Theory of descent and algebraic stacks

Lectures by Nitin Nitsure

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Contents

1 Functor of points

All rings will be commutative with unity, unless otherwise indicated.

Let **Sets**, **Rings** and **Schemes** denote the categories of sets, rings, and schemes. If S is a chosen base scheme, by definition $\mathbf{Rings}_S$ is the category whose objects are all pairs $(R, f : \operatorname{Spec} R \to S)$ where R is a ring and f is a morphism of schemes, and morphisms $\phi : (R, f) \to (R', f')$ are ring homomorphisms $\phi : R \to R'$ such that the resulting morphism $\operatorname{Spec} \phi : \operatorname{Spec} R' \to \operatorname{Spec} R$ is over S.

To any scheme X, we associate a functor

$$h_X : \mathbf{Rings} \to \mathbf{Sets}$$

by putting

$$h_X(R) = Hom_{\mathbf{Schemes}}(\operatorname{Spec} R, X)$$

To any ring homomorphism $\phi : A \to B$ we associate the map $h_X(\phi) : h_X(A) \to h_X(B)$ under which any $f \in h_X(A) = Hom_{\mathbf{Schemes}}(\operatorname{Spec} A, X)$ is mapped to the composite $f \circ \operatorname{Spec} \phi : \operatorname{Spec} B \to X$.

Similarly, for any S-scheme X we have a functor $h_X : \mathbf{Rings}_S \to \mathbf{Sets}$, under which

$$h_X(R) = Hom_{\mathbf{Schemes}_S}(\operatorname{Spec} R, X)$$

for any S-ring R.

The functor h_X is called the **functor of points** of the scheme X, and the set $h_X(R)$ is called the **set of R-valued points** of X.

If $f : X \to Y$ is a morphism of schemes, then we get a morphism of functors (natural transformation)

$$h_f : h_X \to h_Y$$

If $g : \operatorname{Spec} R \to X$ is in $h_X(R)$, then $h_f(g) \in h_Y(R)$ is by definition the composite $f \circ g : \operatorname{Spec} R \to Y$. This defines a functor

$$h : \mathbf{Schemes} \to \mathbf{Fun}(\mathbf{Rings}, \mathbf{Sets})$$

where $\mathbf{Fun}(\mathbf{Rings}, \mathbf{Sets})$ is the 'functor category' whose objects are all functors $\mathfrak{X} : \mathbf{Rings} \to \mathbf{Sets}$ and morphisms are natural transformations.

The above can be similarly defined relative to a given base-schem S, to get a functor $h : \mathbf{Schemes}_S \to \mathbf{Fun}(\mathbf{Rings}_S, \mathbf{Sets})$. If we take $S = \operatorname{Spec} \mathbb{Z}$, then we get the absolute case, showing the relative case to be more general.

Theorem 1.1 (Grothendieck) *The functor $h : \mathbf{Schemes}_S \to \mathbf{Fun}(\mathbf{Rings}_S, \mathbf{Sets})$ is fully faithful.*

As a consequence, the category $\mathbf{Schemes}_S$ is equivalent to a full subcategory of $\mathbf{Fun}(\mathbf{Rings}_S, \mathbf{Sets})$.

2 Examples of functor of points

We say that a functor $\mathfrak{X} : \mathbf{Rings}_S \to \mathbf{Sets}$ is **representable** if $\mathfrak{X}$ is naturally isomorphic to the functor of points h_X of some scheme X over S. If X is a scheme over S and $\alpha : h_X \to \mathfrak{X}$ is a natural isomorphism, then we say that the pair (X, α) **represents** the functor $\mathfrak{X}$. The scheme X is called a **representing scheme** or **moduli scheme** for $\mathfrak{X}$, and the natural isomorphism α is called a **universal family** or a **Poincaré family** over X.

Some representable functors

Example 2.1 Let $\mathfrak{X}(R) = R$ (the set of elements the ring R). This functor is representable: it is the functor h_X where $X = \mathbf{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]$ is the affine line over $\mathbb{Z}$.

Example 2.2 Let $\mathfrak{X}(R) = R^\times$ the set of invertible elements of the ring R. This functor is representable by $X = \mathbf{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

Example 2.3 More generally, for $n \geq 1$, let $\mathfrak{X}_n(R) = GL_n(R)$ the set of $n \times n$ invertible matrices over R. This functor is represented by the group-scheme $GL_{n,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_{i,j}, 1/\det(x_{i,j})]$.

Example 2.4 Let $n \geq 0$ and let $\mathfrak{X}_n(R)$ be the set of all equivalence classes of pairs (L, u) where L is a rank 1 projective module over R (a line bundle over $\operatorname{Spec} R$), and $u : R^{n+1} \to L$ is a surjective R-linear homomorphism. We say that two pairs (L, u) and (L', u') are equivalent if $\ker(u) = \ker(u') \subset R^{n+1}$. Then $\mathfrak{X}$ is represented by $\mathbf{P}^n_{\mathbb{Z}}$.

Some non-representable functors

It is easy to give examples of non-representable functors which are 'very bad' in the sense that they do not satisfy the fpqc sheaf property (which we will study in lecture 2). The following examples are fpqc sheaves, but still they are not representable.

Example 2.5 Let $S = \operatorname{Spec} k$ where k is a field, and define $\mathfrak{X} : \mathbf{Rings}_k \to \mathbf{Sets}$ by putting $\mathfrak{X}(R)$ to be the set of all Zariski-open subsets of $\operatorname{Spec} R$. This is not representable (simple exercise).

Example 2.6 Let $\mathfrak{X}(R) = (R/2R)^\times$ the set of invertible elements of the ring $R/2R$. This functor is not representable. (See arXiv.org/abs/math.AG/0204047)

Example 2.7 Let S be a noetherian scheme, and E a coherent sheaf on S which is not locally free. For any object $(R, f : \operatorname{Spec} R \to S)$ in $\mathbf{Rings}_S$, define $\mathfrak{X}(R)$ to be $H^0(\operatorname{Spec} R, f^*E)$. This functor is not representable.
(see arXiv.org/abs/math.AG/0308036)

3 Proof of Theorem 1.1

The theorem says that for any S-schemes X and Y, the set map $Hom_S(X,Y) \to Hom_{\mathbf{Fun}}(h_X, h_Y)$ is bijective.

h **is faithful** We want to show that for any S-schemes X and Y, the set map

$$Hom_S(X,Y) \to Hom_{\mathbf{Fun}}(h_X, h_Y)$$

is injective. Let $f, g \in Hom_S(X,Y)$ with $h_f = h_g$. Let $U_i = \operatorname{Spec} R_i$ be affine open subschemes of X that cover X, with inclusion map $\theta_i : U_i to X$. Then $h_f(\theta_i) = f \circ \theta_i = f|_{U_i} : U_i \to Y$ and $h_g(\theta_i) = g \circ \theta_i = g|_{U_i} : U_i \to Y$, so from $h_f = h_g$ we get $f|_{U_i} = |_{U_i} : U_i \to Y$ for each U_i. As these cover X, we have $f = g$.

h **is full** We want to show that for any S-schemes X and Y, the set map

$$Hom_S(X,Y) \to Hom_{\mathbf{Fun}}(h_X, h_Y)$$

is surjective. If $\varphi : h_X \to h_Y$ is in $Hom_{\mathbf{Fun}}(h_X, h_Y)$, then we get a set map $\varphi(U_i) : h_X(U_i) \to h_Y(U_i)$. Hence we get a morphism $\varphi(U_i)(\theta_i) : U_i \to Y$.

If $V \subset U \subset X$ are affine open subschemes with inclusions $\theta_V : V \to X$, $\theta_U : U \to X$, then we have

$$\varphi(V)(\theta_V) = (\varphi(U)(\theta_U))|_V$$

We can cover the intersections $U_i \bigcap U_j$ with affine open subschemes of X (if X is separated over S then these intersections are already affine). Hence from the above, the morphisms $\varphi(U_i)(\theta_i) : U_i \to Y$ agree in the intersections $U_i \bigcap U_j$, hence they glue together to define a morphism $f : X \to Y$. ¿From its construction, $f|_{U_i} = \varphi(U_i)(\theta_i)$, and hence we have $h_f = \varphi$, showing surjectivity of $Hom_S(X,Y) \to Hom_{\mathbf{Fun}}(h_X, h_Y)$.

This completes the proof of the Theorem 1.1. □

Remark 3.1 As h is fully faithful, it gives an equivalence of the category of schemes over S with a full subcategory of $\mathbf{Fun}(\mathbf{Rings}, \mathbf{Sets})$. Hence, we can regard any scheme X over S as the corresponding functor h_X from S-rings to sets. Out of all possible functors $\mathfrak{X}$ from S-rings to sets, those of the form h_X where X is an S-scheme are rather special: Grothendieck proved that they necessarily satisfy **faithfully flat quasi-compact descent**, that is, they are **fpqc sheaves**, which is the subject of Lecture 2 (April 1, 2005).

The fpqc sheaf condition is necessary but not sufficient for representability of a functor $\mathfrak{X}$.

4 Coverings

Zariski coverings of schemes

Let S be a chosen base scheme. For simplicity, all schemes are assumed to be locally noetherian unless otherwise indicated.

Definition 4.1 Let X be an S-scheme. A **Zariski covering** of X is a family of S-morphisms $(f_i : V_i \to X)_{i \in I}$ such that

(1) Each f_i is an open embedding, and

(2) $X = \bigcup_{i \in I} \operatorname{im}(f_i)$

Remark 4.2 If instead of open embeddings $f_i : V_i \to X$ we only take open subschemes $V_i \subset X$, then the above gives the usual definition of a Zariski open cover of X.

