CHAPTER 1

Poisson point processes and subordinators.

In this chapter, we introduce basic notions on Poisson point processes and subordinators. Poisson processes and two remarkable families of related martingales are studied. We also introduce the notion of Poisson random measures in order to define the Poisson point process. The last part of this chapter concerns to subordinators and their connection with the Lévy-Kinthine formula.

1. Poisson point processes

1.1. Poisson processes.

DEFINITION 1. *A Poisson process with parameter $c > 0$ is a renewal process where the time between occurrences is exponentially distributed with parameter c. More precisely take a sequence $(\tau_n, n \geq 1)$ of independent exponential random variables with parameter c and introduce the partial sums $S_n = \tau_1 + \cdots + \tau_n$, $n \in \mathbb{N}$. The right-continuous inverse*

$$N_t = \sup\Big\{n \in \mathbb{N} : S_n \leq t\Big\}, \qquad t \geq 0,$$

is called a Poisson process of parameter c.

Let us explain some details of the above definition. We first recall that S_n has the same law as a Gamma distribution with parameters c and n. Therefore, for any fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$

$$\begin{aligned}\mathbb{P}(N_t = k) &= \mathbb{E}\Big(\mathbb{1}_{\{S_k \leq t < S_{k+1}\}}\Big) = \mathbb{E}\Big(\mathbb{1}_{\{S_k \leq t\}}\mathbb{P}(\tau_{n+1} \geq t - S_k)\Big)\\ &= \frac{1}{\Gamma(k)}\int_0^t e^{-c(t-x)}c^k e^{-cx}x^{k-1}\mathrm{d}x = \frac{c^k t^k}{k!}e^{-ct}.\end{aligned}$$

This implies that for any fixed $t > 0$, N_t is a Poisson r.v. with parameter tc, from where this process takes his name.

The lack of memory property of the exponential law implies that for every $0 \leq s \leq t$, the increment $N_{t+s} - N_t$ has the Poisson distribution with parameter cs and is independent of the σ-field generated by $(N_u, 0 \leq u \leq t)$, i.e. that the Poisson process $N = (N_t, t \geq 0)$ is a process with independent and homogeneous increments. In particular, the Poisson process N is a strong Markov process.

Here, we are interested in two families of martingales related to the natural filtration $(\mathcal{F}_t)$ of the Poisson process N. Let us define

$$M_t = N_t - ct, \quad \text{and} \quad \xi^q_t = \exp\Big\{-qN_t + ct(1 - e^{-q})\Big\}, \qquad t \geq 0, \quad q > 0.$$

From the independence and the homogeneity of the increments, we get that

$$\mathbb{E}(N_{t+s}|\mathcal{F}_t) = \mathbb{E}(N_{t+s} - N_t + N_t|\mathcal{F}_s) = N_t + cs,$$

then substracting $c(t+s)$ in the both sides, we get that $M=(M_t, t\geq 0)$ is a martingale related to $(\mathcal{F}_t)$. The additivity of the exponents and similar arguments as above, give us that $\xi^q=(\xi^q_t, t\geq 0)$ is also a martingale related to $(\mathcal{F}_t)$.
Recall that one says that a process $H=(H_t, t\geq 0)$ is called **predictable** if it is measurable in the sigma-field generated by the left-continuous adapted processes. *If this notion is difficult to understand, just think in processes that can be approximated by left-continuous adapted processes.*
Now, let us introduce the stochastic integral related to the Poisson process N by

$$\int_0^t H_s \mathrm{d}N_s = \sum_{s\leq t} H_s \Delta N_s, \quad \text{where} \quad \Delta N_s = N_s - N_{s-}.$$

Note that from the definition of N, we have the following identity

$$\int_0^t H_s \mathrm{d}N_s = \sum_{n=1}^{\infty} H_{\tau_n} \mathbb{1}_{\{\tau_n\leq t\}}.$$

PROPOSITION 1. *Let H be a predictible process with*

$$\mathbb{E}\left(\int_0^t |H_s|\mathrm{d}s\right) < \infty, \qquad \textit{for all} \quad t\geq 0,$$

then the compensated integral

$$\int_0^t H_s \mathrm{d}N_s - c\int_0^t H_s \mathrm{d}s, \qquad t\geq 0,$$

is a martingale related to $(\mathcal{F}_t)$. Moreover, we have the well-known **compensation formula**

$$\mathbb{E}\left(\int_0^t H_s \mathrm{d}N_s\right) = c\mathbb{E}\left(\int_0^t H_s \mathrm{d}s\right).$$

Proof: Let us suppose that H is a simple process, i.e.

