Algebraic stacks and moduli of vector bundles

Frank Neumann
Department of Mathematics
University of Leicester
Leicester LE1 7RH
United Kingdom

Preface

This book constitutes the extended version of lectures notes for an advanced course on algebraic stacks and moduli of vector bundles delivered by the author at the 27º Colóquio Brasileiro de Matemática at IMPA in Rio de Janeiro in July 2009. The aim of the course was to give an introduction to algebraic stacks with a particular emphasis on moduli stacks of vector bundles on algebraic curves. The main goal was to present recent joint work with Ulrich Stuhler on the actions of the various Frobenius morphisms on the l-adic cohomology of the moduli stack of vector bundles of fixed rank and degree on an algebraic curve in positive characteristic [NS05], [NS].

Let us describe briefly the content of these lecture notes. Every chapter basically corresponds to the content of one of the five lectures of the advanced course. In the first chapter we will recall some background material from the theory of vector bundles and principal bundles followed by an informal introduction into moduli problems and a justification for the use of stacks. In the final section we will introduce Grothendieck topologies and sheaves as much as it is necessary for the material presented here. The second chapter gives an overview of the theory of stacks in general and of algebraic stacks in particular. We will discuss the main features and examples. From the third chapter on we shift our attention to the cohomology of algebraic stacks. After giving a quick overview of how to define sheaf cohomology of algebraic stacks and l-adic cohomology, we will outline in the fourth chapter how to determine the l-adic cohomology of the moduli stack of vector bundles of fixed rank and degree on a given algebraic curve. This "stackifies" and unifies previous calculations of Betti numbers for the moduli spaces of stable vector bundles by Harder and Narasimhan [HN75] in positive characteristic and by Atiyah and Bott [AB83] in the complex analytic case of stable holomorphic vector bundles over compact Riemann surfaces. In the last chapter, after setting the stage by briefly discussing the classical Weil Conjectures for complex projective varieties we determine the actions of the various geometric and arithmetic Frobenius morphisms on the l-adic cohomology ring of the moduli stack of vector bundles on an algebraic curve and finally indicate how to prove an analogue of the Weil Conjectures.

These lecture notes can only be a very brief encounter with the theory of algebraic stacks and its applications to moduli of vector bundles on algebraic curves. We tried to give as many references from the literature as necessary to help making these notes more transparent and to indicate to the interested reader where more details can be found. For the theory of algebraic stacks the main source we used here is the monograph by Laumon and Moret-Bailly [LMB00] and the overview article by Gómez [Góm01], which also discusses algebraic stacks in connection with moduli of vector bundles. Our discussion of sheaf cohomology and especially l-adic cohomology can only be very modest and brief here. It would be beyond the scope of these lecture notes to give a complete account. For a full-blown treatment we refer the reader to the recent articles of Laszlo and Olsson [LO08a], [LO08b] and in particular for the general theory of l-adic cohomology to the work of Behrend [Beh93], [Beh03]. Some of the material presented here can also be found in the Diploma thesis and recent lecture notes of Heinloth [Hei98], [Hei09].

Prerequisites. We will assume some basic algebraic geometry and a few notions from category theory. For algebraic geometry, we assume familiarity with basic concepts like algebraic varieties, algebraic curves, vector bundles; basic notions of the theory of schemes, sheaves and cohomology like it is presented for example in the book of Hartshorne [Har77], Chapter 1, the first half of chapter 2 and in the first three sections of Chapter 3. Some of the material can also be found in [EH00], [GH94], [Sha77a] or [Mum99]. From category theory we assume the reader is familiar with the basic notions of categories, functors and natural transformations as can be found for example in MacLane's book [ML98].

Acknowledgements. First of all I am cordially thankful to the organizers of the 27º Colóquio Brasileiro de Matemática for the wonderful opportunity to give this advanced course at IMPA. I am very grateful to Ulrich Stuhler for introducing me to this wonderful circle of ideas and for an ongoing inspiring collaboration. I am much indebted to Joachim Kock, not only for suggesting to give such an advanced course, but also for his very constructive comments and suggestions to improve these lecture notes. I very much like to thank also Nitin Nisure, who shared his insights in this material with me. The final version of these lecture notes were written while I was visiting TIFR in Mumbai and I am extremely grateful for the wonderful hospitality and support during my visit. A very first draft of these notes was produced during my visit at the CRM in Barcelona in 2008 as part of the programme "Homotopy Theory and Higher Categories" and I like to thank the CRM and especially the co-ordinators of the programme Carles Casacuberta and Joachim Kock for their encouragement and support. I finally like to thank the University of Leicester for support.

Remarks on corrected version. These notes here are a corrected version of the published lecture notes. The original lecture notes of the advanced course were published by IMPA as a book prior to the course. I like to thank all the participants of the advanced course for their interest and remarks. I am especially grateful to Filippo Viviani for pointing out several errors and misprints in the first version of these notes and to Henrique Bursztyn for his interesting comments. As much as I tried to improve these notes it is clear that they are still far from perfect. A perfect version however can be found in the library of Babel [Bor44].

Frank Neumann
Mumbai and Rio de Janeiro
May and August 2009

Contents

Chapter 1

Moduli problems and algebraic stacks I

1.1 A primer on vector bundles and principal bundles

In this section we will recall some basic notions from the theory of vector bundles and principal G-bundles. The main resources for the theory of vector bundles we use here is [Har77], [LP97] and especially for vector bundles on algebraic curves see [Fal95]. Building upon the theory of "espaces fibrés algébriques" of Weil, principal G-bundles were first introduced in algebraic geometry by Serre [Ser95] in an analogous way as the theory of general fiber bundles in algebraic topology. We will follow closely in this section the exposition of [Sor00]. We will always work over a field k, which is normally assumed to be algebraically closed.

Definition 1.1. *Let X be a scheme over a field k. A* vector bundle *over X is a scheme $\mathcal{E}$ together with a morphism $\pi : \mathcal{E} \to X$ of schemes such that π is* locally trivial *in the (Zariski) topology, i.e. there is a (Zariski) open covering $\{U_i\}_{i\in I}$ of X and isomorphisms*

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{A}^n_k$$

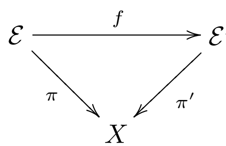
such that for every pair $i, j \in I$ there is a morphism, called transition function

$$\varphi_{ij} : U_i \cap U_j \to GL_n(k)$$

such that $\varphi_i \varphi_j^{-1}(x, v) = (x, \varphi_{ij}(x)v)$ for all $x \in U_i \cap U_j$ and $v \in \mathbb{A}_k^n$. The tuples $(U_i, \varphi_i, \varphi_{ij})$ are called a trivialization *and the integer n is called the* rank *of the vector bundle $\mathcal{E}$ denoted by* $\mathrm{rk}(\mathcal{E})$. *If* $\mathrm{rk}(\mathcal{E}) = 1$ *the vector bundle $\mathcal{E}$ is called a* line bundle.

We also have a natural notion of a morphism between vector bundles, which allows us to define the category of vector bundles.

Definition 1.2. *Let $\pi : \mathcal{E} \to X$ be a vector bundle of rank n with trivializations $(U_i, \varphi_i, \varphi_{ij})$ and $\pi' : \mathcal{E}' \to X$ be a vector bundle of rank n' with trivializations $(U'_i, \varphi'_i, \varphi'_{ij})$. A* morphism of vector bundles *$f : \mathcal{E} \to \mathcal{E}'$ is given by a commutative diagram*



such that for every pair $i, j \in I$ there is a morphism

$$f_{ij} : U'_i \cap U_j \to Mat_{n \times n'}(k)$$

such that $\varphi'_i f \varphi_j^{-1}(x, v) = (x, f_{ij}(x)v)$ for all $x \in U'_i \cap U_j$ and $v \in \mathbb{A}_k^n$. A vector bundle over X is trivial *if it is isomorphic to the vector bundle $pr_1 : X \times \mathbb{A}_k^n \to X$.*

In other words, a morphism of vector bundles $f : \mathcal{E} \to \mathcal{E}'$ over X is a morphism of schemes that commutes with the projections to X and restricts to a linear map on each fiber. Therefore we can extend in a natural way the usual operations on vector spaces from linear algebra to vector bundles. For example, we can speak of subbundles, direct sums, tensor products etc. of vector bundles.

We can also speak of the categories $\mathcal{B}un(X)$ (resp. $\mathcal{B}un^n(X)$) of vector bundles (resp. vector bundles of rank n) over X.

A vector bundle $\mathcal{E}$ over X of rank n is defined if we choose an open covering $\{U_i\}_{i\in I}$ of X together with morphisms $\varphi_{ij} : U_i \cap U_j \to GL_n(k)$ such that $\varphi_{ii} = id_{U_i}$ and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$. An isomorphic vector bundle $\mathcal{E}'$ over X is then simply given by morphisms φ'_{ij} such that $\varphi'_{ij} = f_i \varphi_{ij} f_j^{-1}$ for some morphism $f_i : U_i \to GL_n(k)$. Therefore the set of isomorphism classes of vector bundles $\mathcal{E}$ over X of rank n is in bijective correspondence with the set $H^1_{Zar}(X, GL_n(k))$ [Gro57b].

Definition 1.3. *A* section *of a vector bundle $\pi : \mathcal{E} \to X$ over an open subset $U \subset X$ is a morphism $s : U \to \mathcal{E}$ such that $\pi \circ s = id_U$. A* global section *is a section $s : U \to \mathcal{E}$ with $U = X$.*

The category $\mathcal{B}un^n(X)$ is equivalent to a certain subcategory of the category $\mathcal{C}oh(X)$ of coherent sheaves of $\mathcal{O}_X$-modules, namely to the category of locally free $\mathcal{O}_X$-modules of finite rank.

Definition 1.4. *Let X be a scheme over a field k. A* locally free sheaf *$\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$-modules such that there is a (Zariski) open covering $\{U_i\}_i$ of X with $\mathcal{F}(U_i) \cong \mathcal{O}_X(U_i)^{n_i}$ for every i, i.e. $\mathcal{F}(U_i)$ is a free $\mathcal{O}_X(U_i)$-module. The number n_i is the rank of the free $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$. If all the n_i are equal to a constant number n, then $\mathcal{F}$ is called a* locally free sheaf of rank n. *A locally free sheaf $\mathcal{F}$ on X of rank* 1 *is called an* invertible sheaf*.*

We will denote by $\mathcal{L}oc(X)$ (resp. $\mathcal{L}oc^n(X)$) be the category of locally free sheaves (resp. locally free sheaves of rank n) on X. We have the following correspondence between vector bundles and locally free sheaves.

Theorem 1.5. *Let X be a scheme over a field k. There is an equivalence of categories between the category $\mathcal{B}un^n(X)$ of vector bundles of rank n on X and the category of locally free sheaves $\mathcal{L}oc^n(X)$ of rank n on X.*

Proof. Given a vector bundle $\mathcal{E}$ of rank n on X we can construct a coherent sheaf $\mathcal{F}$ by taking for $\mathcal{F}(U)$ for every (Zariski) open set U in X its set of sections $\Gamma(U, \mathcal{F})$. This is always a locally free sheaf of constant rank n, i.e. such that all stalks $\mathcal{F}_x$ are actually free $\mathcal{O}_{X,x}$-modules of rank $n = \mathrm{rk}(\mathcal{F})$.

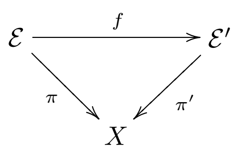
Conversely, let $\mathcal{F}$ be a locally free coherent sheaf of constant rank $\mathrm{rk}(\mathcal{F}) = n$. Then there is an open affine covering $\{U_i\}_{i \in I}$ of X such that $\mathcal{F}(U_i) \cong \mathcal{O}_X(U_i)^n$. We fix isomorphisms $\beta_i : \mathcal{O}_X(U_i)^n \xrightarrow{\cong} \mathcal{F}(U_i)$ and let $\beta_{ij} := \beta_i \beta_j|_{U_i \cap U_j}$. Let $A_i := \Gamma(U_i, \mathcal{O}_X)$ be the coordinate ring of the affine variety U_i, then $A_i[x_1, \ldots, x_n]$ is the coordinate ring of the product $U_i \times \mathbb{A}^n_k$. The morphisms β_{ij} are given as matrices of $GL_n(k)$, i.e. can be identified with morphisms of the form $U_i \cap U_j \to GL_n(k)$. We have $\beta_{ii} = id_{U_i}$ and $\beta_{ij}\beta_{jk} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$, i.e. the morphisms β_{ij} define a vector bundle of rank n on X.

From these construction it follows that the functor associating to a vector bundle of rank n the sheaf of sections as defined above is fully faithful and essentially surjective, i.e. gives the desired equivalence of categories [LP97], 1.8.1. □

We will normally identify these two categories, i.e. we identify a vector bundle with its locally free sheaf of local sections and will use the words "vector bundle" and "locally free sheaf" as synonyms.

Now let us define the notion of a principal bundle over a scheme. This is very much in analogy with similar definitions in algebraic topology.

Definition 1.6. *Let X be a scheme over a field k and G an affine algebraic group over k. A* G-fibration *over X is given by a scheme $\mathcal{E}$, an action $\rho : \mathcal{E} \times G \to \mathcal{E}$ and a G-equivariant morphism $\pi : \mathcal{E} \to X$. A morphism between two G-fibrations $\pi : \mathcal{E} \to X$ and $\pi' : \mathcal{E}' \to X$ is given by a morphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\pi = \pi' \circ f$, i.e. by a commutative diagram*



A G-fibration is called trivial *if it is isomorphic to the G-fibration $pr_1 : X \times G \to X$, where the action is given by*

$$\rho : (X \times G) \times G \to X \times G,\ \rho((x, g), g') = (x, gg').$$

A principal G-bundle is now simply a locally trivial G-fibration. But it is important to specify local triviality with respect to a given topology:

Definition 1.7. *Let X be a scheme over a field k and G an affine algebraic group over k. A* principal G-bundle *in the* Zariski *(resp.* étale*, resp.* smooth *, resp.* fppf*, resp.* fpqc*) topology is a G-fibration which is locally trivial in the Zariski (resp. étale, resp. smooth, resp. fppf, resp. fpqc) topology. This means that for any point $x \in X$ there is a neighborhood U of x such that $\mathcal{E}|_U$ is trivial in the Zariski topology, resp. there is an étale (resp. smooth, resp. flat of finite presentation, resp. flat quasi-compact) covering $U' \xrightarrow{\varphi} U$ such that the fibre product*

$$\varphi^*(\mathcal{E}|_U) \cong U' \times_U \mathcal{E}|_U$$

is trivial.

Again, we can speak of the category $\mathcal{B}un_G(X)$ of principal G-bundles over the scheme X.

Local triviality in the Zariski topology is the strongest, while local triviality in the fpqc topology the weakest condition we can ask for. If the algebraic group G is smooth, then a principal G-bundle with respect to the fpqc topology is also a principal G-bundle with respect to the étale topology [Gro95a], §6.

If $G = GL_n$ or if X is a smooth projective algebraic curve then such a principal G-bundle is always locally trivial in the Zariski topology [Ste65], 1.9. In general this is however not true as Serre showed [Ser95]. We refer to the third section of this chapter for the definition of the various Grothendieck topologies already mentioned here. For a first encounter we advise just to think about Zariski local triviality here. If a principal G-bundle $\mathcal{E}$ has a section, then it follows that $\mathcal{E}$ is trivial.

Let $H^1_{et}(X, G)$ be the set of isomorphism classes of principal G-bundles over a scheme X. This is a pointed set, pointed by the isomorphism class of the the trivial bundle.

Example 1.8 (Associated bundle with fiber F)**.** If F is a quasi-projective scheme over the field k with a left action of G and $\mathcal{E}$ is a principal G-bundle over X, we can form the *associated bundle* $\mathcal{E}(F) =$

$\mathcal{E} \times^G F$ *with fiber* F. It is given as the quotient $\mathcal{E} \times F$ under the right action of G defined by

$$\rho : G \times (\mathcal{E} \times F) \to \mathcal{E} \times F,\ \rho(g, e, f) = (e \cdot g, g^{-1} \cdot f).$$

The following two examples are important special cases of the associated bundle construction.

Example 1.9 (Associated vector bundle)**.** Let V be a vector space over a field k of dimension n and let $G = GL_n(V)$. Furthermore let $\mathcal{E}$ be a principal G-bundle. The algebraic group G acts on V from the left and we can form the associated bundle $\mathcal{V} = \mathcal{E}(V)$. This is a vector bundle of rank n.

Conversely, given any vector bundle $\mathcal{V}$ of rank n the associated frame bundle $\mathcal{E}$ is a principal GL_n-bundle.

Again, it is possible to show that the category $\mathcal{B}un^n(X)$ of vector bundles of rank n and the category $\mathcal{B}un_{GL_n}(X)$ of principal GL_n-bundles over X are equivalent and we will freely make use of this.

Example 1.10 (Extension and reduction of the structure group)**.** Let G, H be algebraic groups and $\rho : G \to H$ be a morphism. Furthermore let $\mathcal{E}$ be a principal G-bundle. The group G acts on H via ρ and we can form the associated bundle $\mathcal{E}(H)$, which is a principal H-bundle. It is also called the *extension of the structure group.* This construction induces a map of pointed sets

$$H^1_{et}(X, G) \to H^1_{et}(X, H).$$

Conversely, given a principal H-bundle $\mathcal{F}$ we call a principal G-bundle $\mathcal{E}$ together with an isomorhism of G-bundles

$$\tau : \mathcal{E}(H) \stackrel{\cong}{\to} \mathcal{F}$$

a *reduction of the structure group.*

From now on we will restrict ourselves to vector bundles $\mathcal{E}$ on a smooth projective irreducible algebraic curve X of genus g, even though some of the construction below make sense for more general schemes X.

There is another important invariant associated to a vector bundle $\mathcal{E}$ over an algebraic curve besides the rank $\mathrm{rk}(\mathcal{E})$, namely the degree $\deg(\mathcal{E})$, which we will define now.

Example 1.11 (Vector bundle associated to a representation)**.** Let X be a smooth projective algebraic curve over a field k and $\mathcal{E}$ be a vector bundle over X with $n = \mathrm{rk}(\mathcal{E})$. Consider a representation of the form

$$\rho : GL_n \to GL_{n'}.$$

As $\mathcal{E}$ is locally free we can choose an open cover $\{U_i\}_{i \in I}$ of the algebraic curve X such that there are isomorphisms

$$\beta_i : \mathcal{E}(U_i) \xrightarrow{\cong} \mathcal{O}_X(U_i)^n.$$

On the intersection $U_i \cap U_j$ we get

$$\beta_{ij} := \beta_i \circ \beta_j^{-1} \in GL_n(\Gamma(U_i \cap U_j, \mathcal{O}_X))$$

with $\beta_{ii} = id_{U_i}$ and $\beta_{ij}\beta_{jk} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$. Therefore we have that $\rho(\beta_{ij}) \in GL_{n'}(\Gamma(U_i \cap U_j, \mathcal{O}_X))$ defines a vector bundle $\rho(\mathcal{E})$ on X, the *vector bundle associated to the representation* ρ.