Definition 4.3 Let A be a ring over S (that is, we are given a structure morphism $\operatorname{Spec} A \to S$). A **Zariski covering** of A in the category $\mathbf{Rings}_S$ is a family of S-morphisms $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} A)_{i \in I}$ such that

(1) Each f_i is an open embedding, and

(2) $\operatorname{Spec} A = \bigcup_{i \in I} \operatorname{im}(f_i)$.

Flatness recalled

If M is a module over a ring A, then M is called a **flat module** if the functor $- \otimes_A M$ is exact, that is, if $0 \to N' \to N \to N'' \to 0$ is exact then $0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$ is exact. If M is flat and moreover if the functor $- \otimes_A M$ is faithful (that is, $Hom_A(N_1, N_2) \to Hom_A(N_1 \otimes_A M, N_2 \otimes_A M)$ is always injective) then M is called a **faithfully flat module**.

If $A \to B$ is a ring homomorphism then B is faithfully flat over A (means, as an A-module) if and only if the following two conditions are satisfied:

(1) B is flat as a module over A, and

(2) The morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

If $A \to B$ is a ring homomorphism such that B is faithfully flat over A, then $A \to B$ is automatically injective.

Étale coverings of schemes

Recall that a morphism $f : Y \to X$ of locally noetherian schemes is called **étale** if f is locally finite-type, flat, and unramified (means the sheaf of relative differentials Ω^1_f is zero).

Definition 4.4 Let X be an S-scheme. An **étale covering** of X is a family of S-morphisms $(f_i : V_i \to X)_{i \in I}$ such that

(1) Each f_i is étale, and

(2) $X = \bigcup_{i \in I} \operatorname{im}(f_i)$.

Definition 4.5 Let A be a ring over S (that is, we are given a structure morphism $\operatorname{Spec} A \to S$). An **étale covering** of A is a family of S-morphisms $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} R)_{i \in I}$ such that

(1) Each f_i is étale.

(2) $\operatorname{Spec} R = \bigcup_{i \in I} \operatorname{im}(f_i)$.

fppf coverings of schemes

Definition 4.6 Let X be an S-scheme, locally noetherian. An **fppf covering** of X is a family of S-morphisms $(f_i : V_i \to X)_{i \in I}$ such that

(1) Each f_i is finite type and flat, and

(2) $X = \bigcup_{i \in I} \operatorname{im}(f_i)$.

Definition 4.7 Let A be a ring over S (that is, we are given a structure morphism $\operatorname{Spec} A \to S$). A **fppf covering** of A is a family of S-morphisms $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} R)_{i \in I}$ such that

(1) Each f_i is finite type and flat, and

(2) $\operatorname{Spec} R = \bigcup_{i \in I} \operatorname{im}(f_i)$

fpqc coverings of schemes

A morphism $f : Y \to X$ of schemes is called a **quasi-compact morphism** if X has an open covering by affine open subschemes U_i such that each $f^{-1}(U_i)$ is a finite union of affine open subschemes of Y.

A morphism $f : Y \to X$ of schemes is called an **fpqc morphism** if f is faithfully flat, and satisfies the following condition: every quasi-compact open subset $U \subset X$ is the image of some quasi-compact open subset $V \subset Y$. Equivalently, f is faithfully flat, and any point $y \in Y$ has an open neighbourhood $y \in V \subset Y$ such that $f(V)$ is open in X and the morphism $f|_V : V \to f(V)$ is a quasi-compact morphism.

Definition 4.8 Let X be an S-scheme. An **fpqc covering** of X is a family of S-morphisms $(f_i : V_i \to X)_{i \in I}$ such that the resulting morphism

$$\coprod_{i \in I} V_i \to X$$

is an fpqc morphism.

If X and each V_i are affine, then the above specialises to the following.

Definition 4.9 Let A be a ring over S (that is, we are given a structure morphism $\operatorname{Spec} A \to S$). A **fpqc covering** of A is a family of S-morphisms $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} R)_{i\in I}$ such that

(1) Each f_i is flat, and

(2) There exists a finite subset $J \subset I$ such that $\operatorname{Spec} R = \bigcup_{j\in J} \operatorname{im}(f_j)$

5 Sheaf conditions

Exact diagram of sets

Let $f : X \to Y$ and $g, h : Y \rightrightarrows Z$ be maps of sets. The diagram $X \to Y \rightrightarrows Z$ is called an **exact diagram of sets**, or f is called an **equaliser** of g and h if the following two conditions are satisfied:

(1) f is injective, and

(2) $\operatorname{im}(f) = \{y \in Y | g(y) = h(y)\}$.

Small Zariski sheaf on X/S

Let X be an S-scheme. Let $\mathbf{C}^{Zar}_{X/S}$ denote the category whose objects are open embeddings $f : U \to X$ over S, and morphisms are S-morphisms. A **small Zariski presheaf of sets** on X/S is a functor $\mathfrak{X} : (\mathbf{C}^{Zar}_{X/S})^{op} \to \mathbf{Sets}$ such that for any Zariski cover $(f_i : V_i \to U)_{i\in I}$ where $U \in Ob(\mathbf{C}^{Zar}_{X/S})$, the following diagram of sets is exact.

$$\mathfrak{X}(U) \to \prod_{i\in I} \mathfrak{X}(V_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathfrak{X}(V_j \times_U V_k)$$

In the above, the map $\mathfrak{X}(U) \to \prod_{i\in I} \mathfrak{X}(V_i)$ is induced by the component maps $\mathfrak{X}(f_i) : \mathfrak{X}(U) \to \mathfrak{X}(V_i)$, which the two maps $\mathfrak{X}(V_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathfrak{X}(V_j \times_U V_k)$ are respectively induced by the component maps $\mathfrak{X}(p_j) : \mathfrak{X}(V_j) \to \mathfrak{X}(V_j \times_U V_k)$ and $\mathfrak{X}(p_k) : \mathfrak{X}(V_k) \to \mathfrak{X}(V_j \times_U V_k)$ where $p_j : V_j \times_U V_k \to V_j$ and $p_k : V_j \times_U V_k \to V_k$ are the two projections.

Small étale sheaf on X/S

Let $\mathbf{C}^{et}_{X/S}$ denote the category whose objects are étale S-morphisms $f : U \to X$ and morphisms are S-morphisms. A **small etale presheaf of sets** on X/S is a functor $\mathfrak{X} : (\mathbf{C}^{et}_{X/S})^{op} \to \mathbf{Sets}$ such that for any étale cover $(f_i : V_i \to U)_{i\in I}$ where $U \in Ob(\mathbf{C}^{et}_{X/S})$, the following diagram of sets is exact.

$$\mathfrak{X}(U) \to \prod_{i\in I} \mathfrak{X}(V_i) \rightrightarrows \prod_{(j,k)\in I\times I} \mathfrak{X}(V_j \times_U V_k)$$

Big fppf sheaf on $\mathbf{Schemes}_S$

This is a functor $(\mathbf{Schemes}_S)^{op} \to \mathbf{Sets}$ such that for any S-scheme U and any fppf cover $(f_i : V_i \to U)_{i\in I}$ over S, the following diagram of sets is exact.

$$\mathfrak{X}(U) \to \prod_{i \in I} \mathfrak{X}(V_i) \rightrightarrows \prod_{(j,k) \in I \times I} \mathfrak{X}(V_j \times_U V_k)$$

Big fppf sheaf on *$Rings_S$*

This is a functor $\mathbf{Rings}_S \to \mathbf{Sets}$ such that for any S-ring A and any fppf cover $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} A)_{i \in I}$ over S, the following diagram of sets is exact.

$$\mathfrak{X}(A) \to \prod_{i \in I} \mathfrak{X}(B_i) \rightrightarrows \prod_{(j,k) \in I \times I} \mathfrak{X}(B_j \otimes_A B_k)$$

Big fpqc sheaf on $\mathbf{Schemes}_S$

This is a functor $(\mathbf{Schemes}_S)^{op} \to \mathbf{Sets}$ such that for any S-scheme U and any fpqc cover $(f_i : V_i \to U)_{i \in I}$ over S, the following diagram of sets is exact.

$$\mathfrak{X}(U) \to \prod_{i \in I} \mathfrak{X}(V_i) \rightrightarrows \prod_{(j,k) \in I \times I} \mathfrak{X}(V_j \times_U V_k)$$

Big fpqc sheaf on *$Rings_S$*

This is a functor $\mathbf{Rings}_S \to \mathbf{Sets}$ such that for any S-ring A and any fpqc cover $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} A)_{i \in I}$ over S, the following diagram of sets is exact.

$$\mathfrak{X}(A) \to \prod_{i \in I} \mathfrak{X}(B_i) \rightrightarrows \prod_{(j,k) \in I \times I} \mathfrak{X}(B_j \otimes_A B_k)$$

We will prove the following fundamental theorem in the next talk.

Theorem 5.1 (Grothendieck) *If X is a scheme over S, then the corresponding functor $h_X : \mathbf{Schemes}_S \to \mathbf{Sets}$ is a big fpqc sheaf on $\mathbf{Schemes}_S$.*

6 Amitsur's theorem

Let $A \to B$ be a ring homomorphism. We have ring homomorphisms $p_1^* : B \to B \otimes_A B : b \mapsto b \otimes 1$ and $p_2^* : B \to B \otimes_A B : b \mapsto 1 \otimes b$. These give the pullback of regular functions on $\operatorname{Spec} B$ under the two projections from $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec}(B \otimes_A B)$ to $\operatorname{Spec} A$.

