CIMAT

A bound on the number of twice-punctured tori in a knot exterior

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The problem

Question (K. Motegi)

Is there an upper bound on the number of JSJ-pieces of manifolds which are obtained by Dehn-surgery on hyperbolic knots in $\mathbb{S}^3$?

Problem

Find upper bounds on the number of non-isotopic essential n-punctured tori in the complement of a hyperbolic knot in $\mathbb{S}^3$.

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The case $n = 1$ was first addressed by Y. Tsutsumi (≤ 7) and later L. Valdez-Sánchez found the optimal bound.

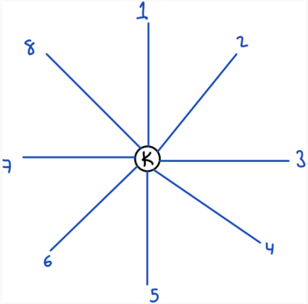
Theorem (L. Valdez-Sánchez 19)

Any genus one hyperbolic knot in $\mathbb{S}^3$ bounds at most five mutually disjoint, non parallel, genus one Seifert surfaces.

Tsutsumi's upper bound

Lemma

Let V be a genus two handlebody, and $J \subset V$ be an essential simple closed curve separating ∂V. Then J bounds at most four disjoint, non-parallel, genus one incompressible surfaces in V.



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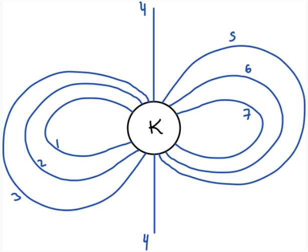
Case n=2

We focused on Motegi's problem when $n = 2$ using Tsutsumi's approach of counting tori in a genus-two handlebody.

Main Result

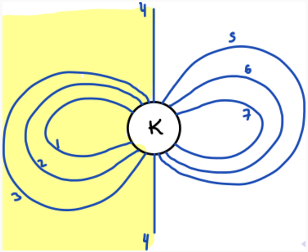
Theorem (R, Aranda, Ramirez–Losada 23)

Let K be a hyperbolic knot in $\mathbb{S}^3$. There are at most six pairwise disjoint, non–isotopic, nested, embedded twice–punctured tori with an integral slope in the complement of K.

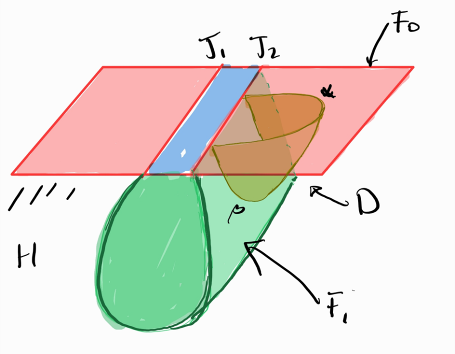


Main Lemma

Let V a handlebody of genus two, and let $J = J_1 \cup J_2$ be two disjoint copies of a non-separating simple closed curve in ∂V . Then, J bounds at most three mutually disjoint, non-parallel, incompressible, separating, twice-punctured tori in V .



Sketch of Main Lemma's Proof



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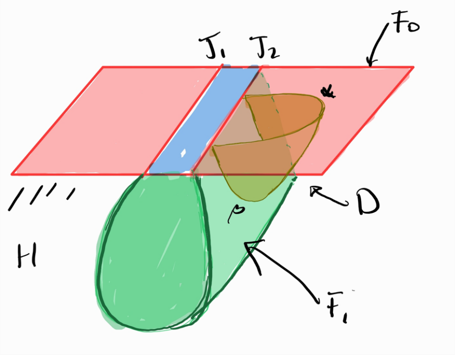

∂–compress along D

$F_1 \to F_1'$

$F_0 \to F_0'$

A becomes $P = A \cup \eta(\alpha)$

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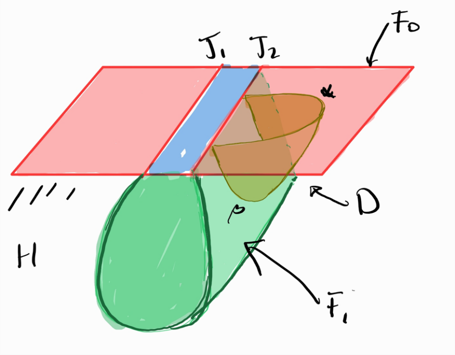