There is an important special case of this general construction we want to mention here:

Example 1.12 (Determinant bundle)**.** Let X be a smooth projective algebraic curve over a field k and $\mathcal{E}$ be a vector bundle over X with $n = \mathrm{rk}(\mathcal{E})$. We have the representation

$$\det : GL_n \to GL_1 = \mathbb{G}_m.$$

The associated vector bundle $\det(\mathcal{E}) = \Lambda^{\mathrm{rk}(\mathcal{E})}(\mathcal{E})$ is a line bundle, called the *determinant bundle of* $\mathcal{E}$.

We can use now the determinant line bundle to define the degree of a vector bundle $\mathcal{E}$:

Definition 1.13. *Let $\mathcal{E}$ be a vector bundle over a smooth projective algebraic curve X over a field k. The* degree $\deg(\mathcal{E})$ *of $\mathcal{E}$ is defined as the degree of the associated determinant bundle of $\mathcal{E}$*

$$\deg(\mathcal{E}) = \deg(\det(\mathcal{E})),$$

i.e. the degree of the divisor corresponding to the line bundle $\det(\mathcal{E})$*.*

For the general theory of divisors and line bundles on algebraic curves we refer to [Har77], II. 6 and [Sha77a].

It is easy to see that there exist vector bundles $\mathcal{E}$ of any given rank n and degree d on the algebraic curve X. We just take the vector bundle $\mathcal{E} = \mathcal{O}_X^{n-1} \oplus \mathcal{L}$, where $\mathcal{L}$ is a line bundle of degree d. Then $\mathcal{E}$ has rank n and degree d.

The rank and degree of a vector bundle $\mathcal{E}$ on X are directly related via the Riemann-Roch theorem (see [Har77], IV. 1):

Theorem 1.14 (Riemann-Roch). *Let $\mathcal{E}$ be a vector bundle over a smooth projective algebraic curve X over a field k. Then we have*

$$\chi(X, \mathcal{E}) = (1-g)\mathrm{rk}(\mathcal{E}) + \deg(\mathcal{E}),$$

where $\chi(X, \mathcal{E}) = \dim H^0(X, \mathcal{E}) - \dim H^1(X, \mathcal{E})$ is the Euler characteristic *of $\mathcal{E}$ and g the genus of the algebraic curve X.*

In the special case that $X = \mathbb{P}^1_k$ is the projective line vector bundles on X can be classified via the following splitting theorem of Grothendieck [Gro57a].

Theorem 1.15 (Grothendieck). *Let k be a field, then every vector bundle $\mathcal{E}$ on $\mathbb{P}^1_k$ is isomorphic to a direct sum of the form*

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1_k}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(d_2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_k}(d_n)$$

where the $d_i \in \mathbb{Z}$ and $n = \mathrm{rk}(\mathcal{E})$. The integers d_i are uniquely determined and $\sum_i d_i = \deg(\mathcal{E})$.

Proof. A proof can be found for example in [Fal95], Ch.1. □

In these lectures we are interested in classifying vector bundles $\mathcal{E}$ on any smooth projective curve X over the field $\mathbb{F}_q$. For a more general algebraic curve X we do not have a splitting theorem as in the case of the projective line $\mathbb{P}^1_k$ and the moduli problem of classifying vector bundles on more general curves or schemes is much harder.

Finally we like to state some auxiliary results for stable and semistable vector bundles on algebraic curves. We refer to [HN75], [Sha77b] and [VLP85] for the general theory and details.

Definition 1.16. *Let X be a smooth projective irreducible algebraic curve over a field k and $\mathcal{E}$ be a vector bundle over X. The* slope *of $\mathcal{E}$ is defined as*

$$\mu(\mathcal{E}) := \frac{\mathrm{rk}(\mathcal{E})}{\deg(\mathcal{E})}.$$

A vector bundle $\mathcal{E}$ on X is called semistable *if for all proper subbundles $\mathcal{F} \subset \mathcal{E}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. A vector bundle $\mathcal{E}$ on X is called* stable *if for all proper subbundles $\mathcal{F} \subset \mathcal{E}$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$.*

We can use the slope to construct a filtration for vector bundles on algebraic curves [HN75].

Theorem 1.17 (Harder-Narasimhan filtration)**.** *Let X be a smooth projective and irreducible algebraic curve over a field k. Every vector bundle $\mathcal{E}$ over X has a unique filtration of proper subbundles $\mathcal{F}_i$ over X*

$$0 = \mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_r = \mathcal{E}$$

such that $\mu(\mathcal{F}_{i+1}) < \mu(\mathcal{F}_i)$ for all i and all successive quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$ are semistable vector bundles.

Proof. For a proof see [HN75] or [VLP85]. □

Let $\mathcal{E}$ be a vector bundle on X of rank n and degree d. We get from the Harder-Narasimhan filtration a sequence of pairs of integers $(n_1, d_1), (n_2, d_2), \ldots (n_r, d_r)$ where n_i is the rank and d_i the degree of the vector bundle $\mathcal{F}_i$. And the theorem says that these numbers are unique. If we plot these pairs (n_i, d_i) as points in the euclidean plane $\mathbb{R}^2$ and join the line segments from (n_i, d_i) to (n_{i+1}, d_{i+1}), we obtain a polygonal curve with origin (n, d) such that the slope of each successive line segment decreases. Such a polygonal curve is called a *Shatz polygon* for (n, d). We can think of Shatz polygons as graphs of functions $f : [0, n] \to \mathbb{R}$ and the set of all these functions is partially ordered. Let us denote by $s(\mathcal{E})$ the Shatz polygon of $\mathcal{E}$. We have the following theorem:

Theorem 1.18 (Shatz)**.** *Let U be locally finite scheme over the field k and and $\mathcal{E}$ be a vector bundle on $U \times X$. Fix a Shatz polygon P with origin (n, d). Then we have:*

(1) The locus $U^{>P} := \{u \in U : s(\mathcal{E}_u) > P\}$ is closed.

(2) *The locus* $U^P := \{u \in U : s(\mathcal{E}_u) = P\}$ *is closed in the open set* $U \backslash U^{>P}$.

Proof. For a proof see [Sha77b] or [VLP85]. □

The Harder-Narasimhan filtration and the Shatz polygon can be used to construct a so-called Shatz stratification of the moduli stack of vector bundles of rank n and degree d on the algebraic curve X via substacks (see [Hei98], [Dhi06]). These stratifications give rise to Gysin sequences in cohomology.

1.2 Moduli problems, moduli spaces and moduli stacks

The main motivation for the introduction of algebraic stacks comes from the elegant treatment they provide for the study of moduli problems in algebraic geometry.

What is a *moduli problem*? Philosophically speaking a moduli problem is a classification problem. In geometry or topology, for example, we like to classify interesting geometric objects like manifolds, algebraic varieties, vector bundles or principal G-bundles up to their intrinsic symmetries, i.e. up to their isomorphisms depending on the particular geometric nature of the objects.

Just looking at the set of isomorphism classes of the geometric objects we like to classify normally does not give much of an insight into the geometry. To solve a moduli problem means to construct a certain geometric object, a *moduli space*, which could be for example a topological space, a manifold or an algebraic variety such that its set of points corresponds bijectively to the set of isomorphism classes of the geometric objects we like to classify.

We could therefore say that a moduli space is a solution space of a given classification problem or moduli problem. In constructing such a moduli space we obtain basically a parametrizing space in which the geometric objects we like to classify are then parametrized by the coordinates of the moduli space.

Constructing a moduli space as the solution space for a given moduli problem is normally not all what we like to ask for. We also would like to have a way of understanding how the different isomorphism classes of the geometric objects can be constructed geometrically in a universal manner. So what we really like to construct is a universal geometric object, such that all the other geometric objects can be constructed from this universal object in a kind of unifying way.

Let's look at a motivating example. We like to study the moduli problem of classifying vector bundles of fixed rank over an algebraic curve over a field k. We will be mainly interested later in the case when k is the finite field $\mathbb{F}_q$ of characteristic p.

Let X be a smooth projective algebraic curve of genus g over a field k. We define the *moduli functor* $\mathcal{M}_X^n$ as the contravariant functor or presheaf of sets from the category (Sch/k) of all schemes over k to the category of sets

$$\mathcal{M}_X^n : (Sch/k)^{op} \to (Sets).$$

On objects the functor $\mathcal{M}_X^n$ is defined by associating to a scheme U in (Sch/k) the set $\mathcal{M}_X^n(U)$ of isomorphism classes of families of vector bundles of rank n on X parameterized by U, i.e. the set of isomorphism classes of vector bundles $\mathcal{E}$ of rank n on $X \times U$.

On morphisms $\mathcal{M}_X^n$ is defined by associating to a morphism of schemes $f : U' \to U$ the map of sets $f^* : \mathcal{M}_X^n(U) \to \mathcal{M}_X^n(U')$ induced by pullback of the vector bundle $\mathcal{E}$ along the morphism $id_X \times f$ as given by the commutative diagram

$$\begin{array}{ccc} (id_X \times f)^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X \times U' & \xrightarrow{id_X \times f} & X \times U \end{array}$$

The moduli problem for classifying vector bundles of rank n and degree d on a smooth projective algebraic curve X is now equivalent to the following question.

Question (Moduli Problem). *Is the moduli functor $\mathcal{M}_X^n$ representable? In other words, does there exist a scheme M_X^n in the* *category (Sch/k) such that for all schemes U in (Sch/k) there is a bijective correspondence of sets*

$$\mathcal{M}_X^n(U) \cong Hom_{(Sch/k)}(U, M_X^n)?$$

If such a scheme M_X^n exists, it is also called a *fine moduli space*.

Let's discuss why the representability of the moduli functor $\mathcal{M}_X^n$ would indeed solve the moduli problem as described before. If a fine moduli space M_X^n exists, we would have in paricular a bijective correspondence

$$\mathcal{M}_X^n(\mathrm{Spec}(k)) \cong Hom_{(Sch/k)}(\mathrm{Spec}(k), M_X^n).$$

But this means that isomorphism classes of vector bundles over X are in bijective correspondence with points of the moduli space M_X^n. If a fine moduli space M_X^n exists, we would also have a bijective correspondence

$$\mathcal{M}_X^n(M_X^n) \cong Hom_{(Sch/k)}(M_X^n, M_X^n).$$

Now let $\mathcal{E}^{univ}$ be the element of the set $\mathcal{M}_X^n(M_X^n)$ corresponding to the morphism $id_{M_X^n}$, i.e. $\mathcal{E}^{univ}$ is a vector bundle of rank n over $X \times M_X^n$.

This vector bundle $\mathcal{E}^{univ}$ over $X \times M_X^n$ is called a *universal family* of vector bundles over X, because representability implies that for any vector bundle $\mathcal{E}$ over $X \times U$ there is a *unique* morphism $f : U \to M_X^n$ such that $\mathcal{E} \cong (id_X \times f)^*(\mathcal{E}^{univ})$ in the pullback diagram

$$\begin{array}{ccc} \mathcal{E} \cong (id_X \times f)^*\mathcal{E}^{univ} & \longrightarrow & \mathcal{E}^{univ} \\ \downarrow & & \downarrow \\ X \times U & \xrightarrow{id_X \times f} & X \times M_X^{n,d} \end{array}$$

Representability of the moduli functor $\mathcal{M}_X^n$ would therefore solve the moduli problem and addresses both desired properties of the solution, namely the existence of a geometric object such that its points correspond bijectively to isomorphism classes of vector bundles on the curve X and the existence of a universal family $\mathcal{E}^{univ}$ of vector bundles such that any family of vector bundles $\mathcal{E}$ over X can be constructed up to isomorphism as the pullback of the universal family $\mathcal{E}^{univ}$ along the classifying morphism.

Unfortunately it turns out that we run into a fundamental problem, which is typical for many moduli problems arising in algebraic geometry.

Problem. *The moduli functor $\mathcal{M}_X^n$ is not representable, because vector bundles have non-trivial automorphisms.*

For example, there are many automorphisms induced by scalar multiplication, so the multiplicative group $\mathbb{G}_m$ is always a subgroup of the automorphism group $Aut(\mathcal{E})$ of a vector bundle $\mathcal{E}$.

We can argue as follows to show that the moduli functor $\mathcal{M}_X^n$ is not representable. Let $\mathcal{E}$ be a vector bundle on $X \times U$ and let $pr_2 : X \times U \to U$ be the projection map. In addition, let $\mathcal{L}$ be a line bundle on U. Define the induced bundle $\mathcal{E}' := \mathcal{E} \otimes pr_2^* \mathcal{L}$. As vector bundles are always locally trivial in the Zariski topology it follows that there exists an open covering $\{U_i\}_{i \in I}$ of the scheme U such that the restriction $\mathcal{L}|_{U_i}$ of $\mathcal{L}$ on U_i is the trivial bundle for all $i \in I$. We will have on $X \times U_i$ therefore that

$$\mathcal{E}|_{X \times U_i} \cong \mathcal{E}'|_{X \times U_i}.$$

Assume now that the moduli functor $\mathcal{M}_X^n$ is representable, i.e. there exists a scheme M_X^n such that for all schemes U in the category (Sch/k) there is a bijective correspondence of sets

$$\mathcal{M}_X^n(U) \cong Hom_{(Sch/k)}(U, M_X^n).$$

Then it follows that there exists morphisms of schemes

$$\alpha, \alpha' : U \to M_X^n$$

corresponding to the two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ on $X \times U$. But from the remarks above on local triviality of vector bundles it follows that the restrictions of α and α' on U_i must be equal for all $i \in I$, i.e.

$$\alpha|_{U_i} = \alpha'|_{U_i}.$$

And from this it would follow immediately that $\alpha = \alpha'$ and therefore $\mathcal{E} \cong \mathcal{E}'$. But in general the two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ are not necessarily globally isomorphic.

As we cannot expect representability of the moduli functor $\mathcal{M}_X^n$ the question arises if there are any ways out of this dilemma?

There are basically two approaches to circumvent the problem of non-representability of the moduli functor:

1. *Restrict the class of vector bundles to be classified to eliminate automorphisms, i.e. rigidify the moduli problem via restriction of the moduli functor to a smaller class of vector bundles and use a weaker notion of representability.*

2. *Record the information about automorphisms by organizing the moduli data differently, i.e. enlarge the category of schemes to ensure representability of the moduli functor.*

The first approach is the approach widely used for the study of moduli problems in algebraic geometry. Classically for vector bundles one would rigidify the moduli problem by restricting to the class of stable or semistable vector bundles and uses the weaker notion of a coarse moduli space instead of a fine moduli space. A coarse moduli space still gives representability on points, but will not directly allow for the construction of a universal family. This approach uses the machinery of Geometric Invariant theory (GIT) as developed by Mumford [MFK94]. Using GIT methods it is possible to construct *coarse moduli spaces* for stable and semistable vector bundles on projective varieties. We will not follow this line of investigation here, but like to refer to the lecture notes by Esteves [Est97] and Gatto [Gat00] which also provide an excellent introduction into the general theory of moduli problems and the construction of moduli spaces.

The second approach is the one using algebraic stacks and which we will pursue here. Following earlier ideas of Grothendieck and Giraud [Gir71] Algebraic stacks were first used in the context of moduli problems by Deligne and Mumford [DM69] to study the moduli problem of algebraic curves of genus g. Nowadays these are referred to as Deligne-Mumford algebraic stacks and can be thought of algebro-geometric analogues of orbifolds, which were introduced before in differential geometry by Satake [Sat56] under the name of V-manifolds. Deligne-Mumford algebraic stacks were later generalized by Artin [Art74] to what is now called an Artin stack. He used them to develop concepts of global deformation theory.

Let us briefly discuss how this second approach applies to our motiviating example, the moduli problem of vector bundles of rank n on a smooth projective algebraic curve X. How can we record the moduli data differently so that we don't lose the information from the automorphisms? Instead of passing to sets of isomorphism classes of vector bundles we will use a categorical approach to record the information coming from the automorphisms.

As above let X be a smooth projective algebraic curve of genus g over a field k. We define the *moduli stack* $\mathscr{B}un_X^n$ as the contravariant "functor" from the category (Sch/k) of schemes over k to the category of groupoids $\mathfrak{Grpds}$

$$\mathscr{B}un_X^n : (Sch/k)^{op} \to \mathfrak{Grpds}.$$

On objects $\mathscr{B}un_X^n$ is defined by associating to a scheme U in (Sch/k) the category $\mathscr{B}un_X^n(U)$ with objects being vector bundles $\mathcal{E}$ of rank n on $X \times U$ and morphisms being vector bundle isomorphism, i.e. for every scheme U the category $\mathscr{B}un_X^n(U)$ is a *groupoid*, i.e. a category in which all its morphisms are isomorphisms.

On morphisms $\mathscr{B}un_X^n$ is defined by associating to a morphism of schemes $f : U' \to U$ a functor $f^* : \mathscr{B}un_X^n(U) \to \mathscr{B}un_X^n(U')$ induced by pullback of the vector bundle $\mathcal{E}$ along the morphism $id_X \times f$ as given by the pullback diagram

$$\begin{array}{ccc} (id_X \times f)^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X \times U' & \xrightarrow{id_X \times f} & X \times U \end{array}$$

Because pullbacks are only given up to natural isomorphisms we also have for any pair of composable morphisms of schemes

$$U'' \xrightarrow{g} U' \xrightarrow{f} U$$

a natural isomorphism between the induced pullback functors

$$\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*.$$

And these natural isomorphisms will be associative with respect to composition.

It is important to observe that $\mathscr{B}un_X^n$ is not really a "functor" in the classical categorical sense as it preserves composition not on the nose, but only up to specified coherent isomorphisms and $\mathscr{B}un_X^n$ is therefore what in general is called a *pseudo-functor*. When dealing with moduli problems the functors f^* are often given as pullback functors and the natural transformations $\epsilon_{f,g}$ as in the case of vector bundles between composition of pullback functors are induced because of uniqueness of pullbacks up to isomorphisms. The coherence they need to satisfy is then automatically satisfied, because pullbacks are characterized by a universal property. As we will see later in the formal definition of a pseudo-functor, these natural transformations need to be specified explicitly in more general situations.

There is also a fundamental difference between the set of isomorphism classes of vector bundles and the groupoid of vector bundles and vector bundle isomorphisms. Sets can be viewed as categories where the only morphisms are simply the identity morphisms, while groupoids are categories where morphisms can be more general isomorphisms. The category of groupoids is of a higher categorical hierachy than the category of sets and is in fact what is called a 2-category. We will recall later the basic categorical terminology of 2-categories and pseudo-functors relevant for us in the context of moduli problems.

An important feature of vector bundles is in addition that they have the special property that they can be defined on open coverings and glued together when they are isomorphic when restricted to intersections. So what we really will get here for $\mathscr{B}un_X^n$ is a pseudo-functor with glueing properties on the category (Sch/k) once we have specified a topology called Grothendieck topology on the category (Sch/k) in order to be able to speak of "coverings" and "glueing ". Such pseudo-functors with glueing properties, like $\mathscr{B}un_X^n$ are called stacks. In some sense we can think of stacks as "sheaves of groupoids", but need to take into account the difference to classical sheaves of sets. Sheaves of sets can be glued together if they agree by equality on coverings, but stacks can be glued if they agree up to coherent isomorphisms.