Theorem 6.1 *Let $A \to B$ be a faithfully flat ring homomorphism. Then the following sequence of A-modules (the Amitsur sequence) is exact.*

$$0 \to A \to B \stackrel{p_1^*-p_2^*}{\to} B \otimes_A B$$

More generally, if M is any A-module, then the following sequence of A-modules (which is obtained by applying $- \otimes_A M$ to the above sequence, and called the Amitsur sequence for M) is again exact.

$$0 \to M \stackrel{1 \otimes \mathrm{id}_M}{\to} B \otimes_A M \stackrel{(p_1^*-p_2^*)\otimes \mathrm{id}_M}{\to} B \otimes_A B \otimes_A M$$

Proof Let $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$ and $\pi : X \to S$ the induced morphism. First consider the special case where there exists a section $s : S \to X$ for the projection $\pi : X \to S$, so that $\pi \circ s = \mathrm{id}_S$. In other words, we have a ring homomorphism $s^* : B \to A$ with $s^*|_A = \mathrm{id}_A$. Then the homomorphism $s^* \otimes \mathrm{id}_M : B \otimes_A M \to A \otimes_A M = M$ has the property that the composite

$$M \stackrel{1 \otimes \mathrm{id}_M}{\to} B \otimes_A M \stackrel{s^* \otimes \mathrm{id}_M}{\to} M$$

is id_M, showing that the first map is injective. Clearly, the composite $M \stackrel{1 \otimes \mathrm{id}_M}{\to} B \otimes_A M \stackrel{(p_1^*-p_2^*)\otimes \mathrm{id}_M}{\to} B \otimes_A B \otimes_A M$ is zero. Now suppose that $\sum b_i \otimes m_i$ is an element of $\ker((p_1^* - p_2^*) \otimes \mathrm{id}_M)$, that is,

$$\sum b_i \otimes 1 \otimes m_i = \sum 1 \otimes b_i \otimes m_i$$

Applying $(s^*, \mathrm{id}_B) \otimes \mathrm{id}_M : B \otimes_A B \otimes_A M \to B \otimes_A M$ to the two sides, we get

$$\sum s^*(b_i) \otimes m_i = \sum b_i \otimes m_i$$

But $\sum s^*(b_i) \otimes m_i = \sum s^*(b_i) m_i$ in $A \otimes_A M = M$. Hence $\sum b_i \otimes m_i = (1 \otimes \mathrm{id}_M)(\sum s^*(b_i) m_i)$ showing exactness at $B \otimes_A M$.
Next, we treat the general case. Applying the functor $B \otimes_A -$ to the complex

$$0 \to M \stackrel{1 \otimes \mathrm{id}_M}{\to} B \otimes_A M \stackrel{(p_1^*-p_2^*)\otimes \mathrm{id}_M}{\to} B \otimes_A B \otimes_A M$$

we get the complex

$$0 \to B \otimes_A M \stackrel{\mathrm{id}_B \otimes (1 \otimes \mathrm{id}_M)}{\to} B \otimes_A B \otimes_A M \stackrel{\mathrm{id}_B \otimes (p_1^*-p_2^*)\otimes \mathrm{id}_M}{\to} B \otimes_A B \otimes_A B \otimes_A M$$

Note that we have a functorial isomorphism

$$(X \times_S X) \times_{p_1, X, p_1} (X \times_S X) = X \times_S X \times_S X$$

In algebraic terms, we have a ring isomorphism

$$(B \otimes_A B) \otimes_{p_1^*, B, p_1^*} (B \otimes_A B) = B \otimes_A B \otimes_A B$$

Hence, putting $B \to C$ to be the ring homomorphism $p_1^* : B \to B \otimes_A B$, and N to be the B-module $B \otimes_A M$, the above complex becomes

$$0 \to N \to C \otimes_B N \to C \otimes_B C \otimes_B N$$

which is the Amitsur complex for N. As $p_1 : X \times_S X \to X$ admits a section, namely, the diagonal $\Delta : X \to X \times_S X$, by the special case proved above, the Amitsur sequence for N is exact.

It is a basic property of faithful flatness that if a sequence becomes exact after tensoring by a faithfully flat module then it is originally exact. Hence, by faithful flatness of B over A, the Amitsur sequence for M is exact. $\square$

Corollary 6.2 *Let $\mathcal{O} = \mathrm{id} : \mathbf{Rings} \to \mathbf{Rings}$ be the functor which associates to any ring A the ring A itself, and to any $A \to B$ the same homomorphism. Then $\mathcal{O}$ is a big fpqc sheaf of rings on* **Rings***.*

Proof Let $(f_i : \operatorname{Spec} B_i \to \operatorname{Spec} A)_{i \in I}$ be an fpqc cover of A. By assumption, there exists a finite subset $J \subset I$ such that $\bigcup_{j \in J} \operatorname{im}(f_j) = \operatorname{Spec} A$. Let $B = \prod_{j \in J} B_j$ be the direct product ring, so that $\operatorname{Spec} B = \coprod_{j \in J} \operatorname{Spec} B_i$ is the disjoint union. This has a morphism $f = (f_j)_{j \in J} : \operatorname{Spec} B \to \operatorname{Spec} A$ which is faithfully flat, being both flat and surjective. It is clear that the functor $\mathcal{O}$ preserves direct products. Hence it just remains to show that the diagram

$$\mathcal{O}(A) \to \mathcal{O}(B) \rightrightarrows \mathcal{O}(B \otimes_A B)$$

is exact. But this is just the exactness of the Amitsur sequence for $A \to B$. $\square$

The above corollary is a special case of the following corollary by taking $T = \mathbf{A}^1$.

Corollary 6.3 *For any affine scheme $T = \operatorname{Spec} R$, consider the corresponding functor of points h_T :* **Rings** $\to$ **Sets** *which associates to any ring A the set $Hom_{\mathbf{Rings}}(R, A)$ and to any homomorphism $\phi : A \to B$ the set map $\phi \circ - : Hom_{\mathbf{Rings}}(R, A) \to Hom_{\mathbf{Rings}}(R, B)$ obtained by composing with ϕ. This functor is a big fpqc sheaf of sets on* **Rings***.*

Proof Clearly, h_T preserves direct products of rings. Hence it remains to prove that if $A \to B$ is faithfully flat, then the following diagram of sets isexact:

$$Hom_{\mathbf{Rings}}(R, A) \to Hom_{\mathbf{Rings}}(R, B) \rightrightarrows Hom_{\mathbf{Rings}}(R, B \otimes_A B)$$

Let **Abgrps** denote the category of abelian groups. By applying the left-exact functor $Hom_{\mathbf{Abgrps}}(R, -)$ to the Amitsur sequence for $A \to B$, we get the exact sequence of abelian groups

$$0 \to Hom_{\mathbf{Abgrps}}(R, A) \to Hom_{\mathbf{Abgrps}}(R, B) \stackrel{p_1^* - p_2^*}{\to} Hom_{\mathbf{Abgrps}}(R, B \otimes_A B)$$

Hence the conclusion follows by considering the inclusion of functors

$$Hom_{\mathbf{Rings}}(R, -) \hookrightarrow Hom_{\mathbf{Abgrps}}(R, -)$$

$\square$

7 Examples

If $\pi : X \to S$ is a map of sets, then any point of $X \times_S X$ is a pair (x, y) of points of X such that $\pi(x) = \pi(y)$. We have $p_1(x, y) = x$ and $p_2(x, y) = y$. If $f : S \to T$ is any map, then $g = f \circ \pi : X \to T$ has the property that $g \circ p_1 = g \circ p_2 : X \times_S X \to T$, that is, g is **constant along fibers** of $\pi : X \to S$. If π is surjective, then any map $g : X \to T$ which is constant along fibers of π factors uniquely through S, defining a map $f : S \to T$ with $g = f \circ \pi$.

In topological spaces, the above is true if and only if π is surjective and the topology on S is the quotient topology.

The Corollary 6.2 of the theorem of Amitsur shows that if $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and π is faithfully flat, then any regular function $b \in B = Hom(X, \mathbf{A}^1)$ with $b \circ p_1 = b \circ p_2$ factors uniquely through a regular function $a \in A = Hom(S, \mathbf{A}^1)$.

We now give two examples of $A \to B$ where $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective but not flat, where the above conclusion of Amitsur's theorem fails.

Example 7.1 Let $A = \mathbb{C}[t^3, t^5] \subset \mathbb{C}[t] = B$. Then in $B \otimes_A B$ we have

$$t^7 \otimes_A 1 = t^2 t^5 \otimes_A 1 = t^2 \otimes_A t^5 = t^2 \otimes_A t^3 t^2 = t^5 \otimes_A t^2 = 1 \otimes t^7$$

However, t^7 does not lie in A.

Example 7.2 Let $A = \mathbb{C}[x, xy, y^3] \subset \mathbb{C}[x, y] = B$. Then

$$xy^2 \otimes_A 1 = y \cdot (xy) \otimes_A 1 = y \otimes_A xy = xy \otimes_A y = 1 \otimes_A xy^2$$

but xy^2 does not lie in A.