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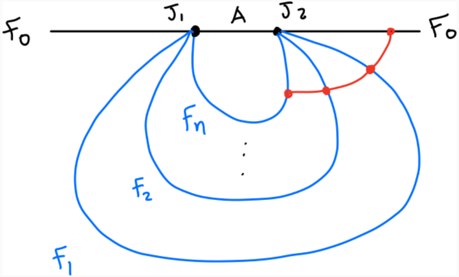
Properties

$\chi(F_1') = \chi(F_0') = \chi(P) = -1$
and $\partial(F_1') = \partial(F_0') = \partial(P)$

Sketch of Main Lemma's Proof

Let $F_0, F_1, \ldots, F_n \subset V$ be $n+1$ twiced-punctured tori. One can ∂–compress them and obtain $n+2$ compact incompressible surfaces with $\chi = -1$ and the same boundary ,



Sketch of Main Lemma's Proof

Conjecture

Given $J' \subset \partial V$, there are at most FOUR compact incompressible surfaces with $\chi = -1$ expanding J' .

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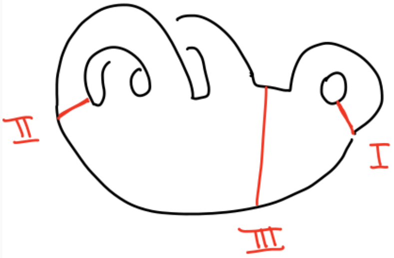
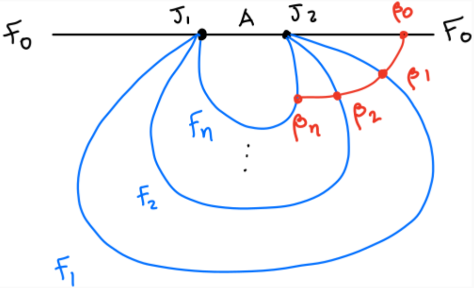
Although it could be a nice generalization of the Tsutsumi's result, it is false.

Sketch: Arc types





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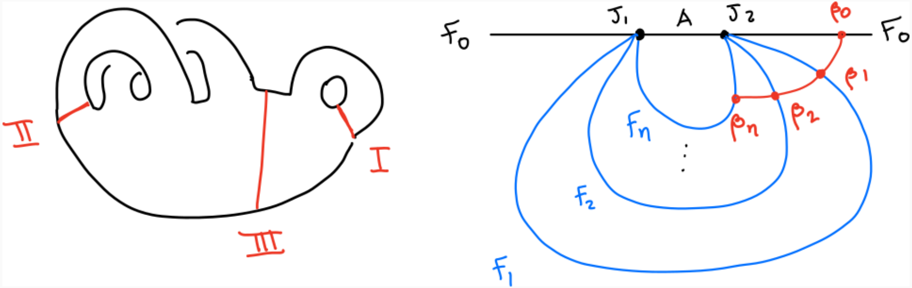

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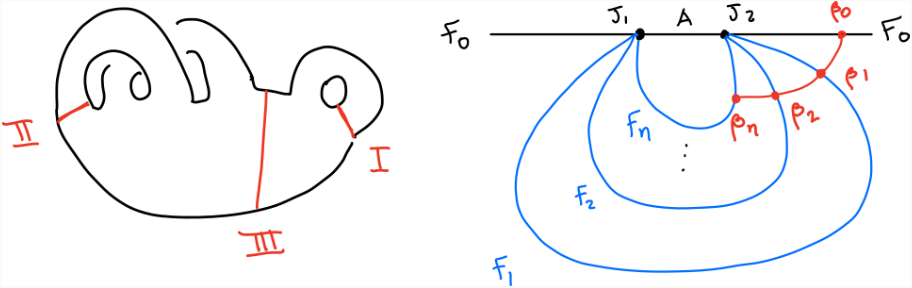

- β_i type I $\iff$ β_{i-1} is type I
- β_i type III $\implies$ β_{i-1} is type III

Sketch: Case division

Cases workout

- Case 1: Only type I arcs.
- Case 1: Only type II arcs
- Case 2: Type II and III arcs.

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- Case 1: Only type II arcs This case reduces to a Pair of pants version of Tsutsumi's result
- Case 2: Type II and III arcs. This case also reduces to Tsutsumi's result, plus some extra work

The pair-pants version

Lemma (Tsutsumi pairs of pants version)

Let J be a submanifold separating ∂V into two pairs of pants. Then J bounds at most four pairwise disjoint, non-parallel, incompressible pairs of pants in V.