$$H_t = H_{t_i} \quad \text{for} \quad t\in(t_i, t_{i+1}]$$

where $t_0 < t_1 < \cdots < t_n$ is a partition of the interval $[0,t]$ and H_{t_i} is $\mathcal{F}_{t_i}$-measurable. We also suppose that H is bounded. Then it is clear

$$\mathbb{E}\left(H_{t_i}\Big[(N_t - ct) - (N_{t_i} - ct_i)\Big]\Big|\mathcal{F}_{t_i}\right) = 0 \quad \text{for} \quad t\in(t_i, t_{i+1}],$$

which implies that $\int_0^t H_s \mathrm{d}M_s$ is a martingale.
Next, we suppose that H is a left-continuous process bounded for some constant $C>0$. For $n\in\mathbb{N}$, let us define

$$H^{(n)}_t = H_{k/2^n}, \qquad \text{for} \quad k2^{-n} < t \leq (k+1)2^{-n}.$$

Therefore, $(H^{(n)}, n\geq 1)$ is a sequence of simple, bounded and left-continuous processes such that

$$H^{(n)}_t \longrightarrow H_t \quad \text{for all} \quad t\geq 0 \quad \text{almost surely.}$$

From the dominated convergence theorem, we deduce that the integral $I^n_t := \int_0^t H^{(n)}_s \mathrm{d}M_s$, where $M_s = N_s - cs$, converges towards $I_t := \int_0^t H_s \mathrm{d}M_s$, almost surely. Since I^n is a martingale then I is also a martingale and the result follows for H a bounded left-continuous adapted process.

In order to extend the result to any bounded predictable process, we will use the monotone class theorem. Define,

$$\mathcal{A} = \left\{ H \text{ bounded and predictable} : \left(\int_0^t H_s \mathrm{d}M_s, t \geq 0 \right) \text{ is a martingale} \right\}.$$

Then, from above, $\mathcal{A}$ contains the bounded left-continuous adapted processes. Moreover from the monotone convergence theorem, we get that if H^n is an increasing sequence in $\mathcal{A}$ which converge towards H with H bounded, then H belongs to $\mathcal{A}$. Therefore, applying the monotone class theorem we obtain that $\mathcal{A}$ contains all bounded predictable processes.
Now, for simplicity we suppose that H is positive and define the following stopping time $T_C = \inf\{t \geq 0 : H_s \geq C\}$. Then,

$$(I_C)_t := \int_0^{T_C \wedge t} H_s \mathrm{d}M_s, \qquad \text{for} \quad t \geq 0,$$

is a martingale. From the optimal stopping theorem, we get the compensation formula.
On the other hand, since T_C goes to ∞, when C increases, we have that I_C converges towards $\int_0^t H_s \mathrm{d}M_s$, almost surely. Hence applying the monotone convergence theorem, we have

$$\mathbb{E}\left(\int_0^{T_C \wedge t} H_s \mathrm{d}N_s \right) \longrightarrow \mathbb{E}\left(\int_0^t H_s \mathrm{d}N_s \right) \quad \text{as} \quad C \to \infty.$$

On the other hand, using again the monotone convergence theorem we get that

$$\mathbb{E}\left(\int_0^{T_C \wedge t} H_s \mathrm{d}s \right) \longrightarrow \mathbb{E}\left(\int_0^t H_s \mathrm{d}s \right) \quad \text{as} \quad C \to \infty,$$

and now the proof is completed. ■

Another important family of martingales related to Poisson processes is definde in the following proposition.

PROPOSITION 2. *Let h be a measurable positive function, then the exponential process*

$$\exp\left\{ -\int_0^t h(s) \mathrm{d}N_s + c \int_0^t (1 - e^{-h(s)}) \mathrm{d}s \right\}, \quad t \geq 0,$$

is a martingale related to $(\mathcal{F}_t)$. In particular, we have the well-known **exponential formula**

$$\mathbb{E}\left(\exp\left\{ -\int_0^t h(s) \mathrm{d}N_s \right\} \right) = \exp\left\{ -c \int_0^t (1 - e^{-h(s)}) \mathrm{d}s \right\}.$$

Proof: Let us suppose that h is a step function. Under this assumption, the result is clear from the exponential martingale ξ^q, taking q be equal to the value of h in one step. In order to extend such result to any h measurable, it is enough to apply the functional version of the Monotone class theorem. ■

We remark that the above result is also true when we replace h by H a bounded predictable process.

