CHAPTER 1

Local times for Markov processes.

The local time at a given state, says x, of a Markov process X is an additive functional of X, which grows only on the random set of times when X visit x. Such functional is a main tool to study the structure of the successive lenghts of the intervals of excursion of X away from x. In general, the local time should be understood as a measure of how often the process visited the point x which stays constant on the excursion intervals.

1. Markov processes.

Here, we will assume that the reader have some knowledge on the basic theory of stopping times, continuous martingales and specially on the theory of Markov processes, as the definition of a transition probability and their construction.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and S be a Polish space (i.e. a separable and complete metric space) with a specified element, here denoted by 0. We consider a stochastic process $X = (X_t, t \geq 0)$ taking values in S, with right-continuous sample paths and satisfying that $\mathbb{P}(X_0 = 0) = 1$. We take $(\mathcal{F}_t, t \geq 0)$ to be a right-continuous complete filtration and we assume that X is adapted to such filtration. Let us suppose that there exists a familly of probability measures $(\mathbb{P}_x, x \in S)$ such that

(MP) For any τ finite stopping time, under the conditional law $\mathbb{P}(\cdot|X_\tau = x)$, the shifted process $X \circ \theta_\tau = (X_{\tau+t}, t \geq 0)$ is independent of $\mathcal{F}_\tau$ and has the law $\mathbb{P}_x$.

A stochastic process satisfying the property (MP) is called a strong Markov process. Here, we only focus on a nice class of strong Markov processes known as Hunt processes.

DEFINITION 1. *We say that the a strong Markov process X is quasi-left-continuous provided that whenever $(T_n, n \geq 1)$ is an increasing sequence of $(\mathcal{F}_t, t \geq 0)$ stopping times with limit T, which is a.s. finite, then a.s. $X_{T_n} \to X_T$ as n goes to ∞. The class of quasi-left continuous strong Markov process is called Hunt processes.*

PROPOSITION 1. *Let B be an open or closed set of S and X a Hunt process. Then*

$$T_B = \inf\{t \geq 0 : X_t \in B\},$$

is a $(\mathcal{F}_t, t \geq 0)$ stopping time.

Proof: Let us first suppose that B is an open set. The right-continuity of the paths implies that

$$\{T_B < t\} = \bigcup_{s \in \mathbb{Q}_+, s<t} \{X_s \in B\} \in \mathcal{F}_t,$$

which implies that T_B is a stopping time, since $(\mathcal{F}_t, t \geq 0)$ is right-continuous.

Now, we assume that B is closed. For $\epsilon > 0$, we define B^ϵ as the ϵ-neighbourhood of B, then T_{B^ϵ} is a stopping time such that $T_{B^\epsilon} \leq T_B$ and when ϵ goes to 0, T_{B^ϵ} increases towards T' which is a stopping time. We will prove the result if we show that $T' = T_B$ a.s. On one hand, we know that $T' \leq T_B$ a.s. On the other hand by the quasi-left-continuity of X, we have that $X_{T_{B^\epsilon}}$ converges towards $X_{T'}$ a.s., as ϵ goes to 0. But $X_{T_{B^\epsilon}} \in \mathrm{cl}(B^\epsilon)$ under the event $\{T_{B^\epsilon} < \infty\}$. Then $X_{T'} \in \mathrm{cl}(B^\epsilon)$ under the event $\{T' < \infty\}$ for all $\epsilon > 0$. Therefore $X_{T'} \in B$ under $\{T' < \infty\}$ since B is closed, which implies that $T' = T_B$. ■

It is important to note that Feller processes form a sub-class of Hunt processes. Indeed, Feller processes are constructed from a specific kind of transition function, known as Feller semigroup, and it can be proved that they have all the preceding properties. Another property of Feller processes which is also satisfied by any Hunt process is the following:

THEOREM 1. *Let X be a Hunt process. Then, almost surely the sample paths of X have left limits on $(0, \infty)$.*

Proof: Take a fixed $\epsilon > 0$ and define the following random time

$$T^{(\epsilon)} = \inf\Big\{t > 0 : d_S(X_t, 0) > \epsilon\Big\},$$

where d_S denotes the metric of the Polish space S. From the previous Proposition, we know that $T^{(\epsilon)}$ is a stopping time.

