X_{F_1+t}, 0 \leq t < R_1 - S_1)$.
The case when 0 is irregular is more simple to describe. Consider the sequence $(R_n, n \geq 0)$ of successive returns times to 0, defined by $R_0 = 0$, $R_{n+1} = \inf\{t > R_n : X_t = 0\}$. We know that $R_1 > 0$ a.s. and on the event $\{R_1 < \infty\}$ the strong Markov property give us that $X \circ \theta_{R_1}$ is independent of $(X_t, 0 \leq t \leq R_1)$ and has law $\mathbb{P}$. Note that $(R_n, n \geq 0)$ is in fact an increasing random walk possible killed at some independent geometric random variable in the transient case. In order to define a local time whose right-continuous inverse is a subordinator, we introduce the sequence of independent exponentially random variable with the same parameter and also independent of X. We define the local time of X at 0 by

$$L(t) = \sum_{i=0}^{m(t)} \tau_i, \qquad m(t) = \max\{i : R_i < t\}.$$

Clearly L increases exactly on $\mathcal{L}$ but is not adapted to the filtration $(\mathcal{F}_t)$ and is only right-continuous. Discontinuity is not a problem and to circumvent the first problem, we simply replace $(\mathcal{F}_t)$ by $(\mathcal{F}'_t)$, with $\mathcal{F}'_t = \mathcal{F}_t \bigvee \sigma(L(s), 0 \leq s \leq t)$. Then, now we have that the right-continuous inverse is a subordinator possibly killed in the transient caseand the excursion process is a Poisson point process possibly stopped at the first point in the set U^∞. Finally, the excursion mesure is again simply proportional to the law of the first excursion $(X_t, 0 \leq t \leq R_1)$.