We conclude this section by proving a criterion for the independence of Poisson processes which will be very important for the sequel. First, we recall the definition of a

Poisson process related to a given filtration. Let $(\mathcal{F}_t)$ be a filtration, an $(\mathcal{F}_t)$-Poisson process with parameter $c > 0$ is a right continuous adapted process, such that $N_0 = 0$ and for every $s < t$, and $k \in \mathbb{N}$,

$$\mathbb{P}(N_t - N_s = k|\mathcal{F}_s) = c^k \frac{(t-s)^k}{k!} \exp(-c(t-s)).$$

PROPOSITION 3. *Let N and N' be two $(\mathcal{F}_t)$-Poisson processes. They are independent if and only if they never jump simultaneously, that is*

$$N_t - N_{t-} = 0 \quad or \quad N'_t - N'_{t-} = 0 \quad for\ all\ t > 0 \quad almost\ surely.$$

It is crucial in Proposition 3 to assume that N and N' are Poisson processes in the same filtration. Otherwise, the result is not true. For example, if we take $N'_t = N_{2t}$ it is clear that N' is still a Poisson process which never jumps at the same times that N and that it is not independent of N.

Proof: First, we suppose that N and N' are independent and let $(\tau_n, n \geq 1)$ be the successive jumps times of N. Then

$$\sum_{s>0} (\Delta N_s)(\Delta N'_s) = \sum_{s>0} (\Delta N'_s) \quad \text{a.s.}$$

On the other hand, since the laws of the jumps are diffuse, it is clear that for every fixed time t, $\Delta N'_t$ is a.s. zero. From the independence of N' and $(\tau_n, n \geq 1)$, we get $\Delta N'_{\tau_n} = 0$, a.s. for every n. Hence the processes N and N' never jumps simultaneously.

For the converse, we need the following lemma.

LEMMA 1. *Let M be a càdlàg martingale of bounded variation and M' a càdlàg bounded martingale in the same filtration. If M and M' never jumps simultaneously then MM' is a martingale.*

Proof: In order to proof this lemma, it is enough to show that for any bounded stopping time T, we have

$$\mathbb{E}(M_T M'_T) = \mathbb{E}(M_0 M'_0).$$

Take, $0 = t_0 < t_1 < \cdots < t_k = A$, a partition of the interval $[0, A]$ with $T \leq A$ a.s. Therefore

$$M_T M'_T - M_0 M'_0 = \sum_{t_i < T} M_{t_i}(M'_{t_{i+1}} - M'_{t_i}) + \sum_{t_i < T} M'_{t_i}(M_{t_{i+1}} - M_{t_i}) + \sum_{t_i < T} (M'_{t_{i+1}} - M'_{t_i})(M_{t_{i+1}} - M_{t_i}).$$

If we take the expectation of the above equality, it is clear that the expectation of the first two terms of the right hand-side are equal to 0, hence

$$\mathbb{E}(M_T M'_T - M_0 M'_0) = \mathbb{E}\left(\sum_{t_i < T} (M'_{t_{i+1}} - M'_{t_i})(M_{t_{i+1}} - M_{t_i})\right).$$

On the other hand, since M' is bounded and M is of bounded variation, we have

$$\left|\sum_{t_i < T} (M'_{t_{i+1}} - M'_{t_i})(M_{t_{i+1}} - M_{t_i})\right| \leq C \left|\sum_{t_i < T} (M_{t_{i+1}} - M_{t_i})\right| \leq C V_0^A(M) \leq C', \tag{1.1}$$

where $V_0^A(M)$ is the total variation of M. When the length of the longest of the subintervals in the partition goes to 0, we have that

$$\sum_{t_i<T}(M'_{t_{i+1}} - M'_{t_i})(M_{t_{i+1}} - M_{t_i}) \longrightarrow \sum_{s<T}(\Delta M_s)(\Delta M'_s) = 0,$$

the convergence follows from the fact that M (also recall that M is of bounded variation) and M' are càdlàg and never jump simultaneously. Hence applying the dominated convergence theorem, we obtain the result. ■

Now, let h and h' be two step functions and define the exponential martingales

$$M_t = \exp\left\{-\int_0^t h(s)\mathrm{d}N_s + c\int_0^t (1-e^{-h(s)})\mathrm{d}s\right\},$$

and

$$M'_t = \exp\left\{-\int_0^t h'(s)\mathrm{d}N'_s + c'\int_0^t (1-e^{-h'(s)})\mathrm{d}s\right\},$$

which are defined on the same filtration. Since N and N' never jumps simultaneously, the martingales M and M' never jumps simultaneously too and their product is also a martingale from the previous lemma. Hence $\mathbb{E}\left(M_t M'_t\right) = 1$, which implies that