And as every scheme S can be considered as a sheaf of sets and therefore also as a "sheaf of groupoids", i.e. a stack, a version of the Yoneda Lemma for stacks will ensure representability of $\mathscr{B}un_X^n$ and from this we will get the desired properties we asked for of a solution of a moduli problem. The difference is that representability of the moduli functor will not take place anymore in the category of schemes, but in the larger category of stacks. In this way we will also obtain automatically a universal family of vector bundles $\mathcal{E}^{univ}$ of rank n, but now over the stack $X \times \mathscr{B}un_X^n$. It will have the desired property that any vector bundle $\mathcal{E}$ of rank n over $X \times U$ can be obtained via pullback from the universal bundle $\mathcal{E}^{univ}$ along a classifying morphism.

In order to make all this rigorous we will formally introduce in the next sections the concept of a stack and derive its main properties necessary for treating moduli problems and to do algebraic geometry with them.

To summarize this informal discussion, we can say that the advantages of using the language of stacks over the classical approach towards constructing moduli spaces are the following:

1. *The moduli problem of classifying vector bundles of rank n over a smooth projective algebraic curve X of genus g over the field k has no solution in the category (Sch/k), but in stacks. The moduli functor $\mathscr{B}un_X^n$ is representable in stacks, i.e. there is an equivalence of categories for any scheme U in (Sch/k)*

$$\mathscr{B}un_X^n(U) \cong Hom_{Stacks}(\underline{U}, \mathscr{B}un_X^n)$$

where $\underline{U} = Hom_{Sch/k}(-, U)$ is the stack associated to U.

2. *There exists a universal vector bundle bundle $\mathcal{E}^{univ}$ of rank n over the stack $X \times \mathscr{B}un_X^n$ such that for any vector bundle $\mathcal{E}$ of*

rank n over the scheme $X \times U$ there is a morphism of stacks

$$\varphi : \underline{U} \to \mathscr{B}un_X^n$$

such that $\mathcal{E}$ is given via the pullback $\mathcal{E} \cong (id_X \times \varphi)^* \mathcal{E}^{univ}$ from the pullback diagram of stacks

$$\begin{array}{ccc} \mathcal{E} \cong (id_X \times \varphi)^* \mathcal{E}^{univ} & \longrightarrow & \mathcal{E}^{univ} \\ \downarrow & & \downarrow \\ \underline{X} \times \underline{U} & \xrightarrow{id_X \times \varphi} & \underline{X} \times \mathscr{B}un_X^n \end{array}$$

These are the desired properties we ask for a solution of a moduli problem, which will allow for many important constructions with stacks.

Another motivation besides moduli problems for the use of the language of stacks are quotient problems.

What is a *quotient problem*? A quotient problem can be characterized as follows. Let X be a scheme over a field k and let G be a linear algebraic group acting on X via

$$\rho : X \times G \to X.$$

Let us for the moment also assume the the action ρ is a *free* action. Then the quotient X/G exists as a scheme and the quotient morphism $\tau : X \to X/G$ is a principal G-bundle. The points of the quotient scheme X/G are given by morphism of schemes $U \to X/G$. For any such morphism $f : U \to X/G$ we have a pullback diagram of the form

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ U & \xrightarrow{f} & X/G \end{array}$$

The morphism f defines therefore a principal G-bundle $\pi : \mathcal{E} \to U$ together with an G-equivariant morphism $\alpha : \mathcal{E} \to X$.

If the action of the algebraic group G on X is not free, the quotient X/G might in general not exist in the category of schemes. Again there are in principle two ways out of this dilemma:

1. *Restrict the class of schemes X or the action of G on X to construct quotients, i.e. rigidify the quotient problem via restriction to a smaller class of schemes and actions.*

2. *Organize the quotient data differently, i.e. enlarge the category of schemes to ensure the existence of the quotient.*

The first approach to the quotient problem uses again Geometric Invariant Theory (GIT), while the second approach can be made precise by working with stacks and considering the quotient simply as a stack.

Let us here briefly outline again the principal idea behind the second approach towards quotient problems.

Let X be a scheme defined over a field k together with an action of a linear algebraic group G.

We define the *quotient stack* $[X/G]$ as the contravariant "functor" from the category (Sch/k) of schemes over k to the category of groupoids $\mathfrak{Grpds}$

$$[X/G] : (Sch/k)^{op} \to \mathfrak{Grpds}.$$

On objects the functor $[X/G]$ is defined by associating to a scheme U in (Sch/k) the category $[X/G](U)$ with objects being principal G-bundles $\pi : \mathcal{E} \downarrow U$ over U together with a G-equivariant morphism $\alpha : \mathcal{E} \to X$ and morphisms being isomorphisms of principal G-bundles commuting with the G-equivariant morphisms, i.e. for every scheme U the category $[X/G](U)$ is a groupoid.

On morphisms $[X/G]$ is defined by associating to a morphism of schemes $f : U' \to U$ a functor $f^* : [X/G](U) \to [X/G](U')$ induced by pullbacks of principal G-bundles.

Again, because pullbacks are only defined up to natural isomorphisms we have for any pair of composable morphisms of schemes

$$U'' \stackrel{g}{\to} U' \stackrel{f}{\to} U$$

a natural isomorphism between the induced pullback functors

$$\epsilon_{f,g} : g^* \circ f^* \cong (f \circ g)^*.$$

These natural transformations will again be associative with respect to composition.

So similar as with the moduli stack $\mathscr{B}un_X^n$, we get a pseudo-functor $[X/G]$, which will again be a stack after defining a suitable topology on the category (Sch/k).

In the special case that $X = \mathrm{Spec}(k)$, i.e. X is just a point with a trivial G-action, the quotient stack $[\mathrm{Spec}(k)/G]$ can be understood as the moduli stack of all principal G-bundles, which is also called the *classifying stack* of the group G denoted by $\mathscr{B}G$. It plays a similar role in algebraic geometry as the classifying space BG in algebraic topology. But while in algebraic topology principal G-bundles are classified by homotopy classes of maps into the classifying space BG in algebraic geometry principal G-bundles are classified by morphisms of stacks into the classifying stack $\mathscr{B}G$.

1.3 Sites, sheaves and spaces

In order to define stacks on the category of all schemes (Sch/S) over a base scheme S we have to introduce a topology on the category (Sch/S). This is done via the general concept of a Grothendieck topology, which is a generalization of the classical topology of open sets of a topological space. We will collect here the basic terminology necessary for our purpose. We refer to [sga72], [Del77], [Tam94], [MLM94] for a more systematic treatment. Especially in relation with descent theory and stacks we will refer also to [Gir71], [Vis05].

Definition 1.19. *Let $\mathcal{C}$ be a category such that all fiber products exist in $\mathcal{C}$. A* Grothendieck topology *on $\mathcal{C}$ is given by a function τ which assigns to each object U of $\mathcal{C}$ a collection $\tau(U)$ consisting of families of morphisms $\{U_i \stackrel{\varphi_i}{\to} U\}_{i\in I}$ with target U such that*

1. *(Isomorphisms) If $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is in $\tau(U)$.*

2. *(Transitivity) If the family $\{U_i \stackrel{\varphi_i}{\to} U\}_{i\in I}$ is in $\tau(U)$, and if for each $i \in I$ one has a family $\{U_{ij} \stackrel{\varphi_{ij}}{\to} U_i\}_{j\in J}$ in $\tau(U_i)$, then the family $\{U_{ij} \stackrel{\varphi_i\circ\varphi_{ij}}{\longrightarrow} U\}_{i\in I, j\in J}$ is in $\tau(U)$.*

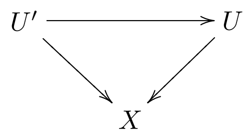
3. *(Base change) If the family* $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ *is in* $\tau(U)$*, and if* $V \to U$ *is any morphism, then the family* $\{V \times_U U_i \to V\}$ *is in* $\tau(V)$*.*

The families in $\tau(U)$ *are called* covering families *for* U *in the* τ*-topology. A* site *is a category* $\mathcal{C}$ *with fiber products together with a Grothendieck topology* τ*, also denoted by* $\mathcal{C}_\tau$*.*

It is good to think of a Grothendieck topology on a category $\mathcal{C}$ simply as a choice of a class of morphisms, which play the role of "open sets" in analogy with a topological space. An open cover of a topological space U can be seen also as choosing a class of morphisms in the category of topological spaces $f_i : U_i \to U$ such that the f_i are open inclusions and the union of their images is U.

Let us also recall here a useful notion from category theory which we will need later.

Definition 1.20. *Let* $\mathcal{C}$ *be a category and* X *an object of* $\mathcal{C}$*. The* slice category $(\mathcal{C}/X)$ *is the category whose objects are morphisms* $U \to X$ *and whose morphisms are given by commutative diagrams of the form*



Our main example of a slice category will be the category (Sch/S) of schemes over a base scheme S. The objects of (Sch/S) are schemes X together with a structure morphism $X \to S$ and the morphisms are given by the obvious commutative diagrams again. We call this category simply the *category of* S*-schemes.* Most important for us in these lectures will be the case when $S = \mathrm{Spec}(k)$ is the spectrum of a field k and we simply write (Sch/k) for the category of all schemes over the base scheme $\mathrm{Spec}(k)$.

Now let us look at some examples of sites used in algebraic geometry to illustrate the concept of a Grothendieck topology.

*Geometric properties of morphisms of schemes.*We will recall some geometric properties of morphisms of schemes. We refer to [GD67a] and [Har77] for a systematic treatment and background material.

A morphism $f : X \to Y$ is *locally of finite type* if for every point x in X with $f(x) = y$ there are affine neighborhoods $U \cong \mathrm{Spec}(B)$ of x and $V \cong \mathrm{Spec}(A)$ of y with $f(U) \subset V$ such that the induced map $A = \Gamma(V, \mathcal{O}_V) \to B = \Gamma(U, \mathcal{O}_U)$ makes B into a finitely generated algebra over A, i.e. $B = A[X_1, \dots X_n]/I$ for some ideal I. $f : X \to Y$ is *locally of finite presentation* if we can find such neighborhoods as before with B of finite presentation over A, i.e. $B = A[X_1, \dots X_n]/I$ with $I = (F_1, \dots F_m)$ for some polynomials $F_i \in A[X_1, \dots X_n]$.

A morphism $f : X \to Y$ of schemes is of *finite type* if every point y of Y has an affine open neighborhood $V \cong \mathrm{Spec}(A)$ such that $f^{-1}(V) \subset X$ is covered by a finite number of affine open sets $U \cong \mathrm{Spec}(B)$ with B a finitely generated algebra over A.

A morphism $f : X \to Y$ of schemes is an *open embedding* if it factors into an isomorphism $X \to Y'$ followed by an inclusion $Y' \hookrightarrow Y$ of an open subscheme Y' of Y. Similar if Y' is a closed subscheme of Y then f is called a *closed embedding.*

A morphism $f : X \to Y$ of schemes is *quasi-compact* if $f^{-1}(U)$ can be covered by a finite number of affine open subsets of X for every affine open subset U of Y. It follows that a morphism is of finite type if and only if it is locally of finite type and quasi-compact.

A morphism $f : X \to Y$ is *separated* if the image of the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed subset of $X \times_Y X$ or equivalently the diagonal morphism Δ is a closed embedding.

A morphism $f : X \to Y$ of schemes is *quasi-separated* if the diagonal morphism Δ is quasi-compact.

A morphism $f : X \to Y$ of schemes is *proper* if it is separated, of finite type and closed (i.e. the image of any closed subset is closed).

A morphism $f : X \to Y$ of schemes is *flat* if for every point x in X the local ring $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

A morphism $f : X \to Y$ of schemes is *faithfully flat* if it is flat and surjective.

A morphism $f : X \to Y$ of schemes is *fppf* if it is faithfully flat and locally of finite presentation.

A morphism $f : X \to Y$ of schemes is *fpqc* if it is faithfully flat, surjective and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

This is the modified definition of an fpqc morphism of Kleiman, which gives rise to the correct sheaf theory. If we would just assume instead that f is surjective and faithfully flat, Zariski covers would not be includes and one would not be able to compare the topologies (see the discussion in [Vis05], 2.3.2).

Let $f : X \to Y$ be a morphism of schemes and y be a point of Y. We have an induced morphism of local rings $f^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{Y,f(x)}$ and $\mathfrak{n} = f^\sharp(\mathcal{O}_{Y,f(x)})$ be the ideal generated by the image of $\mathfrak{m}$ in $\mathcal{O}_{X,x}$. A morphism $f : X \to Y$ of schemes is *unramified* if f is locally of finite presentation and for all points y in Y the ideal $\mathfrak{n}$ is the maximal ideal of $\mathcal{O}_{X,x}$ and the induced map $\mathcal{O}_{Y,f(x)}/\mathfrak{m} \to \mathcal{O}_{X,x}/\mathfrak{n}$ is a finite, separable field extension. This is the geometric generalization of an unramified field extension in algebraic number theory.

A morphism $f : X \to Y$ of schemes is *étale* if it is flat and unramified. We can think about this property as an analogue of an étale map or local diffeomorphism in differential topology.

A morphism $f : X \to Y$ of schemes is *smooth* if it is locally of finite presentation, flat and for any morphism of schemes of the form $\mathrm{Spec}(k) \to Y$ with k a field the geometric fiber $X \times_Y \mathrm{Spec}(k)$ is regular, i.e. all its local rings are regular local rings. Equivalently, every point x in X has a neighborhood U, which is mapped to an open subset V of Y such that there is a commutative diagram

$$\begin{array}{ccc} U & \hookrightarrow & \mathrm{Spec}(A[X_1, \ldots X_n]/(F_1, \ldots F_m)) \\ \downarrow & & \downarrow \\ V & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

where the horizontal morphisms are open embeddings such that on U we have: $\mathrm{rk}(\partial F_i/\partial X_j) = m$. There is a similar local characterisation of étale morphisms, but with $m = n$. We can think again of a smooth morphism as an analogue of a smooth map in differential topology.

Many more characterizations and properties of unramified, étale and smooth morphisms can be found in [GD67a] and [Har77]. Depending on the correct choice of a class of geometric morphisms we can define various Grothendieck topologies on the category (Sch/S)

of S-schemes. We can define small and big sites depending on if the emphasis is on a single scheme or on all schemes.

Example 1.21. Let X be a scheme over a base scheme S and let X_{Zar} be the category whose objects are open embeddings $U \to X$ and whose morphisms are morphisms $U' \to U$ over X, i.e. morphisms are given by commutative diagrams

$$\begin{array}{ccc} U' & \xrightarrow{\varphi} & U \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

A family of morphisms $\{U_i \hookrightarrow U\}_{i \in I}$ is defined to be a covering family for U if the union is U i.e. given by commutative diagrams for every $i \in I$

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_i} & U \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

such that $\bigcup_{i \in I} U_i = U$. The resulting site is called the *small Zariski site* of X denoted by X_{Zar}.

Example 1.22. Let X be a scheme over a base scheme S and let X_{et} be the category whose objects are étale morphisms $U \to X$ and whose morphisms are étale morphisms $U' \to U$ over X, i.e. morphisms are given by commutative diagrams

$$\begin{array}{ccc} U' & \xrightarrow{\varphi} & U \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

A family of morphisms $\{U_i \overset{\varphi_i}{\to} U\}_{i \in I}$ is a covering family if it is a jointly surjective family of morphisms, i.e. if the union of the images is U, i.e. given by commutative diagrams for every $i \in I$

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_i} & U \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

such that $\bigcup_{i \in I} \varphi_i(U_i) = U$. In other words, $\coprod_{i \in I} U_i \to U$ is an étale morphism.The resulting site is the *small étale site* of X denoted by X_{et}.

The étale topology is finer than the Zariski topology on a scheme X, because there are more "open sets".

Example 1.23. Let X be a scheme over a base scheme S. If we replace in the last example "étale" by "smooth" (resp. "fppf", resp. "fpqc"), we will get instead the *small smooth site* X_{sm} (resp. the *small fppf site* X_{fppf}, resp. the *small fpqc site* X_{fpqc}) of X. In the construction of the small sites a bit of care is needed as normally we don't have enough products at hand. One basically defines first a so-called pretopology which then generates a Grothendieck topology.

Example 1.24. Let (Sch/S) be the category of S-schemes. We can give it the Zariski (resp. étale, resp. smooth, res. fppf, resp. fpqc) topology by imposing them for every scheme U of the category (Sch/S). The covering families of any given scheme U are families $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ with $\bigcup_{i \in I} \varphi_i(U_i) = U$ of open embeddings (resp. étale morphisms, resp. smooth morphisms, resp. fppf morphisms, resp. fpqc morphisms). We call the resulting site on (Sch/S) the *big Zariski site* $(Sch/S)_{Zar}$ (resp. the *big étale site* $(Sch/S)_{et}$, resp. the *big smooth site* $(Sch/S)_{sm}$, resp. the *big fppf site* $(Sch/S)_{fppf}$, resp. the *big fpqc site* $(Sch/S)_{fpqc}$).

If we don't want to specify a particular choice of a Grothendieck topology we will sometimes denote by X_τ the small site with respect to a Grothendieck topology τ on the scheme X and with $(Sch/S)_\tau$ the big site with respect to a Grothendieck topology τ on the category of S-schemes. In practice τ will refer to one of the Grothendieck topologies discussed above.

After having defined the concept of a topology on a category we can now define sheaves on it. Let's start with the purely categorical notion of a presheaf.

Definition 1.25. *Let $\mathcal{C}$ be a category. A* presheaf of sets *on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{op} \to (Sets)$. Morphisms of presheaves of sets on $\mathcal{C}$ are given by natural transformation of functors $f : \mathcal{F} \to \mathcal{F}'$. We denote by* $\mathrm{PrShv}(\mathcal{C})$ *the category of presheaves on $\mathcal{C}$.*

The notion of a sheaf reflects in addition glueing properties with respect to covering families of a Grothendieck topology τ chosen on the category $\mathcal{C}$.

Definition 1.26. *Let $\mathcal{C}_\tau$ be a site. A* sheaf *on $\mathcal{C}$ for the Grothendieck topology τ is a presheaf of sets $\mathcal{F} : \mathcal{C}^{op} \to (Sets)$ satisfying the following two glueing axioms:*

1. *For all objects U in $\mathcal{C}$, for all $f, g \in \mathcal{F}(U)$ and all covering families $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ in $\tau(U)$ we have that, if $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.*

2. *For all covering families $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ in $\tau(U)$ and systems $\{f_i \in \mathcal{F}(U_i)\}_{i \in I}$ such that $\mathcal{F}(\varphi_{ij,i})(f_i) = \mathcal{F}(\varphi_{ij,j})(f_j)$ in the set $\mathcal{F}(U_i \times_U U_j)$, there exists an $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$. Here $\varphi_{ij,i} : U_i \times_U U_j \to U_i$ is the pullback of φ_j along φ_i. In other words, $\mathcal{F}(U)$ is the equalizer of the following diagram*

$$\mathcal{F}(U) = \mathrm{Ker}\Big(\prod_{i \in I} \mathcal{F}(U_i) \underset{\mathcal{F}(\varphi_{ij,j})}{\overset{\mathcal{F}(\varphi_{ij,i})}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \times_U U_j)\Big)$$

Morphisms of sheaves on $\mathcal{C}$ are natural transformations of presheaves $f : \mathcal{F} \to \mathcal{F}'$. We denote by $\mathrm{Shv}(\mathcal{C})$ *the category of sheaves on $\mathcal{C}$.*

We are mainly interested in sheaves on the category (Sch/S) of S-schemes together with a Grothendieck topology τ and will restrict ourselves to this particular situation now.