Next, we define $T_0^{(\epsilon)} = 0$, $T_1^{(\epsilon)} = T^{(\epsilon)}$ and inductively for $n \geq 1$,

$$T_{n+1}^{(\epsilon)} = T_n^{(\epsilon)} + T^{(\epsilon)} \circ \theta_{T_n^{(\epsilon)}} = \inf\Big\{t > T_n^{(\epsilon)} : d_S(X_t, X_{T_n^{(\epsilon)}}) > \epsilon\Big\}.$$

Each $T_n^{(\epsilon)}$ is a stopping time relative to $(\mathcal{F}_t, t \geq 0)$ and since the sequence $(T_n^{(\epsilon)})$ is increasing, the limit $\mathcal{T}^{(\epsilon)} = \lim_n T_n^{(\epsilon)}$ is also a stopping time. On the event $\{\mathcal{T}^{(\epsilon)} < \infty\}$, we have that $\lim_n X_{T_n^{(\epsilon)}} = X_{\mathcal{T}^{(\epsilon)}}$ by quasi-left continuity. On the other hand, right continuity of paths implies that $d_S(X_{T_n^{(\epsilon)}+1}, X_{T_n^{(\epsilon)}}) > \epsilon$ a.s. for all n, which precludes the existence of $\lim_n X_{T_n^{(\epsilon)}}$. There would be a contradiction unless $\mathcal{T}^{(\epsilon)} = \infty$ a. s. In the latter event, we have $[0, \infty) = \cup_{n\geq 0}[T_n^{(\epsilon)}, T_{n+1}^{(\epsilon)})$. Note that if $T_n^{(\epsilon)} = \infty$, then $[T_n^{(\epsilon)}, T_{n+1}^{(\epsilon)}) = \emptyset$. In each interval $[T_n^{(\epsilon)}, T_{n+1}^{(\epsilon)})$ the oscillation of X does not exceed 2ϵ by the definition of $T_{n+1}^{(\epsilon)}$. We have therefore proved that for each $\epsilon > 0$, there exists Ω_ϵ with $\mathbb{P}(\Omega_\epsilon) = 1$ such that X does not oscillate by more than 2ϵ in $[T_n^{(\epsilon)}, T_{n+1}^{(\epsilon)})$, where

$$[0, \infty) = \cup_{n\geq 0}[T_n^{(\epsilon)}, T_{n+1}^{(\epsilon)}). \tag{1.1}$$

Let $\tilde{\Omega} = \cap_{m\geq 1}\Omega_{1/m}$; then $\mathbb{P}(\tilde{\Omega}) = 1$. We assert that if $\omega \in \tilde{\Omega}$, then $t \mapsto X_t(\omega)$ must have left limits in $(0, \infty)$. For otherwise there exist $t \in (0, \infty)$ and m such that $X_\cdot(\omega)$ has oscillation bigger than $2/m$ in $(t - \delta, t)$ for every $\delta > 0$. Thus $t \notin [T_n^{(1/m)}, T_{n+1}^{(1/m)})$ for all $n \geq 0$, which is impossible by (1.1) with $\epsilon = 1/m$. ■

If B is an open or closed set, from the strong Mrkov property we can see that the event $\{T_B = 0\}$ is trivial under $\mathbb{P}$, i.e. that $\mathbb{P}(T_B = 0)$ is equal 0 or 1. We say that 0 is regular for B if $\mathbb{P}(T_B = 0) = 1$ and irregular for B if $\mathbb{P}(T_B = 0) = 0$.

In the sequel, we will assume that X is a Hunt process. We will also use the property that for every stopping time T, the first return time to the initial state after T, $R_T = \inf\{t > T : X_t = 0\}$, is a stopping time. From above $\mathbb{P}(R_0 = 0)$ is necessarily equals 0 or 1. When 0 is regular, we introduce the first exit time from 0, $F = \inf\{t \geq 0 : X_t \neq 0\}$. Then F is a stopping time and again $\mathbb{P}(F = 0)$ necessarily equals 0 or 1. We say that 0 is a holding point in the first case, and an instantaneoous point in the second case.

2. Local times.

In this section, we will suppose that 0 is regular and instantaneous. In fact, this is the most interesting case. The definition of the local time is clear when 0 is either irregular or a holding point; however we will discuss these cases at the end of the next chapter. Before, we start with the construction of the local time we must prove some basic properties of the zero set of X, which is denoted by $\mathcal{L} = \{t : X_t = 0\}$. From the right-continuity of X, it is clear that a sequence $(t_n, n \geq 1)$ in $\mathcal{L}$ which decreases to t implies that $t \in \mathcal{L}$, which means that every point in $\mathrm{cl}(\mathcal{L}) \setminus \mathcal{L}$ is isolated from the right. Since $\mathrm{cl}(\mathcal{L})^c$ is open and hence countable union of disjoint open intervals, it follows that $\mathcal{L}^c$ is a countable union of disjoint intervals of the form (u, v) or $[u, v)$. Before, we establish our next result we recall that a set $\mathcal{A}$ is said to be nowhere dense if $(\mathrm{cl}(A))^\circ = \emptyset$.