$$\begin{aligned}\mathbb{E}\Bigg(\exp\Bigg\{-\int_0^t h(s)\mathrm{d}N_s - \int_0^t h'(s)\mathrm{d}N'_s\Bigg\}\Bigg)& \\ &= \exp\left\{-c\int_0^t (1-e^{-h(s)})\mathrm{d}s\right\}\exp\left\{-c'\int_0^t (1-e^{-h'(s)})\mathrm{d}s\right\} \\ &= \mathbb{E}\left(\exp\left\{-\int_0^t h(s)\mathrm{d}N_s\right\}\right)\mathbb{E}\left(\exp\left\{-\int_0^t h(s)\mathrm{d}N_s\right\}\right),\end{aligned}$$

and the independence is proved. ■

1.2. Poisson measures and Poisson point processes.

DEFINITION 2. *Let $(E,\mathcal{E},\mu)$ be a measurable space with μ a σ-finite measure. A Poisson random measure with intensity measure μ is a familly of random variables $M = \{M(A), A\in\mathcal{E}\}$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that*

i) *If $B\in\mathcal{E}$ is such that $\mu(B)<\infty$, the random variable $M(B)$ has a Poisson distiribution with parameter $\mu(B)$, i.e.*

$$\mathbb{P}(M(B)=k) = \frac{\mu(B)^k}{k!}\exp\{-\mu(B)\}, \qquad \textit{for} \quad k=0,1,\dots$$

If $\mu(B)=\infty$, then $M(B)=\infty$ a.s.

ii) *Let $B_1, B_2, \dots, B_n$ be a finite sequence of pairwise disjoint sets of $\mathcal{E}$, the random variables $M(B_1),\dots,M(B_n)$ are independent.*

Example: Let $E=\mathbb{R}_+$, $\mathcal{E}$ its Borel σ-field and $\mu = c\lambda$, where λ is the Lebesgue measure and $c>0$. The process

$$t \longrightarrow M([0,t]),$$

is a Poisson process with parameter c. Conversely, let N be a Poisson process with parameter $c = 1$ and define

$$M(B) = \int_0^\infty \mathbb{I}_B(s)\mathrm{d}N_s,$$

it is not difficult to show (good exercice: use the exponential martingale and the additivity of exponents to prove it), that the familly M is a Poisson random measure.
From their definition and Proposition 3, it is clear that Poisson random measures satisfy the following properties:

- **Superposition property:** Let $(\mu_n, n \geq 1)$ be a sequence of σ-finite measures and define $\mu = \sum_{n\geq 1}\mu_n$. If μ is also a σ-finite measure and $M^{(1)}, M^{(2)}, \dots$ are independent Poisson random measures with intensity measures $\mu_1, \mu_2, \dots$ respectively, then $M = \sum_{n\geq 1} M^{(n)}$ is a Poisson random measure with intensity measure μ.
- **Splitting property:** Let M be a Poisson random measure on $(E, \mathcal{E})$ with intensity μ and (B_i) a sequence of pairwise disjoint sets of $\mathcal{E}$, then the restictions $M|_{B_1}, M|_{B_2}, \dots$ are independent Poisson random measures with intensity $\mu(\cdot \cap B_1), \mu(\cdot \cap B_2), \dots$ respectively.
- **Image property:** Let $f : (E, \mathcal{E}) \to (G, \mathcal{G})$ be a measurable function, μ a σ-finite measure on $(E, \mathcal{E})$ and γ the image measure of μ by f. We suppose that γ is also σ-finite. If M is a Poisson random measure on $(E, \mathcal{E})$ with intensity measure μ and if we define

  $$M \circ f^{-1}(C) = M(f^{-1}(C)), \qquad \text{for } C \in \mathcal{G}.$$

  Then $M \circ f^{-1}$ is a Poisson random measure with intensity measure γ.

We now turn our attention to the construction of Poisson random measures. First, let us suppose that $\mu(E) < \infty$ and define the probability measure

$$\rho(B) = \frac{\mu(B)}{\mu(E)}, \qquad \text{for} \quad B \in \mathcal{E}.$$

Let us take $(\xi_n, n \geq 1)$ a sequence of independent and identically distributed random variables with law ρ and N a Poisson random variable with parameter $\mu(E)$ which is independent of $(\xi_n, n \geq 1)$.
Next, we define the random measure

$$M(\mathrm{d}x) = \sum_{i=1}^{N} \delta_{\xi_i}(\mathrm{d}x),$$

where δ_y is the Dirac measure in y. The random measure M is a counting measure, i.e. for any $B \in \mathcal{E}$

$$M(B) = \text{card}\{i \leq N : \xi_i \in B\},$$

and we claim that M is a Poisson random measure with intensity measure μ (we will prove this fact below). In the σ-finite case, we can construct a Poisson random measure using the superposition property and the splitting property. More precisley, we chose a partition $(B_n, n \geq 1)$ of E such that each element of the partition is measurable and of finite measure. Then, we construct a sequence of independent Poisson random measures whose intensity measures are the restiction of μ on each B_n. Finally, the superposition property give us the desired result.