Definition 1.27. *Let $(Sch/S)_\tau$ be a site over the category (Sch/S) of S-schemes. An S-*space *with respect to the Grothendieck topology*

τ is a sheaf of sets over the site $(Sch/S)_\tau$. We denote by (Spaces/S) *the category of S-spaces.*

We have the following important presheaf on the category (Sch/S) of S-schemes.

Definition 1.28. *Let X be an object of (Sch/S). Its* functor of points *is defined as the Yoneda functor*

$$\begin{aligned} h_X := \mathrm{Hom}_{(Sch/S)}(?, X) : (Sch/S)^{op} &\to (Sets) \\ U &\mapsto \mathrm{Hom}_{(Sch/S)}(U, X) \end{aligned}$$

where $\mathrm{Hom}_{(Sch/S)}(U, X)$ *is the set of morphisms between S-schemes.*

We can ask when the presheaf h_X for a given S-scheme X is actually a sheaf on the category (Sch/S) of S-schemes together with a Grothendieck topology. This can be answered using descent theory.

Theorem 1.29 (Grothendieck)**.** *For any S-scheme X, the functor $h_X : (Sch/S)^{op} \to (Sets)$ is a sheaf for the fpqc-topology.*

Proof. This is [Vis05], Thm. 2.55. See also [Gro95a]. □

Therefore the functor h_X is a sheaf also for the Zariski, étale, smooth and fppf topology as the fpqc is the finest Grothendieck topology among them. In general it is harder to define a sheaf on a finer topology than on a given one as we have more things to glue.

If the functor of points $h_X = \mathrm{Hom}_{(Sch/S)}(?, X)$ is a sheaf with respect to the Grothendieck topology τ, we can view any scheme X in the category (Sch/S) as an S-space via h_X and a morphism between two S-spaces is equivalent to a morphism of schemes via the Yoneda embedding. Then the category (Sch/S) of S-schemes is a full subcategory of the category (Spaces/S) of S-spaces and we will sometimes write $\underline{X}$ if we view an object of the category (Sch/S) as an S-space. We can then define the following concept:

Definition 1.30. *Let (Sch/S) be the category of S-schemes together with a Grothendieck topology τ. An S-space $\mathcal{F}$ is called* representable *if there exists an object X of (Sch/S) such that $\mathcal{F}$ is isomorphic to the functor of points h_X, i.e. we have a natural isomorphism of functors*

$$\mathcal{F}(?) \cong \mathrm{Hom}_{(Sch/S)}(?, X).$$

We will later need a special case of an S-space with respect to the étale topology, called algebraic space which behaves very much like a scheme and which allows us to do algebraic geometry with. Algebraic spaces were introduced by Artin [Art71] and studied in detail by Knutson [Knu71]. They proved to be the right notion for the construction of quotients from étale equivalence relations.

Definition 1.31. *An* equivalence relation *in the category* (Spaces/S) *of S-spaces on the big étale site $(Sch/S)_{et}$ is given by two S-spaces R and X together with a monomorphism of S-spaces*

$$\delta : R \longrightarrow X \times_S X$$

such that for all objects U of (Sch/S) the map

$$\delta(U) : R(U) \to X(U) \times X(U)$$

is the graph of an equivalence relation between sets. A quotient S-space *for such an equivalence relation is given as the coequalizer of the following diagram*

$$R \underset{pr_1 \circ \delta}{\overset{pr_2 \circ \delta}{\rightrightarrows}} X.$$

We define now the notion of an algebraic space [Knu71], [Art71], which is a special S-space on the category (Sch/S) with the étale topology. See also [LMB00], Chap. 1. for a general discussion of algebraic spaces.

Definition 1.32. *An* algebraic space *is an S-space $\mathcal{X}$ on the site $(Sch/S)_{et}$ such that*

1. *For all schemes X, Y and all morphisms of S-spaces $x : X \to \mathcal{X}$, $y : Y \to \mathcal{X}$ the sheaf $X \times_{\mathcal{X}} Y$ is representable by a scheme.*
2. *There exists a scheme X, called an* atlas *and a surjective étale morphism $x : X \to \mathcal{X}$, i.e. for all morphisms of S-spaces $y : Y \to \mathcal{X}$, where Y is a scheme, the projection $X \times_{\mathcal{X}} Y \to Y$ is a surjective étale morphism of schemes.*

Let (AlgSpaces/S) be the full subcategory of algebraic spaces of the category (Spaces/S) of S-spaces. Algebraic spaces can be characterized as being a quotient of a scheme by an étale equivalence relation.

Proposition 1.33. *An S-space $\mathcal{X}$ is an algebraic space if and only if it is the quotient S-space for an equivalence relation with R and X both objects of (Sch/S), $pr_1 \circ \delta$, $pr_2 \circ \delta$ étale morphisms and δ a quasi-compact morphism in (Sch/S).*

Proof. This is [Knu71]. II, 1.3.. Basically let $R = X \times_{\mathcal{X}} X$ where X is an atlas of the algebraic space $\mathcal{X}$. □

We can therefore think of an algebraic space as an S-space that looks locally in the étale topology like an affine scheme, similar as a scheme looks locally like an affine scheme in the Zariski topology.

Later we will see that algebraic spaces are algebraic stacks which are actually sheaves of sets.

Chapter 2

Moduli problems and algebraic stacks II

2.1 Stacks

In this section we will introduce stacks formally. The idea is to define them exactly as those pseudo-functors with glueing properties as discussed before in relation with moduli and quotient problems. We are mainly interested in stacks over the category of schemes, but we will formulate much of the theory as general as possible. Besides in algebraic geometry stacks are nowadays also used in topology, differential geometry or complex geometry. Stacks can be defined equally well over the category of topological spaces, over the category of topological or smooth manifolds or over the category of complex analytic spaces. We refer the interested reader to the articles of Noohi [Noo05] and Metzler [Met03] for stacks over the category of topological spaces or to Heinloth [Hei05] and Metzler [Met03] for stacks over the category of smooth manifolds. Alternative approaches towards stacks via homotopy theory can be found in the article of Hollander [Hol08] and using topos theory in the article of Pronk [Pro96].

Before defining the notion of a stack we will recall the basic notions of 2-categories and 2-functors, which are necessary to make the definition of a stack rigorous. We will follow here [Hak72] and the appendix in [Góm01].

Definition 2.1. *A* 2-category $\mathfrak{C}$ *is given by the following data:*

1. *A class of objects* $ob(\mathfrak{C})$.

2. *For each pair of objects* $X, Y \in ob(\mathfrak{C})$ *a category* $\mathrm{Hom}(X, Y)$.

3. *(Horizontal composition of 1-morphisms and 2-morphisms) For each triple of objects* $X, Y, Z \in ob(\mathfrak{C})$ *a functor*

$$\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

such that

(i) *(Identity 1-morphism) For each object* $X \in ob(\mathfrak{C})$, *there exists an object* $id_X \in \mathrm{Hom}(X, X)$ *such that*

$$\mu_{X,X,Y}(id_X, ?) = \mu_{X,Y,Y}(?, id_Y) = id_{\mathrm{Hom}(X,Y)},$$

where $id_{\mathrm{Hom}(X,Y)}$ *is the identity functor on the category* $\mathrm{Hom}(X, Y)$.

(ii) *(Associativity of horizontal compositions) For each quadruple of objects* $X, Y, Z, W \in ob(\mathfrak{C})$, *we have*

$$\mu_{X,Y,W} \circ (\mu_{X,Y,Z} \times id_{\mathrm{Hom}(Z,W)}) = \mu_{X,Y,W} \circ (id_{\mathrm{Hom}(X,Y)} \times \mu_{Y,Z,W}).$$

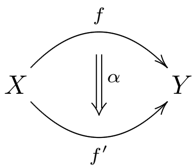
Example 2.2 (Category of categories)**.** Let $\mathfrak{Cat}$ be the 2-category of categories. The objects of $\mathfrak{Cat}$ are categories $\mathcal{C}$ and for each pair $\mathcal{C}, \mathcal{D}$ of categories, $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ is the category with objects being the functors between $\mathcal{C}$ and $\mathcal{D}$ and morphisms the natural transformation between these functors.

Example 2.3 (Groupoids)**.** A groupoid is a category where all morphisms are invertible, i.e. isomorphisms. Let $\mathfrak{Grpds}$ be the 2-category of groupoids. The objects of $\mathfrak{Grpds}$ are groupoids $\mathcal{G}$ and for each pair $\mathcal{G}, \mathcal{H}$ of groupoids, $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ is the category of functors between $\mathcal{G}$ and $\mathcal{H}$ with morphisms being again the natural transformations between these functors.

Definition 2.4. *Let $\mathfrak{C}$ be a 2-category. An object f of the category* $\mathrm{Hom}(X, Y)$ *is called a* 1-morphism *of $\mathfrak{C}$ and is represented by a diagram of the form*

$$X \xrightarrow{f} Y$$

A morphism α of the category $\mathrm{Hom}(X, Y)$ *is called a* 2-morphism *of $\mathfrak{C}$ and is represented by a diagram of the form*



We can now rewrite the axioms of a 2-category using commutative diagrams as in the previous definition.

1. (Composition of 1-morphisms) Given a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

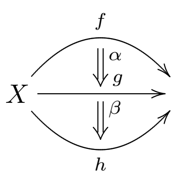
   there is given a composition $X \xrightarrow{g \circ f} Z$ and this composition is associative, i.e. we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

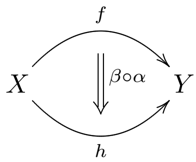
2. (Identity for 1-morphisms) For each object X there is a 1-morphism id_X such that

$$f \circ id_Y = id_X \circ f = f.$$

3. (Vertical composition of 2-morphisms) Given a diagram of the form



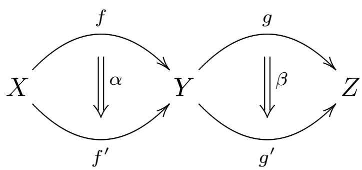
there is given a composition



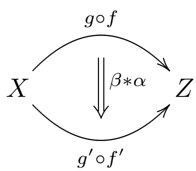
and this composition is associative, i.e. we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

4. (Horizontal composition of 2-morphisms) Given a diagram of the form



there is given a composition



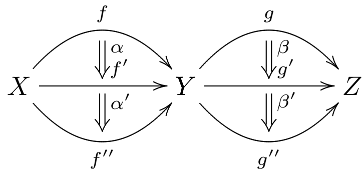
and this composition is associative, i.e. we have

$$(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha).$$

5. (Identity for 2-morphisms) For every 1-morphism f there is a 2-morphism id_f such that $\alpha \circ id_g = id_f \circ \alpha = \alpha$ and $id_g * id_f = id_{g \circ f}$.

6. (Compatibility between horizontal and vertical composition of

2-morphisms) Given a diagram of the form



we have

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha).$$

We also like to be able to say when two objects in a 2-category are actually isomorphic.

Definition 2.5. *Two objects X and Y of a 2-category are* equivalent *if there exist two 1-morphisms $f : X \to Y$, $g : Y \to X$ and two 2-isomorphisms, i.e. invertible 2-morphisms $\alpha : g \circ f \overset{\cong}{\to} id_X$ and $\beta : f \circ g \overset{\cong}{\to} id_Y$*

Example 2.6. Let $\mathcal{C}$ be any 1-category, i.e. an ordinary category. We can make it into a 2-category $\mathfrak{C}$ by just making the set $\text{Hom}(X, Y)$ into a category by taking its elements as objects and adding identity maps to each object.

Giving a 2-category $\mathfrak{C}$ we can obtain a 1-category $\mathcal{C}$ by defining the set of morphisms between two objects X and Y simply as the set of isomorphism classes of objects of the category $\text{Hom}(X, Y)$, where two objects $f, g \in \text{Hom}(X, Y)$ are isomorphic if there exists a 2-isomorphism $\alpha : f \Rightarrow g$ between them.

Example 2.7. The category (Sch/S) of S-schemes can be viewed therefore as a 2-category by adding identity 2-morphisms to all 1-morphisms.

We can now define the concept of a general pseudo-functor $\mathscr{F}$ between two 2-categories $\mathfrak{C}$ and $\mathfrak{D}$, which will be the categorical prototype of a stack.

Definition 2.8. *A* pseudo-functor $\mathscr{F} : \mathfrak{C} \to \mathfrak{D}$ *between 2-categories $\mathfrak{C}$ and $\mathfrak{D}$ is given by the following data:*

1. *For every object* X *in* $\mathfrak{C}$ *an object* $\mathscr{F}(X)$ *in* $\mathfrak{D}$*,*
2. *For each* 1*-morphism* $f : X \to Y$ *in* $\mathfrak{C}$ *we have a* 1*-morphism* $\mathscr{F}(f) : \mathscr{F}(X) \to \mathscr{F}(Y)$ *in* $\mathfrak{D}$*,*
3. *For each* 2*-morphism* $\alpha : f \Rightarrow g$ *in* $\mathfrak{C}$ *we have a* 2*-morphism* $\mathscr{F}(\alpha) : \mathscr{F}(f) \Rightarrow \mathscr{F}(g)$ *in* $\mathfrak{D}$*,*

such that

1. *(Respects identity* 1*-morphism)* $\mathscr{F}(id_x) = id_{\mathscr{F}(X)}$.
2. *(Respects identity* 2*-morphism)* $\mathscr{F}(id_f) = id_{\mathscr{F}(f)}$.
3. *(Respects composition of* 1*-morphism up to* 2*-isomorphism) For every diagram of the form*

   $$X \xrightarrow{f} Y \xrightarrow{g} Z$$

   there is given a 2*-isomorphism* $\varepsilon_{g,f} : \mathscr{F}(g) \circ \mathscr{F}(f) \to \mathscr{F}(g \circ f)$ *with*

   $$\begin{array}{ccccc} & & \mathscr{F}(Y) & & \\ & \overset{\mathscr{F}(f)}{\nearrow} & \Downarrow \varepsilon_{g,f} & \overset{\mathscr{F}(g)}{\searrow} & \\ \mathscr{F}(X) & & \xrightarrow[\mathscr{F}(g\circ f)]{} & & \mathscr{F}(Z) \end{array}$$

   such that

   (i) $\varepsilon_{f,id_X} = \varepsilon_{id_Y,f} = id_{\mathscr{F}(f)}$.

   (ii) ε *is associative, i.e. the following diagram is commutative*

   $$\begin{array}{ccc} \mathscr{F}(h) \circ \mathscr{F}(g) \circ \mathscr{F}(f) & \xRightarrow{\varepsilon_{h,g} * id_{\mathscr{F}(f)}} & \mathscr{F}(h \circ g) \circ \mathscr{F}(f) \\ \Big\Downarrow {\scriptstyle id_{\mathscr{F}(h)} * \varepsilon_{g,f}} & & \Big\Downarrow {\scriptstyle \varepsilon_{h \circ g, f}} \\ \mathscr{F}(h) \circ \mathscr{F}(g \circ f) & \xRightarrow{\varepsilon_{h, g\circ f}} & \mathscr{F}(h \circ g \circ f) \end{array}$$

4. *(Respects vertical composition of* 2*-morphisms) For every pair of* 2*-morphisms* $\alpha : f \to f'$ *and* $\beta : g :\to g'$ *we have*

   $$\mathscr{F}(\beta \circ \alpha) = \mathscr{F}(\beta) \circ \mathscr{F}(\alpha).$$

5. *(Respects horizontal composition of 2-morphisms) For every pair of 2-morphisms $\alpha : f \to f'$ and $\beta : g \to g'$ we have the following commutaive diagram*

$$\begin{array}{ccc} \mathscr{F}(g) \circ \mathscr{F}(f) & \xRightarrow{\mathscr{F}(\beta) * \mathscr{F}(\alpha)} & \mathscr{F}(g') \circ \mathscr{F}(f') \\ \Big\Downarrow \varepsilon_{g,f} & & \Big\Downarrow \varepsilon_{g',f'} \\ \mathscr{F}(g \circ f) & \xRightarrow{\mathscr{F}(\beta * \alpha)} & \mathscr{F}(g' \circ f') \end{array}$$

A bit of care is needed for the general definition of a pseudo-functor $\mathscr{F} : \mathfrak{C} \to \mathfrak{D}$ between 2-categories. There are actually four variants of pseudo-functors. Besides the pseudo-functor defined above, we can revert the direction of 1-morphisms, but leave the 2-morphisms as they are, revert the direction of 2-morphisms, but leave the 1-morphisms as they are or revert the direction of both 1- and 2-morphisms.

We will now restrict ourselves to a very special case of this general 2-categorical framework, namely pseudo-functors over a 1-category $\mathcal{C}$, which is all we need for our application to moduli problems. In fact the category $\mathcal{C}$ in our context will normally just be the 1-category (Sch/S) of schemes over a base scheme S and the 2-category $\mathfrak{D}$ will be the 2-category $\mathfrak{Grpds}$ of groupoids.

Definition 2.9. *Let $\mathcal{C}$ be a category. A* prestack *$\mathscr{X}$ is a pseudo-functor*

$$\mathscr{X} : \mathcal{C}^{op} \to \mathfrak{Grpds}.$$

Unraveling this definition shows the simplification in contrast with the general definition of a pseudo-functor between 2-categories. A prestack $\mathscr{X}$ is simply given by the following data:

1. For every object X in $\mathcal{C}$ an object $\mathscr{X}(X)$ in $\mathfrak{Grpds}$,

2. For each morphism $f : X \to Y$ in $\mathcal{C}$ we have a functor

$$f^* = \mathscr{X}(f) : \mathscr{X}(Y) \to \mathscr{X}(X)$$

   in $\mathfrak{Grpds}$,

3. For each diagram in $\mathcal{C}$ of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad X \xrightarrow{g \circ f} Z$$

we have an invertible natural transformation in $\mathfrak{Grpds}$

$$\varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*,$$

such that the following diagram is commutative

$$\begin{array}{ccc} (h \circ g \circ f)^* & \xRightarrow{\varepsilon_{h,g\circ f}} & (g \circ f)^* \circ h^* \\ \Big\Downarrow{\scriptstyle \varepsilon_{h\circ g,f}} & & \Big\Downarrow{\scriptstyle \varepsilon_{g,f} * id_{h^*}} \\ f^* \circ (h \circ g)^* & \xRightarrow{id_{f^*} * \varepsilon_{h,g}} & f^* \circ g^* \circ h^* \end{array}$$

A stack is now simply a prestack with glueing properties with respect to a Grothendieck topology τ chosen on the category $\mathcal{C}$. In our context $\mathcal{C}$ will normally be the category (Sch/S) together with a Grothendieck topology τ and we will consider stacks over the big site $(Sch/S)_\tau$.