8 Effective descent for closed subschemes

Theorem 8.1 *Let $\pi : X \to S$ be a faithfully flat quasi-compact morphism of schemes. Let $Y \subset X$ be a closed subscheme such that the schematic inverse images $p_1^{-1}Y$ and $p_2^{-1}(Y)$ are identical in $X \times_S X$. Then there exists a unique closed subscheme $Z \subset S$ such that $Y = \pi^{-1}(Z)$.*

Proof The question is clearly local over S. Consider any affine open cover (U_i) of S. By quasi-compactness of $\pi : X \to S$, each $\pi^{-1}(U_i)$ has a finite affine open cover $(V_{i,j})$. Let $V_i = \coprod_j V_{i,j}$, which is affine and faithfully flat over U_i. Hence the theorem follows from the following purely algebraic lemma. □

Lemma 8.2 *Let $A \to B$ be a faithfully flat ring homomorphism, and let $J \subset B$ be an ideal such that $J \otimes_A B = B \otimes_A J \subset B \otimes_A B$. Then there exists a unique ideal $I \subset A$ such that $J = BI$.*

Proof If $I_1 \subset I_2 \subset A$ are two such ideals then $B \otimes_A I_1 = BI_1 = J = BI_2 = B \otimes_A I_2$, in particular, $B \otimes_A (I_1/I_2) = 0$, and so $I_1/I_2 = 0$ by faithful flatness of B, showing uniqueness of I. Next we show existence. Let

$$I = \{b \in J | b \otimes 1 = 1 \otimes b\}$$

By Amitsur's theorem, we indeed have $I \subset A$, and in fact, $I = A \cap J$. Let $i : I \hookrightarrow J$ denote the inclusion. By definition of I, the following sequence is exact:

$$0 \to I \xrightarrow{i} J \xrightarrow{p_1^* - p_2^*} B \otimes_A J$$

(note that p_1^* maps J to $J \otimes_A B$, but by assumption $J \otimes_A B = B \otimes_A J \subset B \otimes_A B$ so we can regard p_1^* as defining a map $J \to B \otimes_A J$.) Hence tensoring with B and using flatness of B, we get an exact sequence

$$0 \to I \otimes_A B \xrightarrow{i \otimes \mathrm{id}_B} J \otimes_A B \xrightarrow{(p_1^* - p_2^*) \otimes \mathrm{id}_B} B \otimes_A J \otimes_A B$$

By Amitsur's theorem applied to the A-module J, we have an exact sequence

$$0 \to J \xrightarrow{1 \otimes \mathrm{id}_J} B \otimes_A J \xrightarrow{(p_1^* - p_2^*) \otimes \mathrm{id}_J} B \otimes_A B \otimes_A J$$

The above two exact sequences fit in a commutative diagram

$$\begin{array}{ccccccc}
0 \to & I \otimes_A B & \xrightarrow{i \otimes \mathrm{id}_B} & J \otimes_A B & \xrightarrow{(p_1^* - p_2^*) \otimes \mathrm{id}_B} & B \otimes_A J \otimes_A B \\
& u \downarrow & & \| & & \| \\
0 \to & J & \xrightarrow{1 \otimes \mathrm{id}_J} & B \otimes_A J & \xrightarrow{(p_1^* - p_2^*) \otimes \mathrm{id}_J} & B \otimes_A B \otimes_A J
\end{array}$$

where the map $u : I \otimes_A B \to J$ sends $x \otimes b \mapsto xb$ for $x \in I$ and $b \in B$.
By five-lemma, the map u is an isomorphism. □

The above can be phrased as follows in the language of sheaves.

Theorem 8.3 *Let C :* $\mathbf{Schemes}^{op} \to \mathbf{Sets}$ *be the functor which associates to any scheme X the set $C(X)$ consisting of all its closed subschemes, and to any morphism $f : Y \to X$ the set map $C(X) \to C(Y)$ which sends a subscheme of X to its schematic inverse image in Y. Then C is a big fpqc sheaf on* **Schemes**.

Remark 8.4 Similar results are also true for open subschemes instead of closed subschemes: they amount to the statement that if $Y \to X$ is faithfully flat quasi-compact then the topology on X is the quotient topology from Y, that is, a subset of X is open if and only if its inverse image in Y is open.

9 Topology of flat morphisms

Exercise 9.1 (see Hartshorne 'Algebraic Geometry', Chapter 3 Exercise 9.1) If $f : Y \to X$ is a locally finite type morphism of locally noetherian schemes, then f is an open map. In particular, X has the quotient topology induced by $f : Y \to X$.

Example 9.2 $\operatorname{Spec}\mathbb{Q}$ is flat over $\operatorname{Spec}\mathbb{Z}$ but not open. This shows that quasi-compact flat morphisms need not be open in general.

Theorem 9.3 (Grothendieck) *If $f : Y \to X$ is a faithfully flat quasi-compact morphism, then a subset $U \subset X$ is open if and only if its inverse image $f^{-1}(U) \subset Y$ is open. Consequently, X has the quotient topology induced by $f : Y \to X$.*

Proof The question is local over X, so we can assume that $X = \operatorname{Spec} A$ is affine. By quasi-compactness of f, Y admits a finite affine open cover (V_i). Then replacing Y by $V = \coprod V_i$, we can assume that $Y = \operatorname{Spec} B$ is also affine. Let $U \subset X$ be a subset such that $f^{-1}(U)$ is open in Y. Let $Z \subset Y$ be the closed subset $Y - f^{-1}(U)$, so that by assumption

$$Z = f^{-1}f(Z)$$

We are required to show that $f(Z)$ is closed in X. Let Z be given the reduced induced subscheme structure from Y, and let $X' \subset X$ be the closed subscheme which is the schematic image of $f|_Z : Z \to X$. In algebraic terms, if Z is defined by the ideal $J \subset B$ then X' is defined by the ideal $I = (f^*)^{-1}(J) \subset A$ (so that $X' = \operatorname{Spec} A/I$). In particular, f^* induces an injective homomorphism

$$A/I \hookrightarrow B/J$$

Let $f' : Y' \to X'$ be the base-change of f under $X' \hookrightarrow X$, which is therefore again faithfully flat quasi-compact. As $B \otimes_A (A/I) = B/IB$, we have $Y' = \operatorname{Spec} B/IB$ and f' is induced by the ring homomorphism $A/I \to B/IB$. As $f^*(I) \subset J$, the quotient homomorphism $B \to B/J$ factors via $B \to B/IB$, giving us a closed subscheme

$$Z \hookrightarrow Y'$$

Hence, we can replace the original morphism $f : Y \to X$ by its base-change $f' : Y' \to X'$, and replace $Z \subset Y$ by $Z \subset Y'$, and thereby assume that the schematic image of Z in X is all of X (in algebraic terms, this means $A \to B/J$ is injective). Let $T = Y \times_X Z$, with projection $p_1 : T \to Y$. In algebraic terms, $T = \operatorname{Spec}(B \otimes_A (B/J))$, with homomorphism

$$p_1^* : B \to B \otimes_A (B/J) : b \mapsto b \otimes 1$$

As $A \to B$ is faithfully flat, base-change preserves the injectivity of $A \to B/J$, so that p_1^* is again injective. Hence the image $p_1(T)$ is dense in Y. Let $t \in T$ be any point and let k denote the residue field at t. A k-valued point of T is a pair (y, z)

where $y \in Y(k)$ and $z \in Z(k)$ such that $f(y) = f(z) \in X(k)$. This shows that $p_1(t) = y \in f^{-1}f(Z)$. Hence $p_1(T) \subset f^{-1}f(Z)$, and as by assumption $f^{-1}f(Z) = Z$, we get following inclusion of sets:

$$p_1(T) \subset Z$$

As $p_1(T)$ is dense in Y and Z is closed in Y, this shows that $Z = Y$. Hence $f(Z) = X$, which is closed in X as we wished to show. $\square$

Example 9.4 If $f : Y \to X$ is faithfully flat but not quasi-compact, then the topology on X is not necessarily the quotient topology from Y. For example, let $X = \operatorname{Spec} \mathbb{Z}$. For each prime number p, let $Y_p = \operatorname{Spec} \mathbb{Z}_{(p)}$, which is flat over X. Let Y denote the disjoint union $\coprod_p Y_p$. Then Y is faithfully flat over X. The inverse image of $\operatorname{Spec} \mathbb{Q} \subset X$ in Y is open, but $\operatorname{Spec} \mathbb{Q}$ is not open in X.

Remark 9.5 The above theorem can be interpreted as saying that the functor C which associates to any scheme the set of all its closed (or open) subsets is a big fpqc sheaf on **Schemes**.

10 fpqc descent for schemes

Theorem 10.1 (Grothendieck) *For any scheme X over a base S, the functor of points h_X : $\mathbf{Schemes}_S \to \mathbf{Sets}$ is a big fpqc sheaf of sets on $\mathbf{Schemes}_S$.*

Proof The functor h_X is clearly a Zariski sheaf, in particular converts coproducts into products. Hence it just remains to show that if $f : V \to U$ is a faithfully flat quasi-compact morphism over S, then the following sequence of sets is exact:

$$h_X(U) \to h_X(V) \rightrightarrows h_X(V \times_U V)$$

Clearly, the first map is injective. Let $\phi \in h_X(V)$ with $p_1^*(\phi) = p_2^*(\phi) \in h_X(V \times_U V)$. Let X be covered by affine open subschemes $W_i = \operatorname{Spec} R_i$, and let $V_i = \phi^{-1}(W_i) \subset V$. Then it follows from the equality of morphisms

$$\phi \circ p_1 = \phi \circ p_2 : V \times_U V \to X$$

that we have the following equality of open subsets:

$$p_1^{-1}(V_i) = p_2^{-1}(V_i) \subset V \times_U V$$

This shows that $V_i = f^{-1}f(V_i)$, that is, V_i is a saturated open subset of V under $f : V \to U$, and hence from the Theorem 9.3 which tells us the the topology on U is a quotient topology under the faithfully flat quasi-compact morphism f, it follows that the set-theoretic image $f(V_i)$ is an open subset U_i of U. We give U_i the structure of an open subscheme of U. Then replacing U by the disjoint union of the U_i and V by the disjoint union of $V_i = f^{-1}(U_i)$, we can assume that the scheme X is affine. We have already proved the desired exactness in that case during the last lecture. This completes the proof of the theorem. $\square$

Corollary 10.2 *Let X and Y be any two schemes over a base S, and let*

$$\underline{hom}(X,Y) : \mathbf{Schemes}_S \to \mathbf{Sets}$$

be the functor which associates to any $U \to S$ the set of all U-morphisms

$$\underline{hom}(X,Y)(U) = Hom_U(X \times_S U, Y \times_S U)$$

Then $\underline{hom}(X,Y)$ is a big fpqc sheaf of sets on $\mathbf{Schemes}_S$.