Let us verify the contruction in the finite case, for simplicity we choose $\mu(E) = 1$. First, we compute the distribution of $M(B)$, for $B \in \mathcal{E}$

$$\mathbb{P}(M(B) = k) = \sum_{j=k}^{\infty} \mathbb{P}(N = j)\mathbb{P}\left(\sum_{i=1}^{j} \mathbb{I}_{\{\xi_i \in B\}} = k\right).$$

Since $(\mathbb{I}_{\{\xi_i \in B\}}, i \geq 1)$ is a sequence of Bernoulli random variables with parameter $\mu(B)$, then

$$\begin{aligned}\mathbb{P}(M(B) = k) &= \sum_{j=k}^{\infty} \frac{e^{-1}}{j!}\binom{k}{j}\mu(B)^k(1-\mu(B))^{j-k} \\ &= \frac{\mu(B)^k}{k!}e^{-1}\sum_{l=0}^{\infty}\frac{(1-\mu(B))^l}{l!} \\ &= e^{-\mu(B)}\frac{\mu(B)^k}{k!}.\end{aligned}$$

Now, we prove the independence (property (ii)). Let B and B' be two disjoint sets in $\mathcal{E}$ and define $X_t = \xi_{N_t}$ where $(N_t, t \geq 0)$ is a Poisson process with parameter 1 (or $\mu(E)$) which is independent of $(\xi_n, n \geq 1)$, and $(\mathcal{F}_t)$ the natural filtration of $X = (X_t, t \geq 0)$.
Next, we define the counting processes

$$N_t^B = \text{card}\{i \leq N_t : \xi_i \in B\} \quad \text{and} \quad N_t^{B'} = \text{card}\{i \leq N_t : \xi_i \in B'\}.$$

The processes N^B and $N^{B'}$ are two $(\mathcal{F}_t)$-Poisson processes and since B and B' are disjoints sets, they never jump simultaneously. Therefore, the independence follows from the above and noting that $M(B) = N_1^B$ and $M(B') = N_1^{B'}$.

PROPOSITION 4 (Campbell's formula). *Lef $f : E \to \mathbb{R}_+$ be a measurable function and M a Poisson random measure with intensity measure μ. Let us define*

$$< M, f >= \int_E f(x)M(\mathrm{d}x),$$

then

$$\mathbb{E}\Big(\exp\{- < M, f > \}\Big) = \exp\left\{-\int_E \big(1 - e^{-f(x)}\big)\mu(\mathrm{d}x)\right\}. \tag{1.2}$$

Proof: Let us first suppose that $\mu(E) < \infty$ and that f is a simple function, i.e.

$$f(x) = \sum_{i=1}^{\infty} c_i \mathbb{I}_{\{x \in B_i\}}, \qquad c_i \geq 0 \quad \text{and} \quad (B_i) \text{ a partition of } E,$$

it is then clear that $< M, f >= \sum_{i=1}^{\infty} c_i M(B_i)$.
From the previous construction, we have $< M, f >= \sum_{i=1}^{N} f(\xi_i)$, where $(\xi, i \geq 1)$ is a sequence of independent and identically distributed r.v.'s with law $\rho(\cdot) = \mu(\cdot)/\mu(E)$ and

N a Poisson r.v. with parameter $\mu(E)$ which is independent of $(\xi_n, n \geq 1)$. Therefore,

$$\begin{aligned}\mathbb{E}\Big(\exp\big\{- < M, f > \big\}\Big) &= \mathbb{E}\left(\exp\left\{-\sum_{i=1}^{N} f(\xi_i)\right\}\right)\\ &= \sum_{k=0}^{\infty}\frac{\mu(E)^k}{k!}e^{-\mu(E)}\mathbb{E}\Big(e^{-f(\xi_1)}\cdots e^{-f(\xi_k)}\Big)\\ &= \sum_{k=0}^{\infty}\frac{\mu(E)^k}{k!}e^{-\mu(E)}\left(\int e^{-f(x)}\frac{\mu(\mathrm{d}x)}{\mu(E)}\right)^k\\ &= \exp\left\{-\int_E \big(1-e^{-f(x)}\big)\mu(\mathrm{d}x)\right\}.\end{aligned}$$

In order to prove Campbell's formula for any positive measurable function, we define the integral $< M, f >$ by approximation . More precisely, let us define

$$f_n(x) = \frac{k}{2^n} \qquad \text{on} \quad \left\{x : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}.$$