First let us simplify the notation. If $\mathscr{X}$ is a prestack, and if x is an object in the groupoid $\mathscr{X}(U)$ and $f : U' \to U$ is a morphism in $\mathcal{C}$, we will denote the object f^*x in the groupoid $\mathscr{X}(U')$ simply by $x|_{U'}$. Especially, if we have given a covering family $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ of U and x an object of $\mathscr{X}(U)$ we denote by $x|_{U_i}$ the pullback f_i^*x. We will denote by $x_i|_{U_{ij}}$ the pullback $f_{ij,i}^* x_i$ where $f_{ij,i} : U_i \times_U \times U_j \to U_i$, x_i is an object in the groupoid $\mathscr{X}(U_i)$ and $U_{ij} = U_i \times_U U_j$.

Definition 2.10. *Let $\mathcal{C}$ be a category together with a Grothendieck topology τ. A* stack *$\mathscr{X}$ is a prestack satisfying the following glueing conditions:*
Let $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ be a covering family in the site $\mathcal{C}_\tau$, then

1. (Glueing of objects) Given objects x_i of $\mathscr{X}(U_i)$ and morphisms $\varphi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ satisfying the cocycle condition

$$\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}},$$

then there exists an object x of $\mathscr{X}(U)$ and an isomorphism $\varphi_i : x|_{U_i} \xrightarrow{\cong} x_i$ in $\mathscr{X}(U_i)$ for each i such that

$$\varphi_{ji} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}.$$

for all $i, j \in I$.

2. *(Glueing of morphisms) Given objects x and y of $\mathscr{X}(U)$ and morphisms $\varphi_i : x|_{U_i} \to y|_{U_i}$ such that*

$$\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}},$$

then there exists a morphism $\eta : x \to y$ such that $\eta|_{U_i} = \varphi_i$.

3. *(Monopresheaf) Given objects x and y of $\mathscr{X}(U)$ and morphisms $\varphi : x \to y$, $\psi : x \to y$ such that*

$$\varphi|_{U_i} = \psi|_{U_i},$$

then $\varphi = \psi$.

We can rephrase this definition in a more elegant way by using some terminolgy.

Definition 2.11. *Let $\mathcal{C}$ be a category together with a Grothendieck topology τ and $\mathscr{X}$ be a prestack. A* descent datum *for $\mathscr{X}$ with respect to a covering family $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ in the site $\mathcal{C}_\tau$ is a system of the form $(x_i, \varphi_{ij})_{i,j \in I}$ with the following properties:*

1. *each x_i is an object of $\mathscr{X}(U_i)$*

2. *each $\varphi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ is a morphism in $\mathscr{X}(U_{ij})$ satisfying the cocycle condition*

$$\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$$

for all $i, j, k \in I$.

A descent datum is effective *if there exists an object x of $\mathscr{X}(U)$ and an isomorphism $\varphi_i : x|_{U_i} \xrightarrow{\cong} x_i$ in $\mathscr{X}(U_i)$ for each i such that*

$$\varphi_{ji} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}.$$

for all $i, j \in I$.

Now we can rewrite the definition of a stack in a more compact way, which is often used in the literature.

Proposition 2.12. *Let $\mathcal{C}_\tau$ be a site. A* stack $\mathscr{X}$ *is a prestack satisfying the following properties*

1. *Every descent datum for $\mathscr{X}$ is effective.*
2. *For every object U of $\mathcal{C}$ and all objects x, y of $\mathscr{X}(U)$ the presheaf*

$$\begin{aligned} \underline{\mathrm{Isom}}_U(x,y) : (\mathcal{C}/U)^{op} &\to (Sets) \\ (U' \xrightarrow{f} U) &\mapsto \mathrm{Hom}_{\mathscr{X}(U')}(f^*x, f^*y). \end{aligned}$$

is a sheaf on the site $(\mathcal{C}/U)_\tau$.

Proof. The first assertion is simply a reformulation of the first property in the definition of a stack using the terminology of descent data as defined above. The second assertion is just a compact way of writing down the second and third property in the definition of a stack. The Grothendieck topology τ on the category $\mathcal{C}$ also makes the slice category $\mathcal{C}/U$ in an obvious way into a site, which is denoted here by $(\mathcal{C}/U)_\tau$. □

Example 2.13 (Sheaves as stacks)**.** Let $\mathcal{C}_\tau$ be a site. Any sheaf

$$\mathcal{F} : \mathcal{C}^{op} \to (Sets)$$

on the site $\mathcal{C}_\tau$ gives a stack. For each object X of $\mathcal{C}$ the set $\mathcal{F}(X)$ can simply be viewed as a groupoid with objects the elements of the set $\mathcal{F}(X)$ and the only morphisms are given by the identity morphisms for each element of $\mathcal{F}(X)$. The glueing conditions are automatically satisfied as $\mathcal{F}$ is a sheaf. Therefore especially any object X of $\mathcal{C}$ gives rise to a stack $\underline{X}$ if the functor

$$\underline{X} = \mathrm{Hom}_{\mathcal{C}}(?, X)$$

is a sheaf with respect to the Grothendieck topology τ on $\mathcal{C}$. The stack $\underline{X}$ is then also called the *stack associated to an object* X. It will always define a prestack on $\mathcal{C}$. If the context is clear we will normally simply write X instead of $\underline{X}$. Obviously any presheaf $\mathcal{F}$ and therefore $\mathrm{Hom}_{\mathcal{C}}(?, X)$ for any object X of $\mathcal{C}$ always define a prestack.

We are especially interested in stacks on the category (Sch/S) of S-schemes together with a Grothendieck topology τ, i.e. pseudo-functors

$$\mathscr{X} : (Sch/S)^{op} \to \mathfrak{Grpds}.$$

Therefore let us just record the following important special case of the preceding example.

Example 2.14 (Sheaves and schemes as stacks)**.** Let (Sch/S) be the category of S-schemes together with a Grothendieck topology τ. Any sheaf

$$\mathcal{F} : (Sch/S)^{op} \to (Sets)$$

on the site $(Sch/S)_\tau$ gives a stack as discussed before and again especially any scheme X gives rise to a stack $\underline{X}$ if the functor

$$\underline{X} = \mathrm{Hom}_{(Sch/S)}(?, X)$$

is a stack for the Grothendieck topology τ. The stack $\underline{X}$ is then called the *stack associated to a scheme* X.

Via descent theory (see Theorem 1.29) we know that the functor $\mathrm{Hom}_{(Sch/S)}(?, X)$ is a sheaf for example for the fpqc topology and therefore for any coarser topology, i.e. for the Zariski, étale, smooth and fppf topology on (Sch/S).

The following are important examples of stacks in algebraic geometry.

Example 2.15 (Moduli stack of quasi-coherent sheaves on a scheme)**.** Let (Sch/S) be the category of S-schemes and X be an S-scheme. The *moduli stack* $\mathscr{Q}coh_X$ *of quasi-coherent* $\mathcal{O}_X$*-modules* is the prestack defined as

$$\mathscr{Q}coh_X : (Sch/S)^{op} \to \mathfrak{Grpds}.$$

On objects the functor $\mathscr{Q}coh_X$ is defined by associating to a scheme U in (Sch/S) the category $\mathscr{Q}coh_X(U)$ with objects being quasi-coherent $\mathcal{O}_{X\times U}$-modules, which are flat over U and morphisms being isomorphisms of $\mathcal{O}_{X\times U}$-modules

On morphisms $\mathscr{Q}coh_X$ is defined by associating to a morphism of schemes $f : U' \to U$ the inverse image functor

$$f^* : \mathscr{Q}coh_X(U) \to \mathscr{Q}coh_X(U')$$

induced by the morphism $id_X \times f$. Then the quasi-coherent sheaves $(g^* \circ f^*)(\mathcal{E}) \cong (f \circ g)^*(\mathcal{E})$ are naturally isomorphic and it can be shown that quasi-coherent sheaves have the required descent properties even with respect to the fpqc topology [Vis05], [sga03]. Therefore the descent properties are also fulfilled in the Zariski (resp. étale, resp. smooth, resp. fppf) topology on (Sch/S). It follows therefore that $\mathscr{Q}coh_X$ is a stack for the fpqc (resp. Zariski, resp. étale, resp. smooth, resp. fppc) topology on (Sch/S) (see for example [Vis05], Thm. 4.23 for a detailed proof). As quasi-coherent sheaves are sheaves in the Zariski topology, which is a much coarser topology the fact that descent holds for the smooth or even fpqc topology is not obvious at all.

Assume that S is a locally noetherian scheme and X locally of finite type over S. Then we can define in a similar way the *moduli stack $\mathscr{C}oh_X$ of coherent $\mathcal{O}_X$-modules.*

Example 2.16 (Moduli stack of vector bundles over a scheme)**.** Let (Sch/S) be the category of S-schemes and X be an S-scheme. The *moduli stack $\mathscr{B}un_X^n$ of vector bundles on X of rank n* is the prestack defined as

$$\mathscr{B}un_X^n : (Sch/S)^{op} \to \mathfrak{Grpds}.$$

On objects the functor $\mathscr{B}un_X^n$ is defined by associating to a scheme U in (Sch/S) the category $\mathscr{B}un_X^n(U)$ with objects being vector bundles $\mathcal{E}$ on $X \times U$ of rank n and with morphisms being vector bundle isomorphisms.

On morphisms $\mathscr{B}un_X^n$ is defined by associating to a morphism of schemes $f : U' \to U$ a functor $f^* : \mathscr{B}un_X^n(U) \to \mathscr{B}un_X^n(U')$ induced by pullback of the vector bundle $\mathcal{E}$ along the morphism $id_X \times f$ as given by the pullback diagram

$$\begin{array}{ccc} (id_X \times f)^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X \times U' & \xrightarrow{id_X \times f} & X \times U \end{array}$$

We can show that $\mathscr{B}un_X^n$ is indeed a stack for the étale (resp. Zariski, resp. smooth, resp. fppf, resp. fpqc) topology on (Sch/S) as the descent properties are valid for these topologies because they hold for quasi-coherent sheaves [LMB00], 2.4.4, 3.4.4, [sga03], VIII, 1.1, 1.2.

Let X now be a smooth projective irreducible algebraic curve of genus g over a field k. We also have the moduli stacks $\mathscr{B}un_X^{n,d}$ of vector bundles of rank n and degree d on X as well as the moduli stacks $\mathscr{B}un_X^{ss,n,d}$ (resp. $\mathscr{B}un_X^{st,n,d}$) of semistable (resp. stable) vector bundles of rank n and degree d on X. On objects the prestack $\mathscr{B}un_X^{n,d}$ for example, is defined by associating to a scheme U in (Sch/S) the category $\mathscr{B}un_X^{n,d}(U)$ with objects being vector bundles $\mathcal{E}$ on $X \times U$ of rank n such that their restrictions to the slices $X \times \{u\}$ for any point u in U have degree d and with morphisms being vector bundle isomorphisms. Similarly, we define the prestacks $\mathscr{B}un_X^{ss,n,d}$ (resp. $\mathscr{B}un_X^{st,n,d}$) by additionally requesting that the restrictions to the slices are semistable (resp. stable) vector bundles. They are again all stacks over the category (Sch/k) with either Zariski (resp. étale, resp. smooth, resp. fppf, resp. fpqc) topology.

Similarly, we can define the moduli stack $\mathscr{B}un_{G,X}$ of principal G-bundles on X, where G is a reductive algebraic group over a field k [Sor00].

Example 2.17 (Quotient stack)**.** Let (Sch/S) be the category of S-schemes and X be a noetherian S-scheme. Let G be an affine smooth group S-scheme with a right action $\rho : X \times G \to X$. The *quotient stack* $[X/G]$ is the prestack defined as

$$[X/G] : (Sch/S)^{op} \to \mathfrak{Grpds}.$$

On objects the functor $[X/G]$ is defined by associating to a scheme U in (Sch/S) the category $[X/G](U)$ with objects being principal G-bundles $\pi : \mathcal{E} \to U$ over U together with a G-equivariant morphism $\alpha : \mathcal{E} \to X$ and morphisms being isomorphisms of principal G-bundles commuting with the G-equivariant morphisms.

On morphisms $[X/G]$ is defined by associating to a morphism of schemes $f : U' \to U$ a functor $f^* : [X/G](U) \to [X/G](U')$ induced by pullbacks of principal G-bundles.

It is possible to verify that $[X/G]$ defines a stack on (Sch/S) with the étale topology [LMB00], 3.4.2., [Edi00], Prop. 2.1. Let us outline the argument here. Let $x, x' \in [X/G](U)$ corresponding to principal

G-bundles $\pi : \mathcal{E} \to U$ and $\pi' : \mathcal{E}' \to U$ with G-equivariant morphisms $\alpha : \mathcal{E} \to X$ and $\alpha' : \mathcal{E}' \to X$. Then $\underline{\mathrm{Isom}}_U(x, x')$ is empty unless $\mathcal{E} = \mathcal{E}'$ and $\alpha = \alpha'$. If $x = x'$, then the isomorphisms correspond bijectively to elements $g \in G$ which preserve α, i.e. $\underline{\mathrm{Isom}}_U(x, x)$ is the subgroup of G, which is the stabilizer of the G-equivariant morphism α (see [MFK94], Chap. 0). There is a functor which associates to every G-equivariant morphism $\alpha : \mathcal{E} \to X$ its stabilizer, which is represented by a scheme $\mathrm{Stab}_X(G)$, given as the stabilizer of the identity morphism id_X. Therefore $\underline{\mathrm{Isom}}_U(x, x')$ is represented by a scheme and is therefore a sheaf on the étale site of $(Sch/S)/U$. As any principal G-bundle is determined by étale descent it follows finally that $[X/G]$ is a stack on the big étale site $(Sch/S)_{et}$.

In the special case that $X = S$ with trivial G-action, the quotient stack $[S/G]$ can be understood as the moduli stack of all principal G-bundles and is therefore called the *classifying stack* of the group S-scheme G denoted by $\mathscr{B}G$.

Classically the first examples of stacks in algebraic geometry are the moduli stacks of algebraic curves and stable algebraic curves discussed by Deligne and Mumford in [DM69].

Example 2.18 (Moduli stack of algebraic curves)**.** Let (Sch/S) be the category of S-schemes. Let U be an S-scheme and $g \geq 2$. An *algebraic curve* or *family of algebraic curves of genus g over U* is a proper and flat morphism $\pi : C \to U$ whose fibers are reduced, connected, 1-dimensional schemes C_u with arithmetic genus g, i.e. $\dim H^1(C_u, \mathcal{O}_{C_u}) = g$.

The *moduli stack of algebraic curves of genus g* is the prestack defined as

$$\mathscr{M}_g : (Sch/S)^{op} \to \mathfrak{Grpds}.$$

On objects the functor $\mathscr{M}_g$ is defined by associating to an S-scheme U in (Sch/S) the groupoid $\mathscr{M}_g(U)$ with objects being families of algebraic curves of genus g over U and morphisms being isomorphisms of such families over U.

On morphisms $\mathscr{M}_g$ is defined by associating to a morphism of schemes $f : U' \to U$ the base change functor $f^* : \mathscr{M}_g(U) \to \mathscr{M}_g(U')$ given by base change of families of algebraic curves of genus g along the morphism f.

The moduli stack $\mathscr{M}_g$ is a stack in the étale topology as the descent properties hold [sga03], VIII, 7.8. We will later see that $\mathscr{M}_g$ is actually given as a quotient stack and so in fact a special case of the last example. There are other variations of moduli stacks of algebraic curves, like the moduli stack $\mathscr{M}_{g,n}$ of algebraic curves of genus g with n distinct ordered points or the moduli stack of algebraic curves of genus g with symmetries [FN03].

Example 2.19 (Moduli stack of stable algebraic curves)**.** Let (Sch/S) be the category of S-schemes. Let U be an S-scheme and $g \geq 2$. An *stable algebraic curve* or *family of stable algebraic curves of genus g over U* is a proper and flat morphism $\pi : C \to U$ whose fibers are reduced, connected, 1-dimensional schemes C_u with arithmetic genus g, i.e. $\dim H^1(C_u, \mathcal{O}_{C_u}) = g$ and such that

1. the only singularities of the fiber C_u are ordinary double points,
2. if D is a non-singular rational component of the fiber C_u, then D meets the other components of C_u in more than two points.

This is [DM69], Def. 1.1. The second condition ensures that stable algebraic curves have always finite automorphism groups.

The *moduli stack of stable algebraic curves of genus g* is the prestack defined as

$$\widetilde{\mathscr{M}_g} : (Sch/S)^{op} \to \mathfrak{Grpds}.$$

On objects the functor $\widetilde{\mathscr{M}_g}$ is defined by associating to an S-scheme U in (Sch/S) the groupoid $\widetilde{\mathscr{M}_g}(U)$ with objects being families of stable algebraic curves of genus g over U and morphisms being isomorphisms of such families over U.

On morphisms $\widetilde{\mathscr{M}_g}$ is defined again by the base change functor $f^* : \widetilde{\mathscr{M}_g}(U) \to \widetilde{\mathscr{M}_g}(U')$ for a given morphism of schemes $f : U' \to U$.

Again $\widetilde{\mathscr{M}_g}$ is a stack in the étale topology as the descent properties hold also here [sga03], VIII, 7.8 and is also given as a quotient stack as we will see later. We also have many variations of this moduli stack like $\widetilde{\mathscr{M}_{g,n}}$.

We will shortly see that stacks over a site $\mathcal{C}_\tau$ actually form a 2-category. For this we need to introduce the notion of 1- and 2-morphisms between stacks.

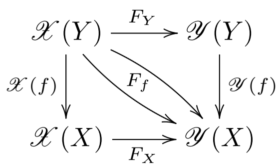
Definition 2.20. *Let $\mathcal{C}$ be a category. A* 1-morphism of prestacks $F : \mathscr{X} \to \mathscr{Y}$ *is a natural transformation of functors of 2-categories, i.e. given by the following data:*

1. *for every object X of $\mathcal{C}$, a functor $F_X : \mathscr{X}(X) \to \mathscr{Y}(X)$*

2. *for every morphism $f : X \to Y$ in $\mathcal{C}$, an invertible natural transformation $F_f : \mathscr{Y}(f) \circ F_X \Rightarrow F_Y \circ \mathscr{X}(f)$, which is compatible with the natural transformations*

$$\varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*,$$

i.e. we have the following commutative square for F_f

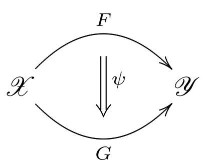


satisfying the following compatibility conditions:

(i) if $f = id_X$, then $F_{id_X} = id_{F(X)}$.

(ii) if f and g are composable morphism, then $F_{g \circ f}$ is the composite of the squares given by F_f and F_g further composed with the composition of the pullback isomorphisms $\varepsilon_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*$ for $\mathscr{X}$ and $\mathscr{Y}$.