Proof Given any fpqc cover $(f_i : V_i \to U)$, we get an fpqc cover $(\mathrm{id}_X \times f_i : X \times_S V_i \to X \times_S U)$. Note that $(X \times_S V_j) \times_{(X \times_S U)} (X \times_S V_k) = X \times_S (V_j \times_U V_k)$. As h_Y is an fpqc sheaf by the above theorem, we get an exact sequence

$$h_Y(X \times_S U) \to \prod h_Y(X \times_S V_i) \rightrightarrows \prod h_Y(X \times_S (V_j \times_U V_k))$$

We have the following equalities:

$$\underline{hom}(X,Y)(U) = Hom_U(X \times_S U, Y \times_S U) = Hom_S(X \times_S U, Y) = h_Y(X \times_S U)$$

Therefore the above exact sequence becomes

$$\underline{hom}(X,Y)(U) \to \prod \underline{hom}(X,Y)(V_i) \rightrightarrows \prod \underline{hom}(X,Y)(V_j \times_U V_k)$$

This completes the proof. $\square$

Remark 10.3 In summary, we have proved fpqc descent for schemes, fpqc descent for morphisms between schemes, effective fpqc descent for closed subschemes, and effective fpqc descent for open subschemes (which is the same as effective descent for open subsets or for closed subsets).

In the next lecture, we will consider the problem of effective fpqc descent for schemes and for quasi-coherent sheaves.

11 Fibered categories

We begin with a **naive definition** of a fibered category.

Definition 11.1 A **fibered category** $\mathcal{C}$ over $\mathbf{Schemes}_S$ consists of the following data, satisfying the given conditions.

(1) For each S-scheme U we are given a category $\mathcal{C}_U$, called the **fiber** of $\mathcal{C}$ over U.

(2) For each S-morphism $f : V \to U$, we are given a functor $f^* : \mathcal{C}_U \to \mathcal{C}_V$ called the **pull-back functor**, such that for the identity morphism $\mathrm{id}_U : U \to U$, we have

$$(\mathrm{id}_U)^* = \mathrm{id}_{\mathcal{C}_U}$$

(3) For any S-morphisms $f : U \to V$ and $g : V \to W$, we have an equality of functors

$$(g \circ f)^* = f^* \circ g^*$$

Remark 11.2 **(Comparision with the correct definition)** The above naive situation rarely arises. What is more realistic is that instead of the equality $(\mathrm{id}_U)^* = \mathrm{id}_{\mathcal{C}_U}$ in (2), we have a natural isomorphism

$$\epsilon_U : (\mathrm{id}_U)^* \to \mathrm{id}_{\mathcal{C}_U}$$

as a part of the defining data, and instead of the equality $(g \circ f)^* = f^* \circ g^*$ in (3), we have a natural isomorphism

$$c_{g,f} : (g \circ f)^* \to f^* \circ g^*$$

as a part of the defining data, such that ϵ_U and $c_{g,f}$ satisfy certain conditions. These conditions are exactly such as to allow us to make a natural identification of $(\mathrm{id}_U)^*$ with $\mathrm{id}_{\mathcal{C}_U}$ using ϵ_U and a natural identification of $(g \circ f)^*$ with $f^* \circ g^*$ using $c_{g,f}$, so that these can be treated as equalities, and it can safely be pretended that we are in the naive situation!

In these lectures, we will pretend for simplicity that the naive conditions of equality (conditions (2) and (3) above) are already fulfilled in our examples. (It is an exercise for the reader to supply suitable isomorphisms ϵ_U and $c_{g,f}$ in our various examples.)

We now give some examples of fibered categories over $\mathbf{Schemes}_S$.

Example 11.3 Let $\mathfrak{Q}coh_U$ be the category of all quasi-coherent sheaves over U, and let f^*F denote the pull-back of a quasi-coherent sheaf F on U under $f : V \to U$, which defines a functor $f^* : \mathfrak{Q}coh_U \to \mathfrak{Q}coh_V$.

Example 11.4 Let $X \to S$ be a chosen scheme, and let for any S-scheme U, $Bun^n(X/S)_U$ be the category whose objects are all rank n locally free sheaves on $X \times_S U$ and morphisms are $\mathcal{O}$-linear maps.

Example 11.5 Let $\mathbf{Schemes}_U$ denote as usual the category of schemes over U, and for any $f : V \to U$ let $f^* : \mathbf{Schemes}_U \to \mathbf{Schemes}_V$ be the functor defined by fibered product.

Note that in all the above examples, it is customary to assume the naive equalities (2) and (3) for simplicity. (For example, in algebraic terms it is customary and harmless to assume an equality between $A \otimes_A M$ and M for any A-module M, and the equality in condition (2) above arises from this.)

12 Descent data

Definition 12.1 Let $\mathcal{C}$ be a fibered category over **Schemes**. Let $f : V \to U$ be a morphism of schemes. We define a category $\mathbf{D}(\mathcal{C}_V, f)$, called **the category of objects of $\mathcal{C}_V$ equipped with descent data under f**, as follows:
An objects of $\mathbf{D}(\mathcal{C}_V, f)$ is any pair (E, ϕ) where E is an object of $\mathcal{C}_V$ and

$$\phi : p_1^*(E) \to p_2^*(E)$$

is an isomorphism (called the **descent datum** or the **transition functions**) in the category $\mathcal{C}_{V\times_U V}$ such that the following equality of morphisms in $\mathcal{C}_{V\times_U V\times_U V}$ (called the **cocycle condition**) is satisfied:

$$p_{2,3}^*(\phi) \circ p_{1,2}^*(\phi) = p_{1,3}^*(\phi) : \pi_1^*(E) \to \pi_3^*(E)$$

where $p_{i,j} : V \times_U V \times_U V \to V \times_U V$ and $\pi_i : V \times_U V \times_U V \to V$ are the projections on the respective factors.
A morphism in $\mathbf{D}(\mathcal{C}_V, f)$ from an object (E, ϕ) to an object (E', ϕ') is a morphism $\theta : E \to E'$ in the category $\mathcal{C}_V$ such that the following square commutes.

$$\begin{array}{ccc} p_1^*(E) & \stackrel{\phi}{\to} & p_2^*(E) \\ {\scriptstyle p_1^*(\theta)} \downarrow & & \downarrow {\scriptstyle p_2^*(\theta)} \\ p_1^*(E') & \stackrel{\phi'}{\to} & p_2^*(E') \end{array}$$

For any morphism of schemes $f : V \to U$, we now define a functor

$$Pull(f) : \mathcal{C}_U \to \mathbf{D}(\mathcal{C}_V, f)$$

called the **pull-back** functor, as follows. For any object E of $\mathcal{C}_U$, we put $Pull(f)E = (f^*E, \phi)$, where $\phi : p_1^*(f^*E) \to p_2^*(f^*E)$ is just the identity isomorphism of the object q^*E of $\mathcal{C}_{V\times_U V}$ where $q = f \circ p_1 = f \circ p_2 : V \times_U V \to U$, under the identifications

$$p_1^*(f^*E) = (f \circ p_1)^*E = (f \circ p_2)^*E = p_2^*(f^*E)$$

Given any morphism $u : E \to E'$ in $\mathcal{C}_U$, we get a morphism

$$Pull(f)u : Pull(f)E \to Pull(f)E'$$

in $\mathbf{D}(\mathcal{C}_V, f)$ defined by the morphism $f^*u : f^*E \to f^*E'$ in $\mathcal{C}_V$. It is clear that $Pull(f)$ so defined is a functor.

Definition 12.2 **(Effectivness of descent)** We say that a given descent datum (F, ϕ) in $\mathbf{D}(\mathcal{C}_V, f)$ is **effective** is there exists an object E of $\mathcal{C}_U$ such that (F, ϕ) is isomorphic in $\mathbf{D}(\mathcal{C}_V, f)$ to the pull-back $Pull(f)E$. We say that such an E is obtained by descending the object F under the morphism $f : V \to U$ using descent datum ϕ.

Definition 12.3 Let $\mathcal{C}$ be a fibered category over **Schemes**. A morphism of schemes $f : V \to U$ is called **an effective epimorphism for** $\mathcal{C}$ if the corresponding pull-back functor $Pull(f) : \mathcal{C}_U \to \mathbf{D}(\mathcal{C}_V, f)$ is an equivalence of categories.

Example 12.4 **(Zariski gluing)** Let U be any scheme, let $U = \bigcup V_i$ be an open covering in the Zariski topology, let $V = \coprod V_i$ be the disjoint union of the open subschemes V_i, and let $f : V \to U$ be the morphism induced by the individual inclusions $V_i \hookrightarrow U$. Then f is an effective epimorphism for $\mathcal{C}$ when $\mathcal{C}$ is any of the fibered categories in Examples 11.3, 11.4, 11.5 above.

Exercise 12.5 **(Vector bundles and transition functions)** Let the notation be as in the above example. Show that any descent data ϕ on a trivial vector bundle $\mathcal{O}_V^n$ on V is the same as a family of group elements $g_{j,k} \in GL_{n,\mathbb{Z}}(V_{j,k})$ (where $V_{j,k} = V_j \cap V_k$) such that restricted to $V_{i,j,k} = V_i \cap V_j \cap V_k$, we have the equality $g_{i,j} g_{j,k} = g_{i,k}$ in $GL_{n,\mathbb{Z}}(V_{i,j,k})$.

13 Effective descent for quasi-coherent sheaves

Theorem 13.1 (Grothendieck) *Let $\mathfrak{Q}coh$ denote the fibered category of quasi-coherent sheaves over* **Schemes**. *Then any faithfully flat quasi-compact morphism $f : V \to U$ is an effective epimorphism for $\mathfrak{Q}coh$.*

Proof Let U_i be an affine open cover of U. By quasi-compactness of $f : V \to U$, each $f^{-1}(U_i)$ has an open cover by finitely many affine opens $V_{i,j}$. By Zariski gluing (Example 12.4), we can replace U by the disjoint union $\coprod U_i$ and V by the disjoint union $\coprod V_{i,j}$, and thereby assume without loss of generality that both $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are affine. Then note that quasi-coherent sheaves on U and V are respectively the same as modules over the rings A and B.