PROPOSITION 2. *The set $\mathcal{L}$ is nowhere dense and has no isolated points.*

Proof: Since 0 is instantaneous, we have that $F = 0$ a.s. Then, applying the strong Markov property, we get that $F \circ \theta_{R_r} = 0$ a.s. on $\{R_r < \infty\}$, and so $R_r \in \mathrm{cl}(\mathcal{L}^c)$ a.s. on the same set. Since $\{R_r, r \in \mathbb{Q}_+\}$ is dense in $\mathrm{cl}(\mathcal{L})$ therefore, $\mathrm{cl}(\mathcal{L}) \subset \mathrm{cl}(\mathcal{L}^c)$, and therefore $\mathrm{cl}(\mathcal{L}^c) = \mathbb{R}_+$ a.s. Since $\mathcal{L}^c$ is a disjoint union of intervals of the form (u, v) or $[u, v)$, we have that $\mathrm{cl}(\mathcal{L}^c) = \mathrm{cl}(\mathcal{L})^c$ which proves that $\mathcal{L}$ is nowhere dense.
Since 0 is regular $R_0 = 0$ a.s. , then $R_0 \circ \theta_{R_r} = 0$ a.s. on $\{R_r < \infty\}$. Since every isolated point of $\mathcal{L}$ is of the form R_r for some $r \in \mathbb{Q}_+$, it follows that $\mathcal{L}$ has no isolated points. ■

The excursion interval (g, d), is an open interval where $X_t \neq 0$ for all $t \in (g, d)$. The left-end point $g \in \mathrm{cl}(\mathcal{L})$, the right-end point $d \in \mathrm{cl}(\mathcal{L}) \cup \{\infty\}$ and $\ell = d - g$ is the length of the excursion interval. The set of excursion intervals is endowed with a natural total order, namely $(g, d) < (g', d')$ if $g < d \leq g' < d'$.
Next, we fix a real number $c > 0$ such that

$$\mathbb{P}(\text{there exist a least one excursion with } \ell > c) > 0.$$

Because X is right-continuous, there is always such a constant provide that X is not identically 0. In fact, the above probability is equal 1. Consider for every $t > 0$ the event

$$\Lambda_t = \{\text{all the excursion intervals with right-end point } d < t \text{ have lenght } \ell \leq c\}$$

and pick t large enough such that $\mathbb{P}(\Lambda_t) < 1$. The stopping time R_t could be infinite, if this is the case then the process has an excursion interval of infinite length, a fortiori it has an excursion interval with lenght $\ell > c$. If it is finite, we apply the strong Markov property at R_t and we get

$$\mathbb{P}(\Lambda_{3t}) \leq \big(\mathbb{P}(\Lambda_t)\big)^2,$$

and by iteration

$$\mathbb{P}(\Lambda_{3^n t}) \leq \big(\mathbb{P}(\Lambda_t)\big)^{2n} \qquad \text{for every} \qquad n \geq 1.$$

Hence $\lim_{s \to \infty} \mathbb{P}(\Lambda_s) = 0$ and the assertion follows.
For every $a > 0$ and integer $n \geq 1$, we denote by $\ell_n(a), g_n(a)$ and $d_n(a)$, the lenght, the left-end point and the right-end point, respectively, of the n-th excursion interval with length $\ell > a$. If the total number of excursion intervals with $\ell > a$ is strictely less than n, we decide that $\ell_n(a) = 0$, $g_n(a) = d_n(a) = \infty$. **Important:** note that $d_n(a)$ is a stopping time.

LEMMA 1. *For all $a \in (0, c)$, we have that*

$$\mathbb{P}(\ell_1(a) > c) > 0.$$

Proof: Let us suppose that such probability is equal 0 for some a. Then $d_1(a) < g_1(c)$ a.s. By the strong Markov property at $d_n(a)$, we deduce that $d_n(a) < g_1(c)$ a.s. for all $n \geq 1$. Thus $g_1(c) = \infty$ a.s. which is impossible. ■

Now, define the function $\overline{\Pi} : (0, \infty] \to (0, \infty)$ as follows

$$\overline{\Pi}(a) = \begin{cases} 1/\mathbb{P}(\ell_1(a) > c) & \text{if } a \leq c \\ \mathbb{P}(\ell_1(c) > a) & \text{if } a > c. \end{cases}$$

Note that $\overline{\Pi}$ is a decreasing and right-continuous function and also that $\overline{\Pi}(c) = 1$.