Then, it is clear that f_n converge towards f and applying the monotone convergence theorem, we get

$$< M, f >= \lim_{n\to\infty} < M, f_n >,$$

which implies (1.2).
Now, we deal with the case $\mu(E) = \infty$. Let $(B_n, n \geq 1)$ a sequence of disjoint sets in $\mathcal{E}$ such that $\mu(B_n) < \infty$ for every $n \geq 1$. Next for every $n \geq 1$, we define $E_n = \cup_{k\leq n} B_n$ and take the restriction $M^n = M|_{E_n}$. Let $f \geq 0$ be measurable, then by definition

$$< M^n, f >= \int_E f(x)\,\mathbb{I}_{E_n}(x) M(\mathrm{d}x),$$

which satisfies Campbell's formula. Since E_n increases towards E as n goes to ∞, it is clear that $f\,\mathbb{I}_{E_n}$ and $(1-e^{-f})\,\mathbb{I}_{E_n}$ increase towards f and $(1-e^{-f})$, respectively. Again applying the monotone convergence theorem, we get

$$< M, f >= \lim_{n\to\infty} < M^n, f > \text{ and } \int_E \big(1-e^{-f(x)}\big)\mu(\mathrm{d}x) = \lim_{n\to\infty}\int_{E_n}\big(1-e^{-f(x)}\big)\mu(\mathrm{d}x)$$

which implies (1.2) in the general case. ■

Now, we introduce M a Poisson random measure on $[0,\infty)\times E$ with intensity measure $\lambda\otimes\mu$, where λ is the Lebesgue measure on $[0,\infty)$ and μ is a σ-finite measure on E.

LEMMA 2. *Almost surely, for all* $t \geq 0$

$$M\Big(\{t\}\times E\Big) = 0 \textit{ or } 1.$$

Proof: First, we suppose that $\mu(E) < \infty$. For $n \geq 1$, by the stationary and independence property of M

$$\mathbb{P}\bigg(\exists k \leq 2^n : M\Big(\big[(k-1)2^{-n}, k2^{-n}\big)\times E\Big) \geq 2\bigg) \leq 2^n\mathbb{P}\Big(M\big([0,2^{-n})\times E\big) \geq 2\Big).$$

On the other hand

$$\mathbb{P}\Big(M\big([0,2^{-n})\times E\big)\geq 2\Big) = 1 - e^{-2^{-n}\mu(E)} - 2^{-n}\mu(E)e^{-2^{-n}\mu(E)} \leq 2\big(2^{-n}\mu(E)\big)^2,$$

which implies that

$$\mathbb{P}\bigg(\exists k \le 2^n:\; M\Big(\big[(k-1)2^{-n}, k2^{-n}\big) \times E\Big) \ge 2\bigg) \le 2^{-(n-1)}\mu^2(E).$$

The result follows taking n goes towards ∞.
In order to prove this result when $\mu(E) = \infty$, we will use a similar argument as in Proposition 4. Let $(E_n, n \ge 1)$ as in the proof of Proposition 4 and define, for each $n \ge 1$, the function $f_n(x) = 1\!\!1_{\{\{t\}\times E_n\}}(x)$ and note that almost surely, for all $t \ge 0$

$$< M, f_n >= M\big(\{t\} \times E_n\big) \le 1,$$

since $\mu(E_n) < \infty$. On the other hand, the sequence $(f_n, n \ge 1)$ is increasing and it converges towards $f = 1\!\!1_{\{\{t\}\times E\}}$. Then applying the monotone convergence theorem, we have that

$$M\big(\{t\} \times E\big) =< M, f >= \lim_{n\to\infty} < M, f_n >= \lim_{n\to\infty} M\big(\{t\} \times E_n\big) \le 1,$$

which is satisfied almost surely for every $t \ge 0$, since

$$A = \Big\{\omega \in \Omega:\; \text{for all } t \ge 0,\; < M, f >\le 1\Big\},$$

satisfies that

$$A = \bigcap_{n\ge 1}\Big\{\omega \in \Omega:\; \text{for all } t \ge 0,\; < M, f_n >\le 1\Big\},$$

and each of this set has probability equal 1. ■

If $M(\{t\} \times E) = 1$, there exists one and only one point $\Delta_t \in E$ such that

$$M|_{\{t\}\times E} = \delta_{(t,\Delta_t)}.$$

If $M(\{t\} \times E) = 0$, then we do not define Δ_t in E.

DEFINITION 3. *The process defined by $\Delta = (\Delta_t, t \ge 0)$ is Poisson point process with characteristic measure μ.*

LEMMA 3. *Let $B \in \mathcal{E}$ such that $0 < \mu(B) < \infty$ and define*

$$T_B = \inf\{t \ge 0:\; \Delta_t \in B\}.$$

Then, T_B and Δ_{T_B} are independent random variables. The distribution of T_B is an exponential r.v. with parameter $\mu(B)$ and that of Δ_{T_B} is given by $\mu(\cdot \cap B)/\mu(B)$.