We now show that the functor $Pull(f)$ is fully faithful. For this, consider any two A-modules M and M'. Let $\mathbf{D}(B, f)$ denote for short the category of descent data on B-modules under f. We wish to show that the induced map

$$Pull(f) : Hom_A(M, M') \to Hom_{\mathbf{D}(B,f)}(B \otimes_A M, B \otimes_A M')$$

is a bijection.

Let $H = Hom_A(M, M')$ be regarded as as an A-module. Then the B-module $Hom_B(B \otimes_A M, B \otimes_A M')$ can be identified with $B \otimes_A H$ as B is faithfully flat over

A. Moreover, $Hom_{\mathbf{D}(B,f)}(B \otimes_A M, B \otimes_A M') \subset Hom_B(B \otimes_A M, B \otimes_A M') = B \otimes_A H$ is exactly the equaliser for the maps

$$p_1^* \otimes \mathrm{id}_H,\, p_2^* \otimes \mathrm{id}_H : B \otimes_A H \rightrightarrows B \otimes_A B \otimes_A H$$

We have already proved the above as the theorem of Amitsur that the sequence

$$0 \to H \to B \otimes_A H \stackrel{(p_1^* - p_2^*)\otimes \mathrm{id}_H}{\to} B \otimes_A B \otimes_A H$$

is exact. This shows that $Pull(f)$ is fully faithful.

Next, we show that the functor $Pull(f)$ is essentially surjective. This amounts to showing the effectivity of descent, that is, showing that given any (N, ϕ) in $\mathbf{D}(B, f)$, there exists some A-module M with $Pull(f)M$ isomorphic to (N, ϕ).

Candidate for M We define the A-module M as a sub-A-module of N as follows:

$$M = \{m \in N \mid \phi(m \otimes_A 1) = 1 \otimes_A m \in N \otimes_A N\}$$

This comes with an inclusion $i : M \hookrightarrow N$ which is A-linear. Hence we have an exact sequence

$$0 \to M \stackrel{i}{\to} N \stackrel{\phi \circ p_1^* - p_2^*}{\to} B \otimes_A N$$

Tensoring with the faithfully flat A-module B, we get an exact sequence

$$0 \to M \otimes_A B \stackrel{i \otimes \mathrm{id}_B}{\to} N \otimes_A B \stackrel{(\phi \circ p_1^* - p_2^*)\otimes \mathrm{id}_B}{\to} B \otimes_A N \otimes_A B$$

Now we have the following commutative diagram with exact rows, which is similar to the diagram we had while proving effective descent for closed subschemes, with use of the extra data ϕ in place of the earlier equalities of ideals. The exactness of the second row is a consequence of Amitsur's theorem applied to the B-module N and the faithfully flat morphism p_1.

$$\begin{array}{ccccccc}
0 \to & M \otimes_A B & \stackrel{i \otimes \mathrm{id}_B}{\to} & N \otimes_A B & \stackrel{(\phi \circ p_1^* - p_2^*)\otimes \mathrm{id}_B}{\to} & B \otimes_A N \otimes_A B \\
& u \downarrow & & \phi \downarrow & & \mathrm{id}_B \otimes \phi \downarrow \\
0 \to & N & \stackrel{1 \otimes \mathrm{id}_N}{\to} & B \otimes_A N & \stackrel{(p_1^* - p_2^*)\otimes \mathrm{id}_N}{\to} & B \otimes_A B \otimes_A N
\end{array}$$

Here, the map $u : M \otimes_A B \to N$ is defined by $m \otimes b \mapsto bm$.

It is a straight-forward consequence of the co-cycle condition on ϕ that the above diagram commutes. The second and third vertical maps are isomorphisms. Hence by five-lemma, the first vertical map $u : M \otimes_A B \to N$ which maps $m \otimes b \mapsto bm$ is an isomorphism.

It is clear that under the isomorphism $u : f^*M \to N$, the given descent datum ϕ corresponds to the identity descent datum on q^*M where $q = f \circ p_1 = f \circ p_2 : V \times_U V \to U$. Hence $Pull(f)M$ is isomorphic to (N, ϕ) in $\mathbf{D}(B, f)$.

This completes the proof of the theorem. $\square$

Remark 13.2 Unlike the above theorem for the fibered category of quasi-coherent sheaves, effective fpqc descent does not hold good for the fibered category of schemes. We leave it as an exercise to the reader to begin with the famous example of Hironaka of the failure of taking quotients (within schemes) of actions by finite groups (see Hartshorne 'Algebraic Geometry'), and manufacture from it an example of the failure of effective descent for the fibered category of schemes. However, effective descent holds for pairs (X, L) consisting of projective schemes equipped with very ample line bundles. This follows by converting this into an algebra problem by taking the homogeneous coordinate rings which are quasi-coherent sheaves with additional structure, and applying the theorem on effective descent for quasi-coherent sheaves.

14 Algebraic spaces

We have already seen that to each scheme X over a base scheme S, we can associate a functor $h_X : \mathbf{Rings}_S \to \mathbf{Sets}$, and this gives a fully faithful functor h from $\mathbf{Schemes}_S$ to the category of all functors $\mathbf{Rings}_S \to \mathbf{Sets}$. Moreover, h_X is an fpqc sheaf, in particular, it is an étale sheaf on $\mathbf{Rings}_S$. We will in future identify h_X with X itself, and therefore say that X is an étale sheaf on $\mathbf{Rings}_S$.

Definition 14.1 An S**-space** X is an étale sheaf of sets on $\mathbf{Rings}_S$.

Definition 14.2 A morphism $f : X \to Y$ of S-spaces is called a **schematic morphism** if for any affine scheme U over S and an S-morphism $g : U \to Y$, the fibered product $U \times_{g,Y,f} X$ is isomorphic to a scheme. Given any property $\mathcal{P}$ which makes sense for morphisms of S-schemes and is stable under base-change (such as separatedness, properness, quasi-compactness, etc.), we say that a schematic morphism $f : X \to Y$ of S-spaces has property $\mathcal{P}$ if for every scheme U over S and every S-morphism $g : U \to Y$, the projection morphism $p_1 : U \times_{g,Y,f} X \to U$ (which is a morphism of S-schemes) has property $\mathcal{P}$.

Definition 14.3 An **algebraic space** X over S is an étale sheaf of sets on $\mathbf{Rings}_S$, such that (1) the diagonal morphism $\Delta : X \to X \times_S X$ is schematic and quasi-compact, and (2) there exists an étale surjection $U \to X$ where U is a scheme over S.

Remark 14.4 The diagonal morphism of an S-space X is schematic if and only if the following property is satisfied: for any $x \in X(A)$ and $y \in Y(B)$ (that is, $x : h_A \to F$ and $y : h_B \to F$ where A and B are in $\mathbf{Rings}_S$), the fibre product $h_A \times_X h_B$ is a scheme.

Remark 14.5 Unlike differential manifolds or complex manifolds or schemes, algebraic spaces do not necessarily come from locally ringed spaces. See Knutson for examples.

Remark 14.6 Algebraic spaces can be regarded as quotients of schemes by étale equivalence relations. Such quotients are not necessarily possible within **Schemes**. However, if we have an étale equivalence relation on an algebraic space, the quotient always exists as an algebraic space.

Theorem 14.7 *Every algebraic space over S is an fpqc sheaf of sets on* $\mathbf{Rings}_S$.

Proof See [L-M] (Laumon and Moret-Bailly, 'Champs algébriques'), Theorem A.4.

15 Sheaves of groupoids

Definition 15.1 A **groupoid** is a category in which every morphism is an isomorphism.

Example 15.2 Each set (or class) X can be regarded as a category in which objects are elements of X and the only morphisms are identities. In this way, we can regard each set (or class) to be a groupoid.

Example 15.3 More generally, a groupoid in which each object has only one automorphism (namely, its identity) is categorically equivalent to a class.

Example 15.4 Each group G can be regarded as a groupoid in which there is only one object $*$, and the automorphisms of this object form the group G.

Remark 15.5 There is a basic change of a catgorical nature when we go from algebraic spaces to stacks: instead of sheaves of sets, we have sheaves of groupoids. The reason is that groupoids arise more naturally, while sets often arise by going modulo isomorphisms, thereby losing vital data.

Definition 15.6 An S-**groupoid** $\mathfrak{X}$ is a fibered category over $\mathbf{Rings}_S$ such that each category $\mathfrak{X}_A$ is a groupoid.

16 Sheaf conditions : pre-stacks and stacks

Let $\mathfrak{X}$ be an S-groupoid. For any ring A over S and objects $\xi, \eta \in \mathfrak{X}_A$, let

$$\underline{Isom}(\xi, \eta) : \mathbf{Rings}_A \to \mathbf{Sets}$$

be the functor defined by putting

$$\underline{Isom}(\xi, \eta)(B) = Hom_{\mathfrak{X}_B}(\xi|_B, \eta|_B)$$

for any A-algebra B. Given any A-algebra homomorphism $B \to C$, we have an obvious induced map $\underline{Isom}(\xi, \eta)(B) \to \underline{Isom}(\xi, \eta)(C)$.

Definition 16.1 An S-groupoid $\mathfrak{X}$ is called a **pre-stack** over S if for any ring A over S and objects $\xi, \eta \in \mathfrak{X}_A$, the functor $\underline{Isom}(\xi, \eta) : \mathbf{Rings}_A \to \mathbf{Sets}$ is an étale sheaf on $\mathbf{Rings}_A$. An S-pre-stack $\mathfrak{X}$ is called a **stack** over S if effective descent holds for the fibered category $\mathfrak{X}$, that is, given any étale cover $(A \to B_i)$ over S, objects $\xi_i \in \mathfrak{X}_{B_i}$, and co-cycle $(g_{i,j})$ of isomorphisms in $\mathfrak{X}_{B_i \otimes_A B_j}$, there exists $\xi \in \mathfrak{X}_A$ which gives $(\xi_i, g_{i,j})$.