LEMMA 2. *The set $\mathcal{L}$ is a.s. bounded or unbounded according as $\overline{\Pi}(\infty)$ is positive or null.*

Proof: From the strong Markov property, $(I_n, n \geq 1)$ the sequence of lenghts of the excursion intervals with length greater than c are i.i.d. The event,

$$B = \{\text{there exist one excursion interval with } \ell > c \text{ of infinite lenght}\},$$

can be written as

$$B = \bigcup_{n \geq 1} \Big\{ I_i < \infty, \text{ for } i \in \{1, \dots, n-1\} \Big\} \cap \{I_n = \infty\}.$$

Now, we denote $p = \mathbb{P}(I_1 = \infty)$, then

$$\mathbb{P}(B) = \sum_{n \geq 1} p(1-p)^{n-1},$$

and such probability is equal 1 if $p > 0$ or 0 if $p = 0$. Then it is enough to note that $p = \overline{\Pi}(\infty)$ and that the event B is equivalent to say that the set $\mathcal{L}$ is unbounded. ■

We say that 0 is transient if $\overline{\Pi}(\infty) > 0$ and recurrent when $\overline{\Pi}(\infty) = 0$.

LEMMA 3. *The function $\overline{\Pi}$ satisfies*

$$\lim_{a \to 0^+} \overline{\Pi}(a) = \infty.$$

Proof: Suposse that this property fails, then with positive probability, there would be no excursion intervals in $[0, g_1(c))$. Because 0 is regular and the paths are right-continuous, X would then stay at 0 on some neighbourhood of the origin of time with positive probability as well. This contradicts the assumption that 0 is instantaneous. ■

LEMMA 4. *For every $a \in (0, \infty)$ and $b \leq a$ such that $\overline{\Pi}(b) > 0$, we have*

$$\mathbb{P}(\ell_1(b) > a) = \frac{\overline{\Pi}(a)}{\overline{\Pi}(b)}.$$

Proof: Let $b = c$ or $a = c$, then

$$\mathbb{P}(\ell_1(c) > a) = \overline{\Pi}(a), \quad \text{or} \quad \mathbb{P}(\ell_1(b) > c) = \frac{1}{\overline{\Pi}(b)}.$$

Now, we suppose that $b < c < a$. Hence,

$$\mathbb{P}(\ell_1(b) > a) = \mathbb{P}(\ell_1(b) > c, \ell_1(c) > a) = \mathbb{P}(\ell_1(c) > a) - \mathbb{P}(\ell_1(b) \leq c, \ell_1(c) > a),$$

and if we apply the strong Markov property at $d_1(b)$, we have that the events $\{\ell_1(b) \leq c\}$ and $\{\ell_1(c) > a\}$ are independent, which implies that

$$\mathbb{P}(\ell_1(b) > a) = \overline{\Pi}(a) - \overline{\Pi}(a)\Big(1 - \mathbb{P}(\ell_1(b) > c)\Big) = \frac{\overline{\Pi}(a)}{\overline{\Pi}(b)}.$$

Next, suppose that $b < a < c$. Hence,

$$\mathbb{P}(\ell_1(b) > c) = \mathbb{P}(\ell_1(b) > a, \ell_1(a) > c) = \mathbb{P}(\ell_1(a) > c) - \mathbb{P}(\ell_1(b) \leq a, \ell_1(a) > c).$$

Again, applying the strong Markov property at $d_1(b)$, we get

$$\mathbb{P}(\ell_1(b) > c) = \mathbb{P}(\ell_1(b) > a, \ell_1(a) > c) = \mathbb{P}(\ell_1(a) > c) - \Big(1 - \mathbb{P}(\ell_1(b) > a)\Big)\mathbb{P}(\ell_1(a) > c),$$

which implies that

$$\frac{1}{\overline{\Pi}(b)} = \frac{1}{\overline{\Pi}(a)}\mathbb{P}(\ell_1(b) > a),$$

and the result follows.

Finally, we assume that $c < b < a$. Hence,

$$\mathbb{P}(\ell_1(c) > a) = \mathbb{P}(\ell_1(c) > b, \ell_1(b) > a) = \mathbb{P}(\ell_1(b) > a) - \mathbb{P}(\ell_1(b) > a, \ell_1(c) \leq b).$$

Now, applying the Markov property at $d_1(c)$, we get

$$\mathbb{P}(\ell_1(c) > a) = \mathbb{P}(\ell_1(b) > a) - \Big(1 - \mathbb{P}(\ell_1(c) > b)\Big)\mathbb{P}(\ell_1(b) > a) = \mathbb{P}(\ell_1(b) > a)\overline{\Pi}(b),$$

and the proof is complete. ■

A natural consequence of the above result is that if we change the constant c by another constant this only alters the function $\overline{\Pi}$ by a constant multiplicative factor.

For every $a > 0$ and every $t > 0$, we introduce the total number of excursion intervals with $\ell > a$ which started strictly before time t,

$$N_a(t) = \sup\{n : g_n(a) < t\}.$$

Note that if $X_t \neq 0$, then the excursion interval stradding t is counted provided that its lenght is larger than a. On the other hand, if t is the left-end point of an excursion interval, then such interval is always discarded.