Proof: Take $A \subset B$. A straightforward calculation give us that

$$\mathbb{P}(T_B \le t, \Delta_{T_B} \in A) = \mathbb{P}(T_A < T_{B\setminus A}, T_A \wedge T_{B\setminus A} \le t).$$

On the other hand, it is clear that T_A is the first jump of the Poisson process N^A defined by

$$N^A_t = \text{card}\{s \le t : \Delta_s \in A\},$$

hence T_A has an exponential distribution with parameter $\mu(A)$. Since A and $B \setminus A$ are disjoint, then T_A and $T_{B\setminus A}$ are independent exponential random variables with parameters $\mu(A)$ and $\mu(B) - \mu(A)$, respectively. Therefore,

$$\mathbb{P}(T_B \le t, \Delta_{T_B} \in A) = \frac{\mu(A)}{\mu(B)}\big(1 - e^{-t\mu(B)}\big),$$

and we get the desired result. ■

2. Subordinators

Subordinators is an extension of the notion of Poisson point processes and generally, they are defined as follows.

DEFINITION 4. *A subordinator is a stochastic process taking values in* $[0, \infty)$ *with càdlàg paths (i.e. right-continuous and with left limits) and such that it has independent and homogeneous increments.*

This class of processes play an important role in the study of Local times and excursion theory of Markov processes. Their connection will be studied in the next two chapters.
It is important to note that subordinators have increasing sample paths. We are also interested in killed subordinators which are a slightly more general than subordinators. More precisely, let $\sigma = (\sigma_t, t \geq 0)$ be a subordinator and $\mathbf{e} = \mathbf{e}_q$ an independent exponential time with parameter $q \geq 0$, the process $\sigma^{(q)}$ taking values in $[0, \infty]$ and defined by

$$\sigma_t^{(q)} = \begin{cases} \sigma_t & \text{if } t \in [0, \mathbf{e}) \\ \infty & \text{if } t \in [\mathbf{e}, \infty) \end{cases}$$

is called a subordinator killed at rate q. In fact any right-continuous non-decreasing process $X = (X_t, t \geq 0)$ taking values in $[0, \infty]$ is a subordinator killed at rate $q > 0$ if and only if $\mathbb{P}(X_t < \infty) = e^{-qt}$ and conditionally on $X_t < \infty$, the increment $X_{t+s} - X_t$ is independent of $(X_u, 0 \leq u \leq t)$ and has the same law as X_s.
Note that a Poisson process is a subordinator whose jumps are equal to one. Another important example of a subordinator is the first passage time of the Brownian motion, i.e.

$$\sigma_x = \inf\{t : B_t \geq x\}, \qquad x \geq 0,$$

where $B = (B_s, s \geq 0)$ is the standard Brownian motion.
Using the decomposition

$$\sigma_n = \sigma_1 + (\sigma_2 - \sigma_1) + \cdots + (\sigma_n - \sigma_{n-1}),$$

and the independence and homogeneity of the increments of σ, we observe that the Laplace transform of σ_n satisfies

$$\mathbb{E}(e^{-q\sigma_n}) = \Big(\mathbb{E}(e^{-q\sigma_1})\Big)^n \qquad q \geq 0.$$

Moreover if t is a rational number, we have

$$\mathbb{E}(e^{-q\sigma_t}) = \Big(\mathbb{E}(e^{-q\sigma_1})\Big)^t \qquad q \geq 0.$$

By right-continuity of the paths, this last equality holds for an arbitrary $t \geq 0$. Now, if

$$\mathbb{E}(e^{-q\sigma_1}) = e^{-\Phi(q)} \qquad \text{for all} \quad q \geq 0,$$

for a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ called the Laplace exponent, then

$$\mathbb{E}(e^{-q\sigma_t}) = e^{-t\Phi(q)} \qquad \text{for all} \quad t, q \geq 0.$$

A natural quesiton is: Which are the functions Φ that appear as the Laplace exponent of a subordinator?

THEOREM 1 (De Finetti, Lévy, Khintchine). *(i) If* Φ *is the Laplace exponent of a subordinator* $\sigma = (\sigma_t, t \geq 0)$*, then there exist a unique pair* (k, d) *of nonnegative real numbers and a unique measure* Π *on* $(0, \infty)$ *with* $\int (1 \wedge x)\Pi(\mathrm{d}x) < \infty$*, such that for every* $\lambda \geq 0$

$$\Phi(\lambda) = k + d\lambda + \int_{(0,\infty)} \big(1 - e^{-\lambda x}\big)\Pi(\mathrm{d}x). \tag{1.3}$$

(ii) Conversely, any function Φ than can be expressed in the form (1.3) is the Laplace exponent of a subordinator.