Definition 16.2 A **morphism** $f : \mathfrak{X} \to \mathfrak{Y}$ **of** S**-stacks** is an indexed collection (f_A) of functors $f_A : \mathfrak{X}_A \to \mathfrak{Y}_A$, well-behaved under ring homomorphisms $A \to B$.

Definition 16.3 (**Fibre product of** S**-stacks**) If $f : \mathfrak{X} \to \mathfrak{Z}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ are morphisms of S-stacks, the fibered product S-stack $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is defined as follows:
Objects of the category $(\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y})_A$ are triples

$$(\xi,\, \eta,\, \varphi)$$

where ξ is an object in $\mathfrak{X}_A$, η is an object in $\mathfrak{Y}_A$, and $\varphi : f_A(\xi) \to g_A(\eta)$ is a morphism in $\mathfrak{Z}_A$.
Morphisms from (ξ, η, φ) to (ξ', η', φ') are pairs (u, v) where $u : \xi \to \xi'$, $v : \eta \to \eta'$ are morphisms in $\mathfrak{X}_A$ and $\mathfrak{Y}_A$ such that the following square commutes:

$$\begin{array}{ccc} f_A(\xi) & \overset{\varphi}{\to} & g_A(\eta) \\ {\scriptstyle f_A(u)} \downarrow & & \downarrow {\scriptstyle g_A(v)} \\ f_A(\xi') & \overset{\varphi'}{\to} & g_A(\eta') \end{array}$$

It can be verified that this data indeed defines an S-stack.

Definition 16.4 A morphism $f : \mathfrak{X} \to \mathfrak{Z}$ of S-stacks is called **representable** if for any $g : Y \to \mathfrak{Z}$ where Y is a scheme, the fibered product S-stack $\mathfrak{X} \times_{\mathfrak{Z}} Y$ is equivalent to an algebraic space.

Remark 16.5 Properties of morphisms of algebraic spaces which are stable under base change make sense also for representable morphisms of stacks.

17 Algebraic stacks: definition

A stack $\mathfrak{X}$ over S is called an **algebraic stack** if the diagonal morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable, separated and quasi-compact, and moreover at least one of the following two conditions holds:
Deligne-Mumford stack There exists a morphism $f : X \to \mathfrak{X}$ from a scheme X such that f (which is representable as by assumption Δ is representable) is étale.
Artin stack There exists a morphism $f : X \to \mathfrak{X}$ from a scheme X such that f (which is representable as by assumption Δ is representable) is smooth and surjective.
Note that a Deligne-Mumford stack is also an Artin stack. The converse is not true.

We therefore have the following inclusions of categories of 'spaces':
Schemes $\subset$ Algebraic Spaces $\subset$ D-M stacks $\subset$ Artin stacks.

Three Basic Examples

Example 17.1 n-pointed projective curves of genus g form a Deligne-Mumford stack $\mathcal{M}_{g,n}$. This was historically the first example, due to Deligne and Mumford.

Example 17.2 Flat families of coherent sheaves on a projective scheme X form an Artin stack. This uses Grothendieck's theory of quot schemes but does not need GIT stability.

Example 17.3 Let G be a smooth group-scheme over S, acting on another S-scheme X. Then a quotient $[X/G]$ exists as an Artin stack. For any S-ring A, objects of the groupoid $[X/G]_A$ are pairs (E, f) where E is a G-torsor over $\operatorname{Spec} A$, locally trivial in étale topology, and $f : E \to X$ is a G-equivaraint morphism.

Now that we have defined algebraic stacks, what about the following?

- Local properties of morphisms.
- Dimension theory.
- Sheaves of $\mathcal{O}$-modules. Cohomology.
- Differentials. Tangents.
- Étale and ℓ-adic sheaves and cohomology.
- Algebraic fundamental group.
- K-theory. Algebraic cycles.
- When of finite-type over complex numbers: singular homology, cohomology, ...

And so on. By now a large body of literature exists, which uses (and sometimes also explains!) the above.

Guide to study A student can go through the basic book 'Champs algébriques' by Laumon and Moret-Bailly. Various lecture notes are available (see Google.) The notes by Tomas Gomez are a good starting point (Proc. Indian Acad. Sci.). One should also start using the language, not waiting to finish learning the grammar.

Next Lecture: We will do in the modern language of stacks the famous 1963 calculation of Mumford, which shows that $Pic(\mathcal{M}_{1,1}) = \mathbb{Z}/(12)$ where $\mathcal{M}_{1,1}$ is the moduli stack of elliptic curves over $\mathbb{C}$.

18 Elliptic curves and their families

We fix a base field k, with characteristic $\neq 2$, 3.

Definition 18.1 An **elliptic curve over** k is a pair (X, P) where X a smooth irreducible projective curve over k of genus 1, and P a chosen k-valued point on X.

Definition 18.2 A **family of elliptic curves over a base** S, where S is a k-scheme, consists of a smooth projective morphism of k-schemes $X \to S$ together with a global section $s : S \to X$ of $X \to S$, such that for each $t \in S$, the base change to $\kappa(t)$ is an elliptic curve over $\kappa(t)$.

Remark 18.3 There exists a unique group-scheme structure on X over S with identity section s.

Example 18.4 Let $S = \operatorname{Spec} k[\lambda, \lambda^{-1}, (1-\lambda)^{-1}] = \mathbf{A}^1 - \{0, 1\}$. Let $X \subset \mathbf{P}^2_S$ defined by

$$y^2 z = x(x - z)(x - \lambda z)$$

Let $s =$ the constant section $(0, 1, 0)$. This is a family of elliptic curves over the above base.

Example 18.5 Let $S = \mathbf{A}^2_k = \operatorname{Spec} k[a, b]$ the affine plane over k with coordinates a, b. Let the divisor $D \subset \mathbf{A}^2$ be defined by $\Delta = 4a^3 + 27b^2$. Let $S = \operatorname{Spec} k[a, b, \Delta^{-1}] = \mathbf{A}^2 - D$. Let $X \subset \mathbf{P}^2_S$ defined by $y^2 z = x^3 + axz^2 - bz^3$. Again, let $s =$ the constant section $(0, 1, 0)$ as in the earlier examples.

Notation In the above examples, let the fiber over the k-valued point (a, b) or over the k-valued point λ will be denoted by $X_{a,b}$ or X_λ.

Fact: Every elliptic curve over k (or over $k = \overline{k}$) is isomorphic to some $X_{a,b}$ (to some X_λ). Neither (a, b) (or λ), nor the isomorphism, are unique.

19 Moduli scheme

We first define a **set-valued moduli functor** M for elliptic curves.

Let $M : \mathbf{Schemes}_k \to \mathbf{Sets}$ be defined by putting $M(S) =$ the set of all isomorphism classes of families (X, s) of elliptic curves over S. To any k-morphism $f : S \to T$, we attach a map $f^* : M(T) \to M(S)$, defined by $f^*(X, s) = (X_T, s_T)$ where $X_T = X \times_S T$ and $s_T = (s \circ f, \mathrm{id}_T) : T \to X_T$.

Exercise 19.1 Every elliptic curve has an automorphism of order 2, which in terms of a defining equation $y^2 = x(x-1)(x-\lambda)$ sends $(x, y) \mapsto (x, -y)$. Use this to construct a non-constant family parametrized by $\mathbf{A}^1 - \{0\}$ all whose fibers are isomorphic to a given elliptic curve, imitating the construction of the 'Mobius band'. Conclude that the functor $M : \mathbf{Schemes}_k \to \mathbf{Sets}$ is not an étale sheaf.

Question Is the étale sheafification of the functor M representable by a scheme?

Answer No!

Reason In general $Aut(X) = \{\pm 1\}$, but some elliptic curves have more automorphisms. For example, if X is defined by $y^2 = x^3 - x$ then $Aut(X) = \mu_4$, and if X is defined by $y^2 = x^3 - 1$, $Aut(X) = \mu_6$.

Ways out (1) Kill these inconvenient automorphisms.

(2) Take proper cognisance of the automorphisms. They are important and have a positive role!

Coarse Moduli Scheme for Elliptic Curves

Assume $k = \overline{k}$. Then each isomorphism class is represented in the family X_λ over $\mathbf{A}^1 - \{0, 1\}$ above. The group S_3 acts on $\mathbf{A}^1 - \{0, 1\} = \mathbf{P}^1 - \{0, 1, \infty\}$ by permuting $0, 1, \infty$. Orbits are isomorphism classes.

The quotient is the spectrum of the subring of invariants. This is the k-algebra $k[j] \subset k[\lambda, \lambda^{-1}, (1 - \lambda)^{-1}]$, where

$$j = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (1 - \lambda)^2}$$

The coarse moduli scheme for elliptic curves is the schematic quotient

$$(\mathbf{A}^1 - \{0, 1\})/S_3 = \operatorname{Spec} k[j] = \mathbf{A}^1$$

Fact There is no family on $\mathbf{A}^1$ which has the correct j invariant at each point. This can be seen in many ways (exercise!).

Therefore, $\mathbf{A}^1$ does not represent the set-valued moduli functor for elliptic curves. However, as known to the reader, it has a weaker property called the 'coarse moduli property'.

20 Moduli Stack $\mathcal{M}_{1,1}$ of Elliptic Curves

To any k-scheme S, we associate the groupoid $\mathcal{M}_{1,1}(S)$, defined as follows.

Objects Families (X, s) of elliptic curves over S.

Morphisms Isomorphisms of families.

Pull-back functor $f^* : \mathcal{M}_{1,1}(T) \to \mathcal{M}_{1,1}(S)$ corresponding to $f : S \to T$ as before.