PROPOSITION 3. *Let $a \in (0, \infty]$ and $b < a$ be such that $\overline{\Pi}(b) > 0$. Then $N_b(g_1(a))$ is independent of $X \circ \theta_{g_1(a)}$ and has a geometric distribution with parameter $1 - \overline{\Pi}(a)/\overline{\Pi}(b)$.*

Proof: Applying the Markov property at $d_n(b)$, we see that for any $F \geq 0$ measurable functional,

$$\begin{aligned}\mathbb{E}\Big(F(X \circ \theta_{d_n(b)}); N_b(g_1(a)) \geq n\Big) &= \mathbb{E}\Big(F(X \circ \theta_{d_n(b)}); d_n(b) < g_1(a)\Big)\\ &= \mathbb{E}\Big(F(X)\Big)\mathbb{P}\big(d_n(b) < g_1(a)\big).\end{aligned}$$

Under $\{d_n(b) < g_1(a)\}$, the process $X \circ \theta_{g_1(a)}$ may be obtained from $X \circ \theta_{d_n(b)}$ by translation of the origin of time to $g_1^n(a)$, the left-end point of the first excursion interval of $X \circ \theta_{d_n(b)}$ with $\ell > a$. Hence, the process $X \circ \theta_{g_1(a)}$ and $N_b(g_1(a))$ are independent.

Now, applying again the Markov property we have

$$\mathbb{P}\big(d_{n+1} < g_1(a) | d_n(b) < g_1(a)\big) = \mathbb{P}\big(d_1(b) < g_1(a)\big) = \mathbb{P}\big(\ell_1(b) \leq a\big),$$

which implies that

$$\mathbb{P}\Big(N_b(g_1(a)) \geq n+1 \Big| N_b(g_1(a)) \geq n\Big) = 1 - \frac{\overline{\Pi}(a)}{\overline{\Pi}(b)}.$$

Then, $N_b(g_1(a))$ has a geometric distribution with parameter $1 - \bar{\Pi}(a)/\bar{\Pi}(b)$. ■

PROPOSITION 4. *For every $u \in (0, \infty]$ such that $\overline{\Pi}(u) > 0$, the process*

$$\frac{N_a(d_1(u))}{\overline{\Pi}(a)}, \qquad a \in (0, u),$$

is a left-continuous uniformly integrable backwards martingale. It converges a.s. and in $L^1(\mathbb{P})$ as $a \to 0$. Its limit has an exponential distribution with parameter $\bar{\Pi}(u)$ and is independent of $\ell_1(u)$.

Proof: Let us denote $d := d_1(u)$. For $a < c$, we define the σ-field generated by the lengths of the excursion intervals with $\ell > a$ which were completed before time d, i.e.

$$\mathbb{L}_a = \sigma\Big(\ell_k(a), k = 1, \dots, N_a(d)\Big).$$

It is clear that $N_a(d)$ is $\mathbb{L}_a$-measurable and that $\mathbb{L}_a \subset \mathbb{L}_b$, for $b < a$, which means that $(\mathbb{L}_a, a \geq 0)$ is a reversed filtration.
On the other hand, we decompose the path of X at the points $d_k(a)$ and use the convention $d_0(a) = 0$. The strong Markov property at $d_k(a)$ give us that conditionally on the event

$$\Big\{N_a(d) = n; \ell_k(a) = \lambda_k, k = 1, \dots, n\Big\}$$

the processes

$$Y^k = \Big(X_{d_{k-1}(a)+t}, 0 \leq t \leq d_k(a) - d_{k-1}(a)\Big), \qquad k = 1, \dots, n$$

are independent, and each Y^k has the same law as the process $(X_t, 0 \leq t \leq d_1(a))$ given that $\ell_1(a) = \lambda_k$. From Proposition 3, we know that for $b < a$,

$$N_b(d_1(a)) = N_b(g_1(a)) + 1$$

and that $N_b(g_1(a))$ is independent of $\ell_1(a)$ and has a geometric distribution with parameter $1 - \overline{\Pi}(a)/\overline{\Pi}(b)$ (one has to add 1 to take account of the excursion interval $(g_1(a), d_1(a))$ which has length $\ell_1(a) > b$). Hence, the distribution of $N_b(d)$ given $\mathbb{L}_a$ is that of

$$(\xi_1 + 1) + (\xi_2 + 1) + \cdot + (\xi_n + 1),$$

with $n = N_a(d)$, where $(\xi_k, 1 \leq k \leq n)$ are independent geometric random variable with parameter $1 - \overline{\Pi}(a)/\overline{\Pi}(b)$. Recall that the mean of a geometric distribution with parameter $1 - \overline{\Pi}(a)/\overline{\Pi}(b)$ satisfies

$$\mathbb{E}(\xi_k) = \frac{1 - \overline{\Pi}(a)/\overline{\Pi}(b)}{\overline{\Pi}(a)/\overline{\Pi}(b)},$$