Definition 2.21. *Let $\mathcal{C}$ be a category. A* 2-morphism *between two 1-morphisms of prestacks $F, G : \mathscr{X} \to \mathscr{Y}$ is given by a diagram*



associating to every object X in $\mathcal{C}$ an invertible natural transformation $\psi_X : F_X \to G_X$ of the form

$$\mathscr{X}(X) \underset{G_X}{\overset{F_X}{\rightrightarrows}} \mathscr{Y}(X), \qquad \Downarrow \psi_X$$

Proposition 2.22. *Let $\mathcal{C}$ be category. Prestacks over $\mathcal{C}$ together with its 1-morphisms and 2-morphims form a 2-category $\mathfrak{PreStacks}(\mathcal{C})$. Let $\mathcal{C}_\tau$ be a site. Stacks over the site $\mathcal{C}_\tau$ together with its 1-morphisms and 2-morphims form a 2-category $\mathfrak{Stacks}(\mathcal{C})$. Moreover, $\mathfrak{Stacks}(\mathcal{C})$ is a full 2-subcategory of $\mathfrak{PreStacks}(\mathcal{C})$.*

Proof. This follows from the above definitions of 1-morphisms and 2-morphisms of prestacks and the general definition of a 2-category. The 1- and 2-morphisms of stacks are given by the 1- and 2-morphisms of the underlying prestacks. □

Example 2.23. Let (Sch/S) be the category of S-schemes together with a Grothendieck topology τ. Stacks over the site $(Sch/S)_\tau$ together with its 1-morphisms and 2-morphims form a 2-category $\mathfrak{Stacks}/S = \mathfrak{Stacks}(Sch/S)$ called the *2-category of S-stacks.*

The definitions of 1- and 2-morphisms of stacks of course make sense for any pseudo-functor between 2-categories. The complicated compatibility conditions will be often automatically satisfied in the algebro-geometric examples we are interested in. The main thing to keep in mind here is that the pullback objects f^*x for every object x of the groupoid $\mathscr{X}(X)$ and every morphism $f : X \to Y$ are well defined only up to isomorphism, i.e. the object f^*x can be arbitrary chosen within its isomorphism class.

Proposition 2.24. *Let $\mathcal{C}$ be a category. The category of presheaves* $\mathrm{PrShv}(\mathcal{C})$ *is a full 2-subcategory of the 2-category $\mathfrak{PreStacks}(\mathcal{C})$ of prestacks over $\mathcal{C}$. In particular, the category $\mathcal{C}$ can be viewed as a full 2-subcategory of the 2-category $\mathfrak{PreStacks}(\mathcal{C})$.*

Proof. The category $\mathrm{PrShv}(\mathcal{C})$ of presheaves of sets on $\mathcal{C}$ can be viewed as a 2-category, where all the 2-morphisms are just identity morphisms. Any 1-morphism between prestacks, which are actually presheaves of sets is simply a morphism between presheaves, as there are only trivial 2-morphisms. Any object X of $\mathcal{C}$ can be viewed again as the presheaf $\underline{X} = \mathrm{Hom}_{\mathcal{C}}(?, X)$ and so also gives a prestack, the prestack associated to the object X of $\mathcal{C}$. □

Corollary 2.25. *Let $\mathcal{C}_\tau$ be a site. The category of sheaves* $\mathrm{Shv}(\mathcal{C})$ *is a full 2-subcategory of the 2-category $\mathfrak{Stacks}(\mathcal{C})$ of stacks over $\mathcal{C}_\tau$. In particular, the category $\mathcal{C}$ can be viewed as a full 2-subcategory of the 2-category $\mathfrak{Stacks}(\mathcal{C})$ if for each object X of $\mathcal{C}$ the functor $\underline{X} =$* $\mathrm{Hom}_{\mathcal{C}}(?, X)$ *is a sheaf for the Grothendieck topology τ.*

In particular this means that we don't need to distinguish between an object X, the sheaf it defines on the site $\mathcal{C}_\tau$ and the stack associated to X.

We have the following version of the Yoneda Lemma for prestacks [Hak72], [Gir71].

Theorem 2.26 (2-Yoneda Lemma for prestacks)**.** *Let $\mathscr{X}$ be a prestack over a category $\mathcal{C}$. Then for any object X of $\mathcal{C}$ there is an equivalence of categories*

$$\Theta : \mathrm{Hom}_{\mathfrak{PreStacks}(\mathcal{C})}(X, \mathscr{X}) \xrightarrow{\cong} \mathscr{X}(X)$$
$$(F : \underline{X} \to \mathscr{X}) \mapsto F(id_X)$$

Proof. This is a variation of the proof of the classical Yoneda Lemma [ML98]. The functor Θ defines for a morphism of prestacks $F : \underline{X} \to \mathscr{X}$ an object $x_F := F(id_X) \in \mathscr{X}(X)$, where $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ is the identity morphism of the object X in the category $\mathcal{C}$. And any isomorphism $F \to F$ defines an isomorphism of x_F. On the other hand we can define a functor

$$\Xi : \mathscr{X}(X) \to \mathrm{Hom}_{\mathfrak{PreStacks}(\mathcal{C})}(X, \mathscr{X}).$$

associating to every object $x \in \mathscr{X}(X)$ the morphism of prestacks $F_x : \underline{X} \to \mathscr{X}$ defined by sending $f \in X(U) = \mathrm{Hom}_{\mathcal{C}}(U, X)$ to the object $f^*(x) \in \mathscr{X}(U)$. And for every isomorphism $\varphi : x' \to x$ in $\mathscr{X}(X)$ we define a natural transformation $\eta_\varphi : F_{x'} \to F_x$ induced by $f^*\varphi : f^*x' \to f^*x$. We see immediately that the composition $\Theta \circ \Xi$ is the identity functor $\Theta \circ \Xi = Id_{\mathscr{X}(X)}$, because we have

$$\Theta \circ \Xi(x) = \Theta(F_x) = F_x(id_X) = id^*(x) = Id_{\mathscr{X}(X)}(x).$$

On the other hand we get $\Xi \circ \Theta(F) = \Xi(x_F) = F_{x_F}$. Here we have $F_{x_F}(U \xrightarrow{f} X) = f^*(x_F)$. And we get a natural isomorphism

$$\Xi \circ \Theta \Rightarrow Id_{\mathrm{Hom}_{\mathfrak{PreStacks}(\mathcal{C})}(X,\mathscr{X})}$$

induced by $F_f : F(U \xrightarrow{f} X) \to f^*(x_F)$. Therefore the functors Θ and Ξ define the desired equivalence of categories. □

Of course we could again have formulated the 2-Yoneda Lemma for more general pseudo-functors between 2-categories [Hak72]. It is this 2-categorical version of the Yoneda Lemma for prestacks that is crucial for the application of the language of stacks to moduli problems. It implies automatically, that a moduli problem has always a fine solution in stacks or in other words, if a stack $\mathscr{X}$ represents a moduli problem, then $\mathscr{X}$ will be itself a fine moduli space for the moduli problem. And from this it will also follow the existence of universal families.

Let $\mathcal{C}_\tau$ be a site. The 2-category $\mathfrak{Stacks}(\mathcal{C})$ of stacks on $\mathcal{C}_\tau$ has 2-fiber products.

Definition 2.27. *Let $\mathscr{X}$, $\mathscr{X}'$ and $\mathscr{S}$ be stacks over the site $\mathcal{C}_\tau$ and $F : \mathscr{X} \to \mathscr{S}$, $F' : \mathscr{X}' \to \mathscr{S}$ be morphism of stacks. The* 2-fiber product *$\mathscr{X} \times_{\mathscr{S}} \mathscr{X}'$ is the stack defined by associating to every object U in $\mathcal{C}$ the category $(\mathscr{X} \times_{\mathscr{S}} \mathscr{X}')(U)$ with*

- *objects*
  (u, u', ϕ) with $u \in \mathscr{X}(U)$, $u' \in \mathscr{X}'(U)$, $\phi \in \mathrm{Hom}_{\mathscr{S}(U)}(F(u), F'(u'))$
- *morphisms*
  $\mathrm{Hom}((u, u', \phi), (v, v', \psi)) = \{(u \xrightarrow{f} v, u' \xrightarrow{f'} v') : \psi \circ F(f) = \phi \circ F'(f')\}$.

Given two morphisms of objects X, X' of $\mathcal{C}$ into a stack $\mathscr{S}$ we can interpret the 2-fiber product in a more concrete way.

Proposition 2.28. *Let X and X' be objects of $\mathcal{C}$ and let $x : X \to \mathscr{S}$ and $x' : X' \to \mathscr{S}$ be morphisms of stacks. Then we have*

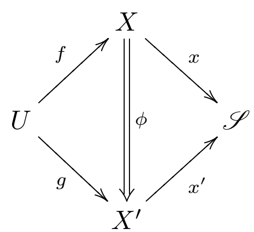
$$X \times_{\mathscr{S}} X' = \mathrm{Hom}_{\mathscr{S}(?)}(pr_1^* x, pr_2^* x'),$$

where

$$\mathrm{Hom}_{\mathscr{S}(?)}(pr_1^* x, pr_2^* x') : (\mathcal{C}/X \times X')^{op} \to (Sets)$$

is the sheaf on $\mathcal{C}/X \times X'$ associating to every object $U \xrightarrow{h} X \times X'$ the set $\mathrm{Hom}_{\mathscr{S}(U)}(h^* pr_1^* x, h^* pr_2^* x')$.

Proof. From the definition we see that $(X \times_{\mathscr{S}} X')(U)$ on objects is given by the set of all diagrams of the form



but this is the same as the set of pairs of morphisms

$$\{(U \overset{h=(f,g)}{\longrightarrow} X \times X', \phi \in \mathrm{Hom}_{\mathscr{S}(U)}(h^* pr_1^* x, h^* pr_2^* x'))\}.$$

This is the set of all morphisms between pullbacks over U of the objects x and y. □

As this is an important special case of a 2-fiber product, we introduce the following notation.

Definition 2.29. *Let X and X' be objects of a category $\mathcal{C}$ and let $x : X \to \mathscr{S}$ and $x' : X' \to \mathscr{S}$ be morphisms of stacks. We let*

$$\underline{\mathrm{Isom}}(X \times X', pr_1^* x, pr_2^* x') = X \times_{\mathscr{S}} X'.$$

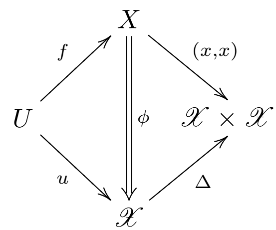
We can also describe the automorphisms of an object as a 2-fiber product of morphisms of stacks.

Proposition 2.30. *Let X be an object of a category $\mathcal{C}$, $\mathscr{X}$ a stack and $x : X \to \mathscr{X}$ be a morphisms of stacks. Then we have*

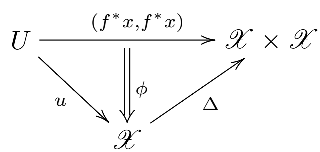
$$X \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X} = \underline{\mathrm{Isom}}_X(x, x) = \mathrm{Hom}_{\mathscr{X}(?)}(x, x),$$

where $\underline{\mathrm{Isom}}_X(x, x) : (\mathcal{C}/X)^{op} \to (Sets)$ *is the sheaf on* $\mathcal{C}/X$ *associating to every object* $U \xrightarrow{f} X$ *the set* $\mathrm{Hom}_{\mathscr{X}(U)}(f^*x, f^*x)$.

Proof. First we see that the groupoid $(X \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X})(U)$ on objects is given by the category of all diagrams of the form



which is equivalent to the category of all pairs of the form (f, D) where $f : U \to X$ is a morphism and D a diagram of the form



which is equivalent to the category of quadruples of the form

$$(f : U \to X, u \in \mathscr{X}(U), \phi_1 : f^*x \to u, \phi_2 : f^*x \to u).$$

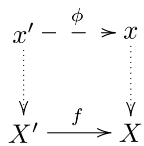
and from this the desired description of the automorphisms of an object follows. □

Sometimes it is better to consider stacks not as pseudo-functors but as categories. Many constructions can be easier formulated in this framework, which we will briefly recall here now. A nice and direct introduction into stacks from scratch using this approach can be found in [Fan01].

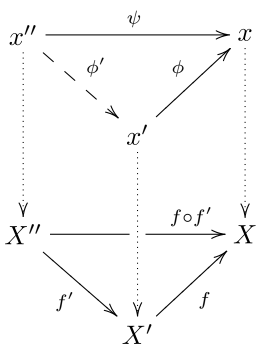
Definition 2.31. *Let $\mathcal{C}$ be a category. A* category fibred over $\mathcal{C}$ *is a category $\mathscr{X}$ together with a projection functor $p_{\mathscr{X}} : \mathscr{X} \to \mathcal{C}$. If x is an object of $\mathscr{X}$ with $p_{\mathscr{X}}(x) = X$, we say that x* lies over X. *If ϕ is a morphism of $\mathscr{X}$ with $p_{\mathscr{X}}(\phi) = f$, we say that ϕ* lies over f.

Definition 2.32. *A* category fibered in groupoids *is a category $\mathscr{X}$ over $\mathcal{C}$ such that*

1. *For every morphism $f : X' \to X$ in $\mathcal{C}$ and every object x in $\mathscr{X}$ with $p_{\mathscr{X}}(x) = X$ there exists an object x' and a morphism $\phi : x' \to x$ such that $p_{\mathscr{X}}(x') = X'$ and $p_{\mathscr{X}}(\phi) = f$, i.e.*



2. *For every diagram of the form*



where $p_{\mathscr{X}}(x'') = X''$, $p_{\mathscr{X}}(x') = X'$, $p_{\mathscr{X}}(x) = X$ and for morphisms $p_{\mathscr{X}}(\phi) = f$, $p_{\mathscr{X}}(\psi) = f \circ f'$, there exists a unique morphism $\phi' : x'' \to x'$ with $\psi = \phi \circ \phi'$ such that $p_{\mathscr{X}}(\phi') = f'$.

The second condition implies that the object x' whose existence is postulated by the first condition is unique up to a canonical isomorphism. Therefore for each object x and each morphism f we choose such an object $f^*x = x'$. It also follows immediately from the second condition that ϕ is an ismorphism if and only if $p_{\mathscr{X}}(\phi) = f$ is an isomorphism.

Definition 2.33. *Let $\mathcal{C}$ be a category. Let $\mathscr{X}$ be a category fibered in groupoids over $\mathcal{C}$ and X be an object of $\mathcal{C}$. The* fiber *$\mathscr{X}(X)$ of $\mathscr{X}$ over X is the subcategory of $\mathscr{X}$ whose objects are the objects of $\mathscr{X}$ lying over X and whose morphisms are the morphisms of $\mathscr{X}$ lying over id_X.*

It follows immediately from this definition that the fiber $\mathscr{X}(X)$ of $\mathscr{X}$ over X is a groupoid as all morphisms lying over id_X are isomorphisms. This explains the choice of the name *category fibered in groupoids.*

Given a category $\mathscr{X}$ fibered in groupoids over a category $\mathcal{C}$ we can associate to it a prestack

$$\widetilde{\mathscr{X}} : \mathcal{C}^{op} \to \mathfrak{Grpds}, \ X \mapsto \mathscr{X}(X)$$

The second condition in the definition ensures the existence of a pullback functor f^* and the existence of the 2-isomorphisms $\varepsilon_{g,f}$ and it is an easy exercise now to check the axioms of a prestack.

On the other hand, given a prestack $\widetilde{\mathscr{X}}$ on a category $\mathcal{C}$ we can associate to it a category $\mathscr{X}$ fibered in groupoids over $\mathcal{C}$. Its objects are given by pairs (x, X), where X is an object of $\mathcal{C}$ and x an object of the groupoid $\widetilde{\mathscr{X}}(X)$. A morphism $(x', X') \to (x, X)$ is given by a pair (f, α), where $f : X' \to X$ is a morphism in $\mathcal{C}$ and $\alpha : f^*x \to x'$ is an isomorphism with $f^* = \widetilde{\mathscr{X}}(f)$. Again, it is easy to check now the axioms for a category fibered in groupoids over $\mathcal{C}$.

This gives the equivalence between prestacks and categories fibered in groupoids over a category $\mathcal{C}$, in fact we would again get an equivalence of the respective 2-categories of prestacks and categories fibered in groupoids. We can give now an equivalent definition of a stack as a category fibered in groupoids satisfying glueing conditions.

Definition 2.34. *Let $\mathcal{C}_\tau$ be a site. A* stack *$\mathscr{X}$ is a category fibered in groupoids over $\mathcal{C}$ such that*

1. *For all objects U in $\mathcal{C}$ and all pairs of objects x, y of $\mathscr{X}$ lying*

over U, the presheaf

$$\underline{\mathrm{Isom}}_U(x,y) : (\mathcal{C}/U)^{op} \to (Sets)$$
$$(U' \xrightarrow{f} U) \mapsto \mathrm{Hom}_{\mathscr{X}(U')}(f^*x, f^*y).$$

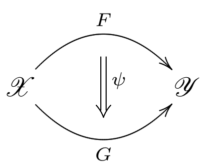
is a sheaf on the site $(\mathcal{C}/U)_\tau$.

2. *Every descent datum is effective.*

Using the language of categories fibered in groupoids, the notions of a 1- and 2-morphism of stacks can now easily be rephrased.

Definition 2.35. *Let $\mathcal{C}_\tau$ be a site. A* 1*-morphism $F : \mathscr{X} \to \mathscr{Y}$ of stacks is a functor between the categories $\mathscr{X}$ and $\mathscr{Y}$ fibered in groupoids over $\mathcal{C}$, which commutes with the projection functors, i.e. $p_{\mathscr{Y}} \circ F = p_{\mathscr{X}}$. Two stacks $\mathscr{X}$ and $\mathscr{Y}$ are* isomorphic *if there is a* 1*-morphism $F : \mathscr{X} \to \mathscr{Y}$ of stacks such that F is an equivalence of categories.*

Definition 2.36. *Let $\mathcal{C}_\tau$ be a site. A* 2-morphism *between two* 1*-morphisms of stacks $F, G : \mathscr{X} \to \mathscr{Y}$ is given by a diagram*



where ψ is a natural transformation over the identity functor $id_{\mathcal{C}}$.

It is also useful to rephrase the notion of a stack associated to an object of a category in the language of categories fibered in groupoids.

Example 2.37 (Stack associated to an object)**.** Let $\mathcal{C}_\tau$ be a site. Let X be an object of $\mathcal{C}$. We consider the slice category $\mathcal{C}/X$ of all objects over X. We define the following functor:

$$p_X : \mathcal{C}/X \to \mathcal{C}$$
$$(U \xrightarrow{f} X) \mapsto U.$$

Then the category $\mathcal{C}/X$ is a category fibered in groupoids over $\mathcal{C}$ with projection functor p_X and it is easy to see that this construction will define the stack $\underline{X}$ associated to the object X.

This will give a way of embedding the category $\mathcal{C}$ into the 2-category $\mathfrak{Stacks}(\mathcal{C})$ of stacks.

The fiber product in $\mathfrak{Stacks}(\mathcal{C})$ also has a nice interpretation in the language of categories fibered in groupoids.