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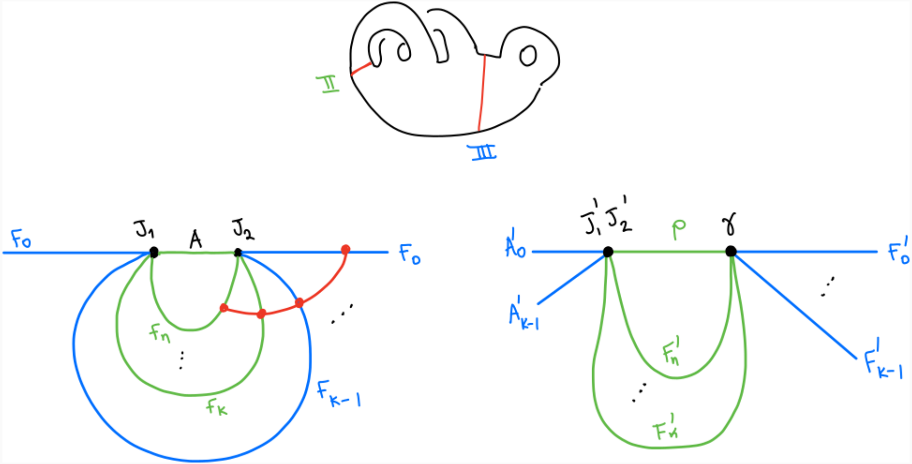
The techniques we used to prove this theorem were based on the tools developed by L. Valdez-Sánchez.

The case III and II types



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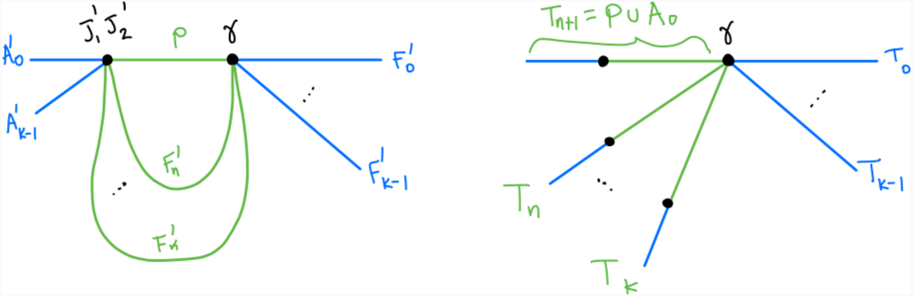

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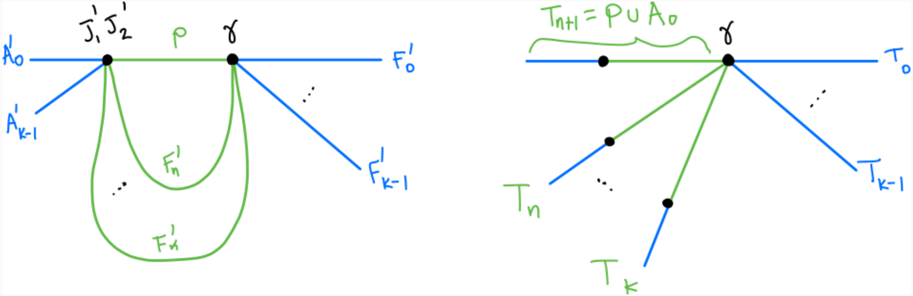

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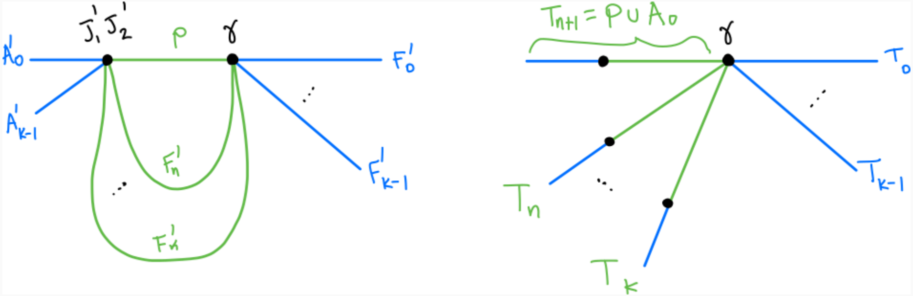

This only proves that $n+1 \leq 4$. We workout the case $n = 3$ separately.

Summary

Theorem (R, Aranda, Ramirez–Losada 23)

Let K be a hyperbolic knot in $\mathbb{S}^3$. There are at most six pairwise disjoint, non–isotopic, nested, embedded twice–punctured tori with an integral slope in the complement of K.

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References

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Thank you!