then $\mathbb{E}(\xi_k + 1) = \overline{\Pi}(b)/\overline{\Pi}(a)$, which implies that

$$\mathbb{E}\Big(N_b(d)\Big|\mathbb{L}_a\Big) = N_a(d)\overline{\Pi}(b)/\overline{\Pi}(a),$$

which is the property of backwards martingale. By construction, we deduce that the paths are left-continuous.
Now, that since the martingale is positive it converges almost surely when a goes to 0. On the other hand, according to Proposition 3, the random variable $N_a(d) = N_a(g) + 1$ is

independent of $(M_{g+t}, t \geq 0)$ which implies that is also independent of $\ell_{N_a(d)}(a) = \ell_1(u)$, the limit is independent of $\ell_1(u)$ as well. Since $N_a(d) - 1$ has a geometric distribution with parameter $1 - \overline{\Pi}(u)/\overline{\Pi}(a)$, for $\lambda > 0$ we have

$$\mathbb{E}\Big(\exp\big\{ -\lambda(N_a(d) - 1)/\overline{\Pi}(a)\big\}\Big) = \frac{\overline{\Pi}(u)/\overline{\Pi}(a)}{1 - e^{-\lambda/\overline{\Pi}(a)}(1 - \overline{\Pi}(u)/\overline{\Pi}(a))},$$

which converges towards $\overline{\Pi}(u)/(\lambda + \overline{\Pi}(u))$ as a goes to 0. Hence

$$\lim_{a\to 0} \frac{N_a(d)}{\overline{\Pi}(a)},$$

has an exponential distribution with parameter $\overline{\Pi}(u)$ and in particular

$$\lim_{a\to\infty} \mathbb{E}\left(\frac{N_a(d)}{\overline{\Pi}(a)}\right) = \frac{1}{\overline{\Pi}(u)},$$

which implies convergence in $L^1(\mathbb{P})$ (by Scheffé's Lemma) and therefore that the martingale is uniformly integrable. ■

THEOREM 2. *The following assertions hold a.s.:*

i) *For all* $t \geq 0$, $N_a(t)/\overline{\Pi}(a)$ *converges as* a *tends to* 0, *the limit is denoted by* $L(t)$.

ii) *The mapping* $t \to L(t)$ *is increasing and continuous, it is called the local time of* X *at* 0.

iii) *The support of the Stieltjes measure* $\mathrm{d}L$ *is cl*$(\mathcal{L})$.

Proof: From our previous Proposition, we know that $N_a(t)/\overline{\Pi}(a)$ converges towards L_t, for $t = d_1(u)$. From the strong Markov property, applied to each $d_k(u)$, $k \geq 1$ and then letting u goes to 0, we see that the convergence holds for all t which belongs to

$$\mathbb{D} = \big\{d_k(u), u > 0 \text{ and } k \geq 1\big\}.$$

The mapping $L : \mathbb{D} \to [0, \infty)$ is increasing. We will show that there is a unique increasing extension on $[0, \infty)$. Let $\epsilon > 0$

$$\Lambda = \Big\{L(d_k(a)) - L(d_{k-1}(a)) \leq \epsilon, \text{ for all } k \leq N_a(d_1(c))\Big\}.$$

On the one hand, we know that $N_a(d_1(c)) - 1$ has a geometric distribution with index $1 - \overline{\Pi}(c)/\overline{\Pi}(a)$. On the other hand, from Proposition 4 and the strong Markov property, we have that conditionally on $\{N_a(d_1(c)) = n\}$, the random variables $L(d_k(a)) - L(d_{k-1}(a))$, for $k = 1, \dots, n$, are independent and have an exponential distribution with parameter $\overline{\Pi}(u)$, Hence,

$$\begin{aligned}
\mathbb{P}(\Lambda) &= \mathbb{E}\left(\prod_{k=1}^{N_a(d_1(c))} \mathbb{I}_{\{L(d_k(a)) - L(d_{k-1}(a)) \leq \epsilon\}}\right) \\
&= \sum_{n=1}^{\infty}\prod_{k=1}^{n} \mathbb{P}(L(d_k(a)) - L(d_{k-1}(a)) \leq \epsilon)\mathbb{P}(N_a(d_1(c)) = n) \\
&= \sum_{n=0}^{\infty} \frac{\overline{\Pi}(c)}{\overline{\Pi}(a)}\left(1 - \frac{\overline{\Pi}(c)}{\overline{\Pi}(a)}\right)^n \left(1 - e^{-\epsilon\overline{\Pi}(a)}\right)^{n+1} \\
&= \left(1 - e^{-\epsilon\overline{\Pi}(a)}\right)\frac{\overline{\Pi}(c)}{\overline{\Pi}(a)}\left[1 - \left(1 - \frac{\overline{\Pi}(c)}{\overline{\Pi}(a)}\right)\left(1 - e^{-\epsilon\overline{\Pi}(a)}\right)\right]^{-1}.
\end{aligned}$$