Example 2.38. (Fiber product of stacks) Let $\mathscr{X}$, $\mathscr{X}'$ and $\mathscr{S}$ be stacks over the category $\mathcal{C}$ and $F : \mathscr{X} \to \mathscr{S}$, $F' : \mathscr{X}' \to \mathscr{S}$ be morphism of stacks. We define a new category fibered in groupoids $\mathscr{X} \times_{\mathscr{S}} \mathscr{X}'$ over $\mathcal{C}$ and projection functors $p_{\mathscr{X}}$ and $p_{\mathscr{Y}}$ with

- objects
  (x, x', α) with $x \in \mathscr{X}$, $x' \in \mathscr{X}'$ lying over the same object X and $\alpha : F(x) \to F'(x')$ is a vertical isomorphism in $\mathscr{S}$ with respect to the projection to $\mathcal{C}$, or in other words $p_{\mathscr{S}}(\alpha) = id_X$.
- morphisms
  $(\phi, \phi') : (x, x', \alpha) \to (y, y', \beta)$ is given by morphisms $\phi : x \to y$ and $\phi' : x' \to y'$ lying over the same morphism $f : X \to Y$ in $\mathcal{C}$ such that $\beta \circ F(\phi) = F'(\phi') \circ \alpha$.

We could have formulated again all this equally well more generally for fibred categories, but we like to restrict ourselves to stacks now. For a general treatment of fibered categories and descent see the article by Vistoli [Vis89] or the original sources by Grothendieck [Gro95a], [Gro62].

Finally, let us discuss briefly a third way of introducing stacks using groupoids. Let us recall the definition of a groupoid internally in a category $\mathcal{C}$.

Definition 2.39. *Let $\mathcal{C}$ be a category with fiber products. A* groupoid *$[X_1 \rightrightarrows X_0]$ in $\mathcal{C}$ consists of sets X_0 (set of objects), and X_1 (set of morphisms) together with five maps of sets s, t, e, m, i given as:*

$$X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0 \xrightarrow{e} X_1$$

$$X_1 \times_{s, X_0, t} X_1 \xrightarrow{m} X_1 \qquad X_1 \xrightarrow{i} X_1.$$

Furthermore, the maps of sets satisfy the following properties:

(1.) (Compatibility) We have:

$$s \circ e = t \circ e = id_{X_1},\ s \circ i = t,\ t \circ i = s,\ s \circ m = s \circ pr_2,\ t \circ m = t \circ pr_1.$$

(2.) (Associativity) $m \circ (m \times id_{X_1}) = m \circ (id_{X_1} \circ m)$.

(3.) (Identity) The compositions

$$X_1 = X_1 \times_{s,X_0} X_0 = X_0 \times_{X_0,t} X_1 \stackrel{id_{X_1} \times e}{\longrightarrow} X_1 \times_{s,X_0,t} X_1 \stackrel{m}{\longrightarrow} X_1$$

$$X_1 = X_1 \times_{s,X_0} X_0 = X_0 \times_{X_0,t} X_1 \stackrel{e \times id_{X_1}}{\longrightarrow} X_1 \times_{s,X_0,t} X_1 \stackrel{m}{\longrightarrow} X_1$$

are both equal to the identity map id_{X_1}.

(4.) (Inverse) We have:

$$m \circ (i \times id_{X_1}) = e \circ s,\ m \circ (id_{X_1} \times i) = e \circ t.$$

Following [LMB00], 2.4.3 and [DM69] we can now define a groupoid space.

Definition 2.40. *Let* C_τ *be a site. A* groupoid space *is a groupoid* $[X_1 \rightrightarrows X_0]$ *in the category* $Shv(\mathcal{C})$.

Given a groupoid space $[X_1 \rightrightarrows X_0]$ we can define a category $\widetilde{[X_1 \rightrightarrows X_0]}$ fibered in groupoids over the category $\mathcal{C}$ as follows: The objects of $\widetilde{[X_1 \rightrightarrows X_0]}$ over an object U of $\mathcal{C}$ are given by the elements of the set $X_0(U)$ and the morphisms over id_U are given by the elements of $X_1(U)$. We have an obvious projection functor $p : \widetilde{[X_1 \rightrightarrows X_0]} \to \mathcal{C}$. Again, there is an equivalence between the 2-categories of groupoid spaces $\mathfrak{Grpds}(\mathcal{C})$ and categories fibred in groupoids.

Given a morphism $f : U' \to U$ in $\mathcal{C}$ we can also define a functor

$$f^* : \widetilde{[X_1 \rightrightarrows X_0]}(U) \to \widetilde{[X_1 \rightrightarrows X_0]}(U')$$

induced via restriction [LMB00], 2.4.3.

In general, the category $\widetilde{[X_1 \rightrightarrows X_0]}$ will only be a prestack, but it is possible to "stackify" this construction, i.e. we can associate a stack $[X_0/X_1]$ to $[\widetilde{X_1 \rightrightarrows X_0}]$ (see [LMB00], 3.4.3).

We are especially interested here in the 2-category $\mathfrak{Grpds}/S = \mathfrak{Grpds}(Sch/S)$ of S-groupoid spaces, i.e. groupoids in the category (Spaces/S) of S-spaces on a site $(Sch/S)_\tau$. Let us look at the quotient of a group action on a scheme again [Góm01], Ex. 2.28.

Example 2.41. (Quotient groupoid) Let (Sch/S) be the category of S-schemes and X be a noetherian S-scheme. Let G be an affine smooth group S-scheme with a right action $\rho : X \times G \to X$. As usual we think of X, S and G as S-spaces via their functor of points. Using the group action we get an S-groupoid space $[X \times G \rightrightarrows X]$, where $t = \rho$, $s = pr_1$ and m is the multiplication in the group G. The map e is given by the identity and i is given by the inverse. The objects of the fiber $[\widetilde{X \times G \rightrightarrows X}](U)$ over an S-scheme U are morphisms $f : U \to X$ or equivalently we can say that they are trivial principal G-bundles $U \times G \downarrow U$ together with the morphism $\rho \circ f : U \times G \to X$. As above we get a prestack and the associated stack $[X/X \times G]$ is isomorphic to the quotient stack $[X/G]$.

Example 2.42. (Equivalence relation) Given an equivalence relation $\delta : R \longrightarrow X \times_S X$ in the category $(Spaces/S)$ of S-spaces we can always construct an S-groupoid space $[R \rightrightarrows X]$, where $s = pr_1 \circ \delta$ and $t = pr_2 \circ \delta$ are the projections. The associated stack $[X/R]$ is sometimes also called the *stacky quotient* of the equivalence relation.

All three ways of defining stacks are useful depending for which purpose the language of stacks is used for. The groupoid approach mentioned here at the end is especially of use if one likes to talk about equivalence relations and quotients. And it is this approach towards stacks which is often used in other areas like differential or symplectic geometry.

2.2 Algebraic stacks

We like to do algebraic geometry on stacks, so it is necessary to be able to extend the geometry of schemes to a geometry of stacks. In order to do this we will restrict ourselves to a special class of stacks, called algebraic stacks. Algebraic stacks behave a lot like schemes and many geometric properties of schemes can be extended to algebraic stacks.

In this section we will always consider the category (Sch/S) of S-schemes over a base scheme S together with a Grothendieck topology τ, which normally will be the étale topology and we will consider stacks over the big étale site $(Sch/S)_{et}$.

Definition 2.43. *A stack $\mathscr{X}$ is* representable *by an algebraic space (resp. by a scheme) if there exists an algebraic space $\mathcal{X}$ (resp. scheme X) such that the stack associated to $\mathcal{X}$ (resp. X) is isomorphic to the stack $\mathscr{X}$. A morphism of stacks $F : \mathscr{X} \to \mathscr{Y}$ is* representable *by algebraic spaces (resp. schemes) if for each object Y of (Sch/S) and each morphism $Y \to \mathscr{Y}$ the fiber product stack $Y \times_{\mathscr{Y}} \mathscr{X}$ is representable by an algebraic space (resp. scheme), i.e. in the cartesian diagram*

$$\begin{array}{ccc} Y \times_{\mathscr{Y}} \mathscr{X} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathscr{Y} \end{array}$$

Proposition 2.44. *Let $\mathscr{X}$ be a stack and $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ be the diagonal morphism. Then the following are equivalent:*

1. *The diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by algebraic spaces.*

2. *For all schemes X, Y and morphisms $x : X \to \mathscr{X}$ and $y : Y \to \mathscr{X}$ the sheaf $X \times_{\mathscr{X}} Y = \underline{\mathrm{Isom}}(X \times Y, pr_1^* x, pr_2^* y)$ is representable by algebraic spaces i.e. isomorphic to an algebraic space.*

3. *For all algebraic spaces $\mathcal{X}$, every morphism $\mathcal{X} \to \mathscr{X}$ is representable by algebraic spaces.*

4. *For all schemes X, every morphism $X \to \mathscr{X}$ is representable by algebraic spaces.*

Proof. We prove the equivalence of 1. and 2. using appropriate cartesian diagrams. We first show that 2. follows from 1. We have the following cartesian diagram

$$\begin{array}{ccc} X \times_{\mathscr{X}} Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \mathscr{X} & \xrightarrow[\Delta]{} & \mathscr{X} \times \mathscr{X} \end{array}$$

We have the following isomorphisms

$$X \times_{\mathscr{X}} Y \cong \mathrm{Hom}_{\mathscr{X}(?)}(pr_1^* x, pr_2^* y) \cong (X \times Y) \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X}.$$

If the diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by algebraic spaces, then $X \times_{\mathscr{X}} Y = \underline{\mathrm{Isom}}(X \times Y, pr_1^* x, pr_2^* y)$ is an algebraic space.

Now to show that 1. follows from 2. let $f : X \to \mathscr{X}$ be a morphism with X a scheme. We have the following diagram

$$\begin{array}{ccc} (X \times_{\mathscr{X}} X) \times_{X \times X} X & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X \times_{\mathscr{X}} X & \longrightarrow & X \times X \\ \downarrow & & \downarrow f \times f \\ \mathscr{X} & \xrightarrow[\Delta]{} & \mathscr{X} \times \mathscr{X} \end{array}$$

The small squares are cartesian and therefore also the big one and we get

$$\mathscr{X} \times_{\mathscr{X} \times \mathscr{X}} X = (X \times_{\mathscr{X}} X) \times_{X \times X} X.$$

By hypothesis we know that $X \times_{\mathscr{X}} X$ is representable by algebraic spaces and therefore also $\mathscr{X} \times_{\mathscr{X} \times \mathscr{X}} X$ as a fiber product of algebraic spaces (in the upper square). So we see that the diagonal Δ is representable by algebraic spaces.

The equivalence of 2. and 3. and of 3. and 4. follows from the definitions (see [LMB00], 3.11, 3.13). □

We have the following basic properties for representable morphisms of stacks.

Proposition 2.45. *1. The property of being representable for a morphism of stacks is stable under arbitrary base change.*

2. A stack $\mathscr{X}$ of $\mathfrak{Stacks}/S$ is representable if and only if the structure morphism $\mathscr{X} \to S$ is representable.

3. Compositions and products of representable morphisms are representable

Proof. Let $F : \mathscr{X} \to \mathscr{Y}$ be a representable morphism of stacks and $F' : \mathscr{X}' \to \mathscr{Y}$ be an arbitrary morphism of stacks defining the base change. To prove 1. look at the following cartesian diagram

$$\begin{array}{ccccc} X \times_{\mathscr{X}'} (\mathscr{X}' \times_{\mathscr{Y}} \mathscr{X}) & \longrightarrow & \mathscr{X}' \times_{\mathscr{Y}} \mathscr{X} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow & & \downarrow F \\ X & \longrightarrow & \mathscr{X}' & \xrightarrow{F'} & \mathscr{Y} \end{array}$$

We have that $X \times_{\mathscr{X}'} (\mathscr{X}' \times_{\mathscr{Y}} \mathscr{X}) \cong X \times_{\mathscr{Y}} \mathscr{X}$, which is representable by hypothesis and therefore also the morphism F'.

The statement 3. is proved via a similar argument involving cartesian diagrams.

To prove 2. observe that the structure morphism $\mathscr{X} \to S$ maps the groupoid $\mathscr{X}(X)$ for any object X of (Sch/S) to the structure morphism $X \to S$ of the scheme X. □

Geometric properties of morphisms of schemes or algebraic spaces, which are local on the target and stable under arbitrary base change can be extended to respresentable morphisms of stacks, which will then allow us to do geometry with them.

Definition 2.46. *Let (Sch/S) be the category of S-schemes together with a Grothendieck topology τ. A property $\mathbb{P}$ of morphisms of schemes is called* stable under base change and local in the topology τ on the target *if the following hold:*

(1.) If the morphism $f : X \to Y$ has property $\mathbb{P}$, then for any morphism $U \to Y$ the morphism $X \times_Y U \to U$ induced by base change has property $\mathbb{P}$.

(2.) If $f : X \to Y$ is a morphism and $\{U_i \to Y\}_{i \in I}$ is a covering of Y with respect to the topology τ, then f has property $\mathbb{P}$ if and only if for every i the morphism $X \times_Y U_i \to U_i$ induced by base change has property $\mathbb{P}$.

Analogously we can define more generally when a property $\mathbb{P}$ of morphisms of algebraic spaces is stable under base change and local in the topology τ on the target [Knu71].

For the étale topology, examples of properties $\mathbb{P}$ *stable under base change and local in the étale topology on the target* include:
étale, surjective, flat, smooth, locally of finite presentation, locally of finite type, quasi-compact, open embedding, closed embedding, affine, quasi-affine, proper, unramified, separated... (see [LMB00], 3.10, [GD67a] for a more complete list).

Now we can naturally extend the notion of morphisms of schemes or algebraic spaces of having a property $\mathbb{P}$ stable under base change and local in the topology on the target to representable morphisms of stacks.

Definition 2.47. *Let (Sch/S) be the category of S-schemes together with the étale topology and let $\mathbb{P}$ be a property of morphisms of schemes which is stable under base change and local in the étale topology on the target. A representable morphism $F : \mathscr{X} \to \mathscr{Y}$ of stacks* has property $\mathbb{P}$, *if for every morphism $Y \to \mathscr{Y}$, where Y is an object of (Sch/S) the induced morphism $\bar{F} : Y \times_{\mathscr{Y}} \mathscr{X} \to Y$ of schemes has property $\mathbb{P}$, i.e. in the cartesian diagram*

$$\begin{array}{ccc} Y \times_{\mathscr{Y}} \mathscr{X} & \longrightarrow & \mathscr{X} \\ {\scriptstyle \mathbb{P}}\Big\downarrow{\scriptstyle \bar{F}} & & {\scriptstyle \mathbb{P}}\Big\downarrow{\scriptstyle F} \\ Y & \longrightarrow & \mathscr{Y} \end{array}$$

As for schemes we have the following basic properties for representable morphism of stacks.

Proposition 2.48. *1. Having property $\mathbb{P}$ for representable morphism of stacks is stable under arbitrary base change.*

2. *$\mathbb{P}$ as a property of representable morphisms of stacks is local on the target.*

3. *If two representable and composable morphisms of stacks $F : \mathscr{X} \to \mathscr{Y}$ and $G : \mathscr{Y} \to \mathscr{Z}$ have property $\mathbb{P}$ then also $G \circ F : \mathscr{X} \to \mathscr{Z}$ has property $\mathbb{P}$ if the same is true for composable morphisms of (Sch/S) having property $\mathbb{P}$.*

Proof. This follows immediately from the definition and the analog statements for morphisms in the category (Sch/S) of S-schemes [GD67b]. □

Let $(Sch/S)_{et}$ be the category of S-schemes with the étale topology. We can now extend the following geometric properties of morphisms in (Sch/S) to representable morphisms of stacks (cf. [GD67b] §2.7 and §17.7):
étale, surjective, flat, smooth, locally of finite presentation, locally of finite type, quasi-compact, open embedding, closed embedding, affine, quasi-affine, proper, unramified, separated...

For example, we can define what it means to be an étale covering of a stack $\mathscr{X}$ by a scheme X. It is simply a representable surjective étale morphism $X \to \mathscr{X}$. Stacks having such a covering by a scheme are the ones we can do algebraic geometry with by using the covering. These stacks are the algebraic stacks we like to discuss now.

Algebraic stacks were first introduced as what is now called Deligne-Mumford stacks by Deligne and Mumford [DM69] to give a new proof of the irreducibility of the coarse moduli space of algebraic curves. Later Artin [Art74] defined a more general notion of algebraic stacks, now called Artin stacks to develop a global framework for deformation theory.

Definition 2.49 (Artin stack)**.** *Let (Sch/S) be the category of S-schemes together with the étale topology. A stack $\mathscr{X}$ over the site $(Sch/S)_{et}$ is an* Artin algebraic stack *if*

1. *The diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by algebraic spaces and quasi-compact.*

2. *There exist a scheme X, called an* atlas *and a surjective smooth morphism $X \to \mathscr{X}$.*

We will often refer to Artin algebraic stacks simply as algebraic stacks. We could have defined Artin algebraic stacks equally well over the fppf site $(Sch/S)_{fppf}$ or the smooth site $(Sch/S)_{sm}$ instead of using the étale topology on (Sch/S). But a theorem of Artin about comparison of topologies [Art74], 6.1 implies that we will get the same notion of an Artin algebraic stack as defined above for the étale topology, i.e. the 2-categories of algebraic stacks will be equivalent. Sometimes though it is technically more convenient to work with the smooth or fppf topology, for example when dealing with cohomology.

Definition 2.50 (Deligne-Mumford stack)**.** *Let (Sch/S) be the category of S-schemes together with the étale topology. A stack $\mathscr{X}$ over the site $(Sch/S)_{et}$ is a* Deligne-Mumford algebraic stack *if*

1. *The diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by schemes, quasi-compact and separated.*
2. *There exist a scheme X, called an* atlas *and a surjective étale morphism $X \to \mathscr{X}$.*

In the first condition we could have asked for representability of the diagonal by algebraic spaces, like we did in the case of Artin algebraic stacks. But from a theorem about algebraic spaces [Knu71], Thm. 6.16 it follows that any algebraic space, which is quasi-finite and separated over a scheme is itself a scheme. So especially for a Deligne-Mumford algebraic stack we would get that the diagonal is representable by schemes. If we would have asked representability of the diagonal morphism by schemes in the definition of Artin algebraic stacks above, we would just simply get a smaller class of algebraic stacks.

The definition of a Deligne-Mumford algebraic stack is similar to the definition of an algebraic space, but in S-stacks instead of S-spaces.

Obviously a Deligne-Mumford algebraic stack is also an Artin algebraic stack, as every étale atlas is also a smooth atlas, but the converse is in general not true, as the example of quotient stacks will show us.

Proposition 2.51. *Let $(Sch/S)_{et}$ be the category of S-schemes together with the étale topology. Algebraic stacks form a strict full*

2-subcategory $\mathfrak{AlgStacks}/S$ *of the* 2*-category* $\mathfrak{Stacks}/S$ *of* S*-stacks.* $\mathfrak{AlgStacks}/S$ *is closed under forming* 2*-fiber products.*

The category (AlgSpaces/S) *of algebraic spaces is a strict full* 2*-subcategory of the category* $\mathfrak{AlgStacks}/S$ *of algebraic stacks.*

Proof. The first statement follows from the definitions. Let us show that $\mathfrak{AlgStacks}/S$ is closed under 2-fiber products. Assume we have the following 2-cartesian diagram of morphisms of stacks

$$\begin{array}{ccc} \mathscr{X} \times_{\mathscr{S}} \mathscr{X}' & \longrightarrow & \mathscr{X}' \\ \downarrow & & \downarrow{\scriptstyle F'} \\ \mathscr{X} & \xrightarrow{F} & \mathscr{S} \end{array}$$

Let $X \to \mathscr{X}$ and $X' \to \mathscr{X}'$ be atlases, then

$$X \times_{\mathscr{S}} X' \to \mathscr{X} \times_{\mathscr{S}} \mathscr{X}'$$

is again an atlas. This is either an algebraic space or a scheme, but an algebraic space has an étale covering by a scheme itself, therefore we get an atlas for the fiber product. The diagonal is representable, because we have

$$Y \times_{\mathscr{X} \times_{\mathscr{S}} \mathscr{X}'} Y' \cong (Y \times_{\mathscr{X}} Y') \times_{Y \times_{\mathscr{S}} Y'} (Y \times_{\mathscr{X}'} Y')$$

for any morphisms $Y \to \mathscr{X}$ and $Y' \to \mathscr{X}'$, where Y and Y' are schemes and the right hand side is again a scheme.