Equation (1.3) will be referred to as the *Lévy-Khintchine formula*; one calls k the killing rate, d the drift and Π the Lévy measure of σ.

Proof: (i) For any $t > 0$, we have that

$$\mathbb{E}\Big(e^{-q\sigma_t}\Big) = \exp\{-t\Phi(q)\}.$$

Note that Φ can be expressed as follows

$$\Phi(q) = \lim_{n\to\infty} n\big(1 - e^{-\Phi(q)/n}\big) \quad \text{and} \quad 1 - e^{-\Phi(q)/n} = \mathbb{E}\Big(1 - e^{-q\sigma_{1/n}}\Big).$$

Now, define $\bar{F}_n(x) = n\mathbb{P}(\sigma_{1/n} > x)$, $x \geq 0$ and note that

$$\frac{\Phi(q)}{q} = \lim_{n\to\infty} \int_{(0,\infty)} e^{-qx} \bar{F}_n(x)\mathrm{d}x.$$

Hence $(\bar{F}_n(x)\mathrm{d}x, n \geq 1)$ converge vaguely as n goes to ∞ towards a given measure on $[0, \infty[$. Since each function $\bar{F}_n$ decreases, the limit has necessarily the form

$$d\delta_0(\mathrm{d}x) + \bar{\Lambda}(x)\mathrm{d}x,$$

where $d \geq 0$, $\bar{\Lambda}$ is a non-increasing function. Thus

$$\frac{\Phi(q)}{q} = d + \int_{(0,\infty)} e^{-qx}\bar{\Lambda}(x)\mathrm{d}x$$

which after integrating by parts proves (1.3) with $k = \bar{\Lambda}(\infty)$ and $\Pi(\mathrm{d}x) = -d\bar{\Lambda}(x)$ on $(0, \infty)$. In order to finish the proof, let us verify the condition of the measure Π established before (1.3), but this is clear since

$$\int_{(0,\infty)} (1 \wedge x)\Pi(\mathrm{d}x) = \int_0^1 \bar{\Lambda}(x)\mathrm{d}x < \infty.$$

(ii) Take a measure Π satisfying that $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$ and construct a Poisson point process $(\Delta_t, t \geq 0)$ with characteristic measure $\Pi + k\delta_\infty$.
Now, let us define $T_\infty = \inf\{t : \Delta_t = \infty\}$ and for all $t \leq T_\infty$, $\Sigma_t = \sum_{0\leq s\leq t} \Delta_s$. From Campbell's formula, we get

$$\mathbb{E}\left(\exp\big\{-q\Sigma_t\big\}\right) = \exp\left\{-tk - t\int_{(0,\infty)}(1 - e^{-qy})\Pi(\mathrm{d}y)\right\}, \quad q, t \geq 0.$$

From our hypothesis, we have

$$\int_{(0,\infty)} (1 - e^{-qy})\Pi(\mathrm{d}y) < \infty.$$

Since

$$\lim_{q\to 0} \int_{(0,\infty)} (1 - e^{-qy})\Pi(\mathrm{d}y) = 0,$$

we deduce that $\Sigma_t < \infty$, whenever $t < T_\infty$ a.s. and

$$\mathbb{E}\Big(\exp\{-q\Sigma_t\}\Big) = \exp\{-t\Phi(q)\}, \quad \text{with} \quad d = 0.$$

Therefore, since Δ is a Poisson point process Σ is a càdàg process with lifetime T_∞ and that its increments are independent and stationary on $[0, T_\infty)$. Now, if we take $d > 0$ and we define $\Sigma_t^{(d)} = dt + \Sigma_t$, it is easy to see that it is still a killed subordinator and that

$$\Phi(q) = k + dq + \int_{(0,\infty)} \left(1 - e^{-qx}\right) \Pi(\mathrm{d}x).$$

The proof is now completed. ■

The proof of the Lévy-Khintchine formula gives us also a probabilistic interpretation.

COROLLARY 1 (Lévy-Itô decomposition). *One has a.s., for every $t \geq 0$:*

$$\sigma_t = dt + \sum_{0 \leq s \leq t} \Delta_s,$$

where $\Delta = (\Delta_s, s \geq 0)$ is a Poisson point process with values in $(0, \infty]$ and characteristic measure $\Pi + k\delta_\infty$. The lifetime of σ is the given by $\zeta = \inf\{t : \Delta_t = \infty\}$.