From Lemma 3 and the above equality, we see that $\mathbb{P}(\Lambda) = 1$ as a goes to 0, which shows that there is no gap larger than ϵ in the range of L on the set $\mathbb{D} \cap [0, d_1(c)]$, a.s. Letting ϵ tend to 0 and applying the Markov property sucessively at the $d_k(c)$'s, this implies that the range of L on $\mathbb{D}$ is dense a.s. Therefore, we have .a.s

$$\inf\{L(t) : t \in \mathbb{D}, t > s\} = \sup\{L(t) : t \in \mathbb{D}, t < s\} \quad \text{for all} \quad s \in [0, \infty). \tag{1.2}$$

Denoting the quantity in (1.2) by $L(s)$, we see that $L : [0, \infty) \to [0, \infty)$ is continuous and the unique increasing extension of $L : \mathbb{D} \to [0, \infty)$.
A monotonicity argument shows that

$$\sup\{N_a(t) : t \in \mathbb{D}, t < s\} \leq N_a(s) \leq \inf\{N_a(t) : t \in \mathbb{D}, s < t\},$$

and from (1.2), Proposition 4 and the continuity of L we see that $N_a(s)/\bar{\Pi}(a)$ converge towards $L(s)$ for all $s \in [0, \infty)$, a.s.
From (i) it is clear that supp $\mathrm{d}L \subset \mathrm{cl}(\mathcal{L})$. We still have to verify that a.s., for all $0 < s < t$, $L(s) < L(t)$ whenever X visits 0 in the open interval (s, t). With no loss of generality, we may restric our attention to the case when s and t vary in some countable dense set (for instance the rational numbers) and thus it is sufficient to show that for any fixed $0 < s < t$,

$$\mathbb{P}\Big(L(s) = L(t) \text{ and } X_u = 0 \text{ for some } u \in (s, t)\Big) = 0.$$

From the strong Markov property at the first return time to 0 after time s, all that is needed is to verify that $\mathbb{P}(L(v) = 0) = 0$ for every $v > 0$. Now, since 0 is instantaneous point, for any $\epsilon > 0$ there is $a > 0$ such that $\mathbb{P}(d_1(a) < v) > 1 - \epsilon$ and from Proposition 4, we know that $\mathbb{P}(L(d_1(a)) = 0) = 0$. Since L, increases, it follows that $\mathbb{P}(L(v) = 0) < \epsilon$. ■

It is important to note that the above construction depends on the constant $c > 0$ and if we change such constant this will affects L by a deterministic multiplicative factor. The construction of the local time of X starting from 0 may be extended to the case when X starts from an arbitrary point in a similar way.
For every stopping time $T < \infty$, denote by $L' = (L'(t), t \geq 0)$ the local time at 0 of the shifted process $X \circ \theta_T$. From Theorem 2, we deduce that a.s. $L(T + t) = L(T) + L'(t)$ for all $t \geq 0$, which is the property of an additive functional. The process L is adapted with respect to $(\mathcal{F}_t)$ and in particular if T is a stopping time such that $X_T = 0$ a.s. on the event $\{T < \infty\}$, by the strong Markov property, we have that under $\mathbb{P}(\cdot|T < \infty)$ the shifted process $(X_{T+t}, L(T + t) - L(T); t \geq 0)$ is independent of $\mathcal{F}_T$ and has the same law as (X, L). The following proposition says that L is the unique continuous adapted processes that increases only on the closure of $\mathcal{L}$ and which has the additive property.

PROPOSITION 5. *Let $A = (A_t, t \geq 0)$ be an increasing and continuous process adapted to $(\mathcal{F}_t, t \geq 0)$ such that*

i) *The support of the Stieltjes measure $\mathrm{d}A$ is included in cl($\mathcal{L}$), a.s.*
ii) *For every stopping time with $X_T = 0$ a.s. on $\{T < \infty\}$, the shifted process $(X_{T+t}, A_{T+t} - A_T; t \geq 0)$ is independent of $\mathcal{F}_T$ under $\mathbb{P}(\cdot|T < \infty)$ and has the same law as (X, A) under $\mathbb{P}$.*

Then, there exist a constant $k \geq 0$ such that $A = kL$ a.s.