This shows $\mathcal{M}_{1,1}$ is a fibered category in groupoids over k-schemes.

Pre-stack property $\underline{Isom}((X, s), (X', s'))$ is an fpqc sheaf on $Schemes/k$ for any two families over S. This follows from fpqc descent for schemes.

Stack property Flat descent of elliptic curves is effective, using the very ample line bundles $\mathcal{O}_X[3s]$.

Algebraicity of the stack

Representability and quasi-compactness of the diagonal $\Delta : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$

Proof $\mathcal{O}_X[3s]$ is very ample relative to S. Its direct image is a rank 3 vector bundle on S, with distinguished subbundle $\mathcal{O}_S$. Well behaved under base-change and isomorphisms.

Given two families, linear isomorphisms between the associated rank 3 vector bundles form a scheme finite-type and affine over S, of which the functor of isomorphisms of elliptic curves is a closed subscheme.

Étale cover The family $y^2 = x(x-1)(x-\lambda)$ with section $s = (0, 1, 0)$ defines an étale cover
$\mathbf{A}^1 - \{0, 1\} \to \mathcal{M}_{1,1}$.

(Degree of this cover is twelve. Bad curves have fewer copies but proportionally more automorphisms, making up the correct number!)

Therefore $\mathcal{M}_{1,1}$ is a Deligne-Mumford stack.

21 Line bundles on $\mathcal{M}_{1,1}$

Line bundles on an algebraic stack $\mathfrak{X}$

A line bundle on $\mathfrak{X}$ means a line bundle L_ξ on S for each object ξ of $\mathfrak{X}_S$, an isomorphism $L_\xi \to L_\eta$ for each $\xi \to \eta$ in $\mathfrak{X}_S$, and an isomorphism $f^*L_\xi \to L_{f^*\xi}$ on T for each $f : S \to T$, well-behaved under composition.

Example The trivial line bundle $\mathcal{O}_\mathfrak{X}$ is defined by associating $\mathcal{O}_S$ to each ξ in $\mathfrak{X}_S$ and $\mathrm{id} : \mathcal{O}_S \to \mathcal{O}_S$ to each $\xi \to \eta$ in $\mathfrak{X}_S$.

The Picard group $Pic(\mathfrak{X})$ of a stack

We can take tensor product of line bundles $L \otimes M$ on $\mathfrak{X}$ by $(L \otimes M)_\xi = L_\xi \otimes_{\mathcal{O}_S} M_\xi$, etc. This defines a group $Pic(\mathfrak{X})$ with identity $\mathcal{O}_\mathfrak{X}$.

Some candidate line bundles on $\mathcal{M}_{1,1}$

Any line bundle on $\mathcal{M}_{1,1}$ will in particular associates to each (X, P) a 1-dimensional vector space, in a functorial manner. Here are some candidates for such associations (which indeed come from line bundles on $\mathcal{M}_{1,1}$):

(0) $H^0(X, \mathcal{O}_X)$ corresponds to $\mathcal{O}_{\mathcal{M}_{1,1}}$.

(1) $H^0(X, \Omega^1_X)$

(2) $H^1(X, \mathcal{O}_X)$ (dual line bundle to (1))

(3) $T_{X,P}$ (isomorphic to (2)).

(4) $\det H^0(X, \mathcal{O}_X(mP))$ for $m \geq 0$

(5) $\det H^1(X, \mathcal{O}_X(mP))$ for $m < 0$.

Proposition 21.1 (Cohomological flatness) *For any smooth proper family of curves $X \to S$ with geometrically irreducible fibers, the natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_X$ is an isomorphism. Also, the sheaves $\pi_*\mathcal{O}_X$, $R^1\pi_*\mathcal{O}_X$, $\pi_*\Omega^1_{X/S}$ and $R^1\pi_*\Omega^1_{X/S}$ are locally free, and base-change correctly.*

This proposition shows, for example, that there exists a line bundle L on $\mathcal{M}_{1,1}$ which associates to any elliptic family X over S the line bundle $\pi_*\Omega^1_{X/S}$ on S.

Theorem 21.2 *The line bundle L on $\mathcal{M}_{1,1}$, which associates to any elliptic family X over S the line bundle $\pi_*\Omega^1_{X/S}$ on S, has order 12 and generates $Pic(\mathcal{M}_{1,1})$. Therefore,*

$$Pic(\mathcal{M}_{1,1}) = \mathbb{Z}/(12)$$

Rigidified elliptic curves

(X, P, ω) where (X, P) is an elliptic curve, and $\omega \in H^0(X, \Omega^1)$ is non-zero.

Family over S : (X, s, ω) where ω generates the line bundle $\pi_*\Omega_{X/S}$.

Example: $X_{a,b}$ defined by $y^2 = x^3 + ax - b$, with $P = (0, 1, 0)$ and $\omega = dx/y$ (over the open set $z = 1$).

This in fact gives a rigidified family over $\mathbf{A}^2 - D$ where divisor D is defined by $\Delta = 4a^3 + 27b^2$.

Proposition 21.3 *The above is a universal family, showing $\mathbf{A}^2 - D$ is fine moduli for rigidified elliptic curves.*

Proof The line bundle $\mathcal{O}_X[3s]$ has section z defined by the rational function 1.

The schematic kernel of $2 : X \to X$ is a disjoint union of s with a divisor E of degree 3. Upto invertible scalar multiple, there is a unique rational function y with $div(y) = E - 3s$.

y is unique when we moreover require that the section it defines of I_s^3/I_s^4 equals $\omega^{\otimes 3}$ where $I_s \subset \mathcal{O}_X$ is ideal sheaf.

Finally, let x be the rational function on X for which

(i) $dx/y = \omega$ (this determines x upto an additive scalar from $H^0(X, \mathcal{O}_X)$), and

(ii) the values of x at the points of E add to zero (this fixes the additive scalar, and makes x unique).

$E \to S$ is finite étale of degree 3. Even when E is not split, the condition makes sense using trace.

This gives basis (x, y, z) of the rank 3 vector bundle $\pi_*\mathcal{O}_X[3s]$ on S, and an embedding $X \hookrightarrow \mathbf{P}^2_S$.

Define $a, b \in H^0(S, \mathcal{O}_S)$ where $b =$ norm of y on E/S and $a =$ the degree 1 coefficient of the characteristic polynomial of y on E/S.

Then it can be seen that (X, s, ω) is the pull-back under $(a, b) : S \to \mathbf{A}^2 - D$.

This completes the proof. $\square$

22 $\mathcal{M}_{1,1}$ as the quotient stack $[(\mathbf{A}^2 - D)/\mathbf{G}_m]$

Theorem 22.1 *The moduli stack $\mathcal{M}_{1,1}$ of elliptic curves is isomorphic to the quotient stack $[(\mathbf{A}^2 - D)/\mathbf{G}_m]$ where the action of $\mathbf{G}_m$ is given in terms of coordinates by*

$$(a, b) \cdot t = (at^4, bt^6)$$

Proof We have an isomorphism of elliptic curves $X_{a,b} \to X_{at^4,bt^6}$ defined by

$$(x, y, z) \mapsto (xt^2,\ yt^3,\ z)$$

Note that $(0, 1, 0) \mapsto (0, 1, 0)$, as required. The form dx/y changes to dx/yt. The theorem follows. $\square$

Calculation of $Pic[(\mathbf{A}^2 - D)/\mathbf{G}_m]$

Let $U = \mathbf{A}^2 - D$. The morphism $U \to [U/\mathbf{G}_m]$ is a $\mathbf{G}_m$-torsor. All line bundles on U are trivial as U is the spectrum of $k[a, b, (4a^3 + 27b^2)^{-1}]$ which is unique factorisation domain as it is a localisation of the unique factorisation domain $k[a, b]$. Therefore, all line bundles on the quotient stack $[U/\mathbf{G}_m]$ are descended from the trivial line bundle on U. By the general description of descent under a torsor, these form the cohomology group

$$Pic[U/\mathbf{G}_m] = H^1(\mathbf{G}_m,\ \mathbf{G}_m(U)) = \frac{Z^1(\mathbf{G}_m,\ \mathbf{G}_m(U))}{B^1(\mathbf{G}_m,\ \mathbf{G}_m(U))}$$

In the above, $Z^1(\mathbf{G}_m,\ \mathbf{G}_m(U))$ (the group of 1-cocyles) is by definition the abelian group of consisting of all morphisms $\varphi : U \times \mathbf{G}_m \to \mathbf{G}_m$ such that

$$\varphi(u, t)\varphi(u \cdot t, s) = \varphi(u, ts)$$

The group structure is given by point-wise multipliction of the images in $\mathbf{G}_m$. Elements of Z^1 are traditionally called **factors of automorphy**.

Its subgroup $B^1(\mathbf{G}_m,\ \mathbf{G}_m(U))$ (the group of 1-coboundaries) by definition consists of all morphisms $\partial(f) : U \times \mathbf{G}_m \to \mathbf{G}_m$, where $f : U \to \mathbf{G}_m$ is any morphism, and

$$\partial(f)(u, t) = f(u)/f(u \cdot t)$$

is its coboundary.

Any invertible function on U is of the form $f(a, b, t) = t^m \cdot (4a^3 + 27b^2)^n$. Therefore Z^1 consists of all powers t^m.

As $\partial(f) = t^{12n}$, the subgroup $B^1 \subset Z^1$ consists of all powers of t^{12}.

Hence it follows that $H^1 = Z^1/B^1 = \mathbb{Z}/(12)$.

The line bundle L defined by $X \mapsto H^0(X, \Omega^1)$ corresponds to the factor of automorphy $1/t \in Z^1$, as dx/y gets multiplied by $1/t$ under the action of t. Therefore L generates $Pic(\mathcal{M}_{1,1})$, as claimed.

This completes the proof of the theorem. $\square$