Proof: Let $b > 0$, we will prove that $A_{d_1(b)}$ has an exponential distribution. First, we fix $s, t > 0$ and define $T = \inf\{u : A_u > t\}$. For every $u \geq 0$, we have

$$\{T < u\} = \{A_u > t\} \in \mathcal{F}_u,$$

since A is increasing, adapted and continuous. Therefore T is a stopping time. Now, we work conditionally on the event $\{T<\infty\}$. Since A is continuous and increases only on $\mathcal{L}$, we have that $A_T=t$ and $X_T=0$. By the condition (ii), the process $(X_{T+u},A_{T+u}-t;u\geq 0)$ has the same law as (X,A) under $\mathbb{P}$. Let us denote $A'=(A_{T+u}-t,u\geq 0)$. Then note that

$$\Big\{A_{d_1(b)}>t+s\Big\}=\Big\{A_{d_1(b)}>t,A'_{d'_1(b)}>s\Big\},$$

where $d'_1(b)$ is the right-end point of the first excursion interval of $X\circ\theta_T$ with $\ell>b$. Therefore by condition (ii) and the Markov property, we have

$$\begin{aligned}\mathbb{P}\Big(A_{d_1(b)}>t+s\Big)&=\mathbb{P}\Big(A_{d_1(b)}>t\Big)\mathbb{P}\Big(A'_{d'_1(b)}>s|T<d_1(b)\Big)\\&=\mathbb{P}\Big(A_{d_1(b)}>t\Big)\mathbb{P}\Big(A_{d_1(b)}>s\Big),\end{aligned}$$

which shows that $A_{d_1(b)}$ has an exponential distribution. Denote its parameter by $\lambda(b)$ and note that for all $b\in(0,c]$ it must be finite because otherwise A would be identically zero and there would be nothing to prove.

Now, we prove that λ and $\overline{\Pi}$ are proportional. Then we fix $a>b$ and we apply condition (ii) to the right-end points of the $N_b(g_1(a))$ excursion intervals with $\ell>a$ which were completed before $g_1(a)$, we deduce that

$$A_{d_1(a)}=\xi_1+\cdots+\xi_n,\qquad n=N_b(g_1(a))+1 \tag{1.3}$$

where ξ's are independent exponetial random variables with parameter $\lambda(b)$ which are also independent of $N_b(g_1(a))$. Now, taking the expectation of $A_{d_1(a)}$, we deduce

$$\frac{1}{\lambda(a)}=\mathbb{E}\Big(A_{d_1(a)}\Big)=\mathbb{E}\Big(\xi_1+\cdots+\xi_{N_b(g_1(a))+1}\Big)=\frac{1}{\lambda(b)}\mathbb{E}\Big(N_b(g_1(a))+1\Big)=\frac{\overline{\Pi}(b)}{\lambda(b)\overline{\Pi}(a)},$$

which implies that $k\lambda=\overline{\Pi}$ for some constant $k>0$.

Now, note that

$$\begin{aligned}\mathbb{E}\Big(\Big|A_{d_1(a)}-N_b(d_1(a))/\lambda(b)\Big|^2\Big)&=\mathbb{E}\Big(\Big|\xi_1+\cdots+\xi_{N_b(g_1(a))+1}-N_b(d_1(a))/\lambda(b)\Big|^2\Big)\\&=\frac{1}{\lambda^2(b)}\sum_{n=1}^{\infty}\mathbb{E}\Big(\Big|(\xi_1+\cdots+\xi_n)\lambda(b)-n\Big|^2\Big)\mathbb{P}\big(N_b(d_1(a))=n\big)\\&=\frac{1}{\lambda^2(b)}\sum_{n=1}^{\infty}n\mathbb{E}\Big(\Big(\xi_1\lambda(b)-1\Big)^2\Big)\mathbb{P}\big(N_b(d_1(a))=n\big)\\&=\frac{1}{\lambda^2(b)}\sum_{n=1}^{\infty}n\mathbb{P}\big(N_b(d_1(a))=n\big)=\frac{\overline{\Pi}(b)}{\lambda^2(b)\overline{\Pi}(a)}\end{aligned}$$

The right-hand side of the above equality converge towards 0 as b goes to 0 and from Theorem 2 part (i), we deduce that $A_{d_1(a)}=kL(d_1(a))$. A similar argument in the proof of Theorem 2 shows that $A_t=kL(t)$ for all $t\in\mathbb{D}$. Since L and A only increase in $\mathcal{L}$, this identity extends to $\mathcal{L}^c$. But this set is dense (since 0 is instantaneous) and since A and L are both continuous, we have $A_t=kL(t)$, everywhere. ■

COROLLARY 1. *There exist a constant $\mathfrak{d}\geq 0$ such that a.s.*

$$\int_0^t \mathbb{I}_{\{X_s=0\}}\mathrm{d}s=\int_0^t \mathbb{I}_{\{s\in cl(\mathcal{L})\}}\mathrm{d}s=\mathfrak{d}L(t)\qquad \textit{for all}\quad t\geq 0.$$

Proof: The set $\textsc{cl}(\mathcal{L})$ differ from $\mathcal{L}$ by at most countably many points, hence both integrals coincide. The condition (i) of proposition 5 is clearly fulfilled, and (ii) follows readily from the Markov property of X and the additivity property of the integral. ■