The last statement follows immediately from the definition of an algebraic space, because an algebraic space is a Deligne-Mumford algebraic stack which is even a sheaf of sets on the site $(Sch/S)_{et}$. □

Given an Artin algebraic stack $\mathscr{X}$ with an atlas $x : X \to \mathscr{X}$ we obtain an S-groupoid space $[X_1 \rightrightarrows X_0]$ by setting $X_0 := X$ and $X_1 := X \times_{\mathscr{X}} X$ given by the fiber product of x with itself. The maps $s, t : X_1 \to X_0$ are for example given by the two projections. It can be shown that there is a natural isomorphism

$$\mathscr{X} \cong [X_0/X_1]$$

between $\mathscr{X}$ and the associated stack $[X_0/X_1]$ of the S-groupoid. $[X_0/X_1]$ in this case is again an algebraic stack. And every Artin algebraic stack (resp. Deligne-Mumford algebraic stack) can be obtained as the associated stack $[X_0/X_1]$ of an S-groupoid space $[X_1 \rightrightarrows X_0]$ where X_0 and X_1 are algebraic spaces, the maps s, t are smooth (resp. étale) morphisms and $(s,t) : X_1 \to X_0 \times X_0$ is a quasi-compact (resp. quasi-compact and separated) morphism (see [LMB00], 4.3.1).

Proposition 2.52. *If $\mathscr{X}$ is an Artin (resp. Deligne-Mumford) algebraic stack, then the diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is of finite type (resp. unramified).*

Proof. For Deligne-Mumford algebraic stacks this is proved in [Vis89], prop. 7.15 and in general for Artin algebraic stacks in [LMB00], lem. 4.2. □

Corollary 2.53. *Let $\mathscr{X}$ be a Deligne-Mumford algebraic stack and X be a quasi-compact scheme. If x is an object of the groupoid $\mathscr{X}(X)$, then x has only finitely many automorphisms.*

Proof. (see [Vis89]). Let $\underline{x} : X \to \mathscr{X}$ be the morphism corresponding to x and $\Delta \circ \underline{x} : X \to \mathscr{X} \times \mathscr{X}$ be the composition with the diagonal morphism. As $\mathscr{X}$ is a Deligne-Mumford stack, the fiber product

$$X \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X} = \underline{\mathrm{Isom}}_X(x,x)$$

is a scheme unramified over X. As X is quasi-compact the morphism of schemes $\underline{\mathrm{Isom}}_X(x,x) \to X$ has only finitely many sections. Therefore x has only finitely many automorphisms. □

We have the following characterization of Deligne-Mumford stacks among Artin algebraic stacks.

Proposition 2.54. *Let $\mathscr{X}$ be an Artin algebraic stack. Then the following properties are equivalent:*

(1) $\mathscr{X}$ is a Deligne-Mumford algebraic stack.

(2) The diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is unramified.

(3) No object has non-trivial infinitesimal automorphisms.

Proof. For a proof see [Art74], Sec. 5 or [LMB00], Chap. 8. □

In fact, for Deligne-Mumford algebraic stacks the stabilizer groups of points are finite groups, where for Artin algebraic stacks the stabilizer groups of points can be more general algebraic groups, as for example it is the case of moduli stacks of vector bundles in contrast to moduli stacks of algebraic curves.

We have defined geometric properties for representable stacks and representable morphisms of stacks. We can also define geometric properties for algebraic stacks and their morphisms by using the atlas. Let us mention just some examples. For a systematic discussion we refer to [Góm01], 2.5 and [LMB00].

Definition 2.55. *An Artin algebraic stack $\mathscr{X}$ is called* smooth *(resp.* reduced*, resp.* locally of finite presentation*, resp.* locally noetherian*, resp.* normal*, resp.* regular*) if there exists an atlas $x : X \to \mathscr{X}$ with the scheme X being smooth (resp. reduced, resp. locally of finite presentation, resp. locally noetherian, resp. normal, resp. regular).*

Definition 2.56. *Let $\mathbb{P}$ be a property of morphisms of schemes $f : X \to Y$ such that f has property $\mathbb{P}$ if and only if for some smooth surjective morphism $Y' \to Y$ the induced morphism $f' : X \times_Y Y' \to Y$ has property $\mathbb{P}$. A* representable morphism $F : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks has property $\mathbb{P}$ *if for some atlas $y : Y \to \mathscr{Y}$ the induced morphism $\bar{F} : Y \times_{\mathscr{Y}} \mathscr{X} \to Y$ of schemes has property $\mathbb{P}$, i.e. in the cartesian diagram*

$$\begin{array}{ccc} Y \times_{\mathscr{Y}} \mathscr{X} & \longrightarrow & \mathscr{X} \\ {\scriptstyle \mathbb{P}}\Big\downarrow{\scriptstyle \bar{F}} & & {\scriptstyle \mathbb{P}}\Big\downarrow{\scriptstyle F} \\ Y & \longrightarrow & \mathscr{Y} \end{array}$$

In this way we can define geometric properties for algebraic stacks and for *representable* morphisms of algebraic stacks by testing them on the atlas. This can be used for example to define the geometric properties of morphisms like *closed* or *open embedding*, *affine*, *finite*, *proper* etc. In particular we get the notion of an open or closed substack of an algebraic stack. For schemes this gives nothing new as all these properties can be checked on a smooth covering.

We can also define geometric properties for *arbitrary* morphisms of algebraic stacks, if they can be checked locally in the source and target of the morphism.

Definition 2.57. *Let $\mathbb{P}$ be a property of morphisms of schemes $f : X \to Y$ such that f has property $\mathbb{P}$ if and only if there is a commutative diagram of the form*

$$\begin{CD} X' @>{f'}>> Y' \\ @V{p}VV @VV{q}V \\ X @>{f}>> Y \end{CD}$$

with p and q smooth surjective morphisms, such that f' has property $\mathbb{P}$. Then a morphism $F : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks has property $\mathbb{P}$ *if there exists atlases, i.e. smooth surjective morphisms from schemes $x : X \to \mathscr{X}$ and $y : Y \to \mathcal{Y}$ and a commutative diagram of the form*

$$\begin{CD} X @>{f}>> Y \\ @V{x}VV @VV{y}V \\ \mathscr{X} @>{F}>> \mathscr{Y} \end{CD}$$

such that the induced morphism f of schemes has property $\mathbb{P}$.

Examples include the geometric properties *smooth*, *flat*, *locally of finite presentation* etc. Let us just look at the particular example of smoothness here. We can check smoothness for an algebraic stack locally on an atlas using the lifting criterion for smooth morphisms of schemes.

Proposition 2.58. *(Lifting criterion for smooth morphisms) A morphism of schemes $f : X \to Y$ is smooth if and only if f is locally of finite presentation and for all local Artin algebras A with ideal $I \subset A$ with $I^2 = (0)$ there is a lifting*

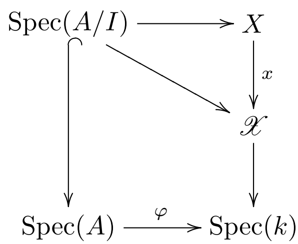
$$\begin{array}{ccc} \operatorname{Spec}(A/I) & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec}(A) & \longrightarrow & Y \end{array}$$

Proof. See [GD67b], IV, §17 for smoothness. □

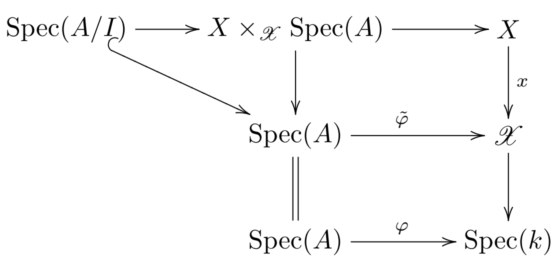
We can now test smoothness on an algebraic stack using the lifting property for schemes (see for example [Hei09]).

Proposition 2.59. *Let* $\mathscr{X}$ *be an algebraic stack, which is locally of finite presentation over* $\mathrm{Spec}(k)$ *such that the structure morphism* $\mathscr{X} \to \mathrm{Spec}(k)$ *satisfies the lifting criterion for smoothness. Then* $\mathscr{X}$ *is smooth.*

Proof. Let $x : X \to \mathscr{X}$ be an atlas, i.e. a smooth surjective morphism. We will show that the scheme X is smooth, i.e. that the lifting criterion holds for the structure morphism $X \to \mathrm{Spec}(k)$. Assume we have the following commutative diagram:



As the structure morphism satisfies the lifting criterion by hypothesis, we can lift the morphism φ to a morphism $\tilde{\varphi} : \mathrm{Spec}(A) \to \mathscr{X}$. As the atlas x is smooth and representable, we see that the morphism $X \times_{\mathscr{X}} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is also smooth and we can complete the following diagram



which finishes the proof. □

It is also possible to extend notions like *separatedness* or *properness* to algebraic stacks and general morphism of algebraic stacks and to prove valuative criteria for separatedness and properness like in the theory of schemes. We refer to [LMB00] and [DM69] for a systematic treatment. Let us just record here for completeness the topological notions of a substack, connectedness and irreducebility of an algebraic stack.

Definition 2.60. *Let $\mathscr{X}$ be an algebraic stack. An algebraic stack $\mathscr{Y}$ is called an* open *(resp.* closed*)* substack *of $\mathscr{X}$ if there exists a representable morphism $\mathscr{Y} \hookrightarrow \mathscr{X}$ of stacks which is an open (resp. closed) embedding.*

Definition 2.61. *An algebraic stack $\mathscr{X}$ is called* connected *if it is not isomorphic to the disjoint union of two non-empty algebraic substacks. An algebraic stack $\mathscr{X}$ is called* irreducible *if it is not the union of two non-empty closed substacks.*

We have also a well-defined notion of connected components for algebraic stacks.

Proposition 2.62. *A locally noetherian algebraic stack is in one and only one way the disjoint union of connected algebraic stacks, called the* connected components.

Proof. This is [LMB00], Prop. 4.9. See also [DM69], 4.13. □

Let us now discuss some of the main examples of algebraic stacks related to moduli and quotient problems.

Theorem 2.63. *Let (Sch/S) be the category of S-schemes and X be a noetherian S-scheme. Let G be an affine smooth group S-scheme with a right action $\rho : X \times G \to X$. The quotient stack $[X/G]$ is an Artin algebraic stack. In particular, the classifying stack $\mathscr{B}G$ is an Artin algebraic stack.*

Proof. Let $\mathcal{E}$ be an object of $[X/G](U)$ and $\mathcal{E}'$ of $[X/G](U')$. Let $u : U \to [X/G]$ and $u' : U' \to X/G$ be the associated morphisms via representability. Using trivializations of the principal G-bundles $\mathcal{E}$ and $\mathcal{E}'$ on some coverings, it follows that the sheaf $\underline{\mathrm{Isom}}(U \times U', pr_1^* u, pr_2^* u')$

is representable by the scheme $(U \times U' \times G) \times_{X \times X} X$ which is quasi-affine over $U \times U'$ and therefore the diagonal morphism Δ is representable (Prop. 2.44) and quasi-affine, so in particular quasi-compact and also separated.

Now we have to construct a smooth atlas for $[X/G]$. The trivial principal G-bundle $X \times G \to X$ over X with the right action $\rho : X \times G \to X$ gives an object in the groupoid $[X/G](X)$ and by representability it defines a morphism of stacks $x : X \to [X/G]$. The morphism x is representable, because for any scheme U and any morphism $u : U \to [X/G]$ let $\pi : \mathcal{E} \to U$ be the corresponding principal G-bundle with G-equivariant morphism $\mu : \mathcal{E} \to X$, then $U \times_{[X/G]} X \cong \mathcal{E}$ and is therefore a scheme and x is representable. We have a cartesian diagram of the form

$$\begin{array}{ccc} \mathcal{E} \cong U \times_{[X/G]} X & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle x} \\ U & \xrightarrow{u} & [X/G] \end{array}$$

From this we see that the morphism x is surjective and smooth, because π is surjective and smooth for every morphism u. Therefore x gives an atlas for $[X/G]$.

In the special case that $X = S$ with trivial G-action, it follows therefore that the classifying stack $\mathscr{B}G = [S/G]$ is an Artin algebraic stack. □

For more general actions of S-group schemes, even though x might not be an atlas anymore, it is still possible to show that $[X/G]$ is an Artin algebraic stack [LMB00], ex. 4.6.1 and prop. 10.13.1.

Theorem 2.64. *Let (Sch/S) be the category of S-schemes and X be a noetherian S-scheme. Let G be an smooth affine group S-scheme with a right action $\rho : X \times G \to X$. If either G is étale over S or the stabilzers of the geometric points of X are finite and reduced, then $[X/G]$ is a Deligne-Mumford algebraic stack.*

In particular, if either G is étale over S or the stabilzers of the geometric points of S are finite and reduced, the classifying stack $\mathscr{B}G$ is a Deligne-Mumford algebraic stack.

Proof. This is a consequence of the theorem above and the discussion in [DM69], ex. 4.8 and [Vis89], ex. 7.17. □

Now we will proof that the moduli stacks of algebraic curves are in fact Deligne-Mumford algebraic stacks [DM69].

Theorem 2.65 (Deligne-Mumford)**.** *Let $g \geq 2$. The moduli stacks $\mathscr{M}_g$ of algebraic curves of genus g and $\widetilde{\mathscr{M}_g}$ of stable algebraic curves of genus g are Deligne-Mumford algebraic stacks.*

Proof. This follows from the fact that we can realize the stacks $\mathscr{M}_g$ and $\widetilde{\mathscr{M}_g}$ as quotient stacks of a scheme by a smooth algebraic group. Let us briefly present the main line of argument. For the details and main properties of these stacks we refer to the original article by Deligne and Mumford [DM69] and the article of Edidin [Edi00]. Let $\pi : C \to U$ be a stable curve of genus g. The morphism π is a local complete intersection morphism, because it is flat and its geometric fibers area local complete intersections. Using duality theory it follows that there exits a canonical invertible dualizing sheaf $\omega_{C/U}$ on C. In the case that π is smooth it follows that $\omega_{C/U}$ is the relative cotangent bundle. Deligne and Mumford proved that $\omega_{C/U}^{\otimes n}$ is relative very ample for $n \geq 3$ and the direct image sheaf $\pi_*(\omega_{C/U}^{\otimes n})$ is locally free of rank $(2n-1)(g-1)$ [DM69], p. 78.

Therefore, every stable algebraic curve can be realized as a curve in the projective space $\mathbb{P}^N$ with $N = (2n-1)(g-1)-1$ with prescribed Hilbert polynomial $P_{g,n}(t) = (2nt-1)(g-1)$. Furthermore, there exists a subscheme $\widetilde{H}_g$ of the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^N}^{P_{g,n}}$ parametrizing n-canonically embedded stable algebraic curves. Similarly, there is a subscheme $H_{g,n}$ of $\widetilde{H}_{g,n}$ parametrizing only n-canonically embedded smooth algebraic curves. Any morphism $U \to \widetilde{H}_{g,n}$ corresponds to a stable algebraic curve $\pi : C \to U$ of genus g together with an isomorphism of $\mathbb{P}(\pi_*(\omega_{C/U}^{\otimes n}))$ with $\mathbb{P}^N \times U$. The projective linear group PGL_{N+1} acts naturally on the schemes $H_{g,n}$ and $\widetilde{H}_{g,n}$. Given a stable algebraic curve $\pi : C \to U$, let $\mathcal{E} \to U$ be the associated principal PGL_{N+1}-bundle over U of the projective bundle $\mathbb{P}(\pi_*(\omega_{C/U}^{\otimes n}))$. We have a cartesian diagram of the form

$$\begin{array}{ccc} C \times_U \mathcal{E} & \xrightarrow{\pi'} & \mathcal{E} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\pi} & U \end{array}$$

The pullback of the projective bundle to $\mathcal{E}$ is trivial and isomorphic to $\mathbb{P}(\pi'_*(\omega^{\otimes n}_{C\times_U \mathcal{E}/\mathcal{E}}))$. This defines a PGL_{N+1}-equivariant morphism $\mathcal{E} \to \widetilde{H}_{g,n}$ and therefore a morphism of stacks

$$\widetilde{\mathscr{M}_g} \to [\widetilde{H}_{g,n}/PGL_{N+1}].$$

which takes $\mathscr{M}_g$ to the quotient substack $[H_{g,n}/PGL_{N+1}]$ and it can be shown that these are in fact isomorphisms of stacks. From this presentation as quotient stacks of a scheme by a smooth algebraic group, it follows that these quotient stacks have a smooth atlas. And because over an algebraically closed field every stable algebraic curve has an automorphism group which is finite and reduced it follows that the diagonal morphisms for both stacks is unramified. Then the alternative characterization of Deligne-Mumford algebraic stacks in [DM69], Thm. 4.21 or [Edi00], Thm. 2.1 implies the statement. $\square$

Deligne and Mumford [DM69] further showed that $\widetilde{\mathscr{M}_g}$ is smooth, proper and irreducible over $\mathrm{Spec}(\mathbb{Z})$ and the complement $\widetilde{\mathscr{M}_g} \backslash \mathscr{M}_g$ is a divisor with normal crossings in $\widetilde{\mathscr{M}_g}$.

Now let us discuss the moduli stacks of coherent sheaves and vector bundles of fixed rank. It turns out that these are in fact Artin algebraic stacks.

Theorem 2.66. *Let (Sch/S) be the category of S-schemes with S a noetherian scheme. Assume that X is a projective scheme with structure morphism $f : X \to S$ such that for any base change $S' \to S$ we have $f'_*\mathcal{O}_{X'} \cong \mathcal{O}_{S'}$. The moduli stacks $\mathscr{C}oh_X$ of coherent $\mathcal{O}_X$-modules and $\mathscr{B}un^n_X$ of vector bundles of rank n over X are Artin algebraic stacks which are locally of finite type.*

Proof. This is [LMB00], Thm. 4.6.2.1. The proof uses the machinery of Hilbert and Quot schemes of Grothendieck [Gro62]. Besides the original article of Grothendieck we refer for the construction and general theory of Hilbert and Quot schemes to the article of Nitsure in [Nit05] or the book of Huybrechts and Lehn [HL97]. $\square$

We are especially interested in these lectures in the moduli stack $\mathscr{B}un_X^{n,d}$ of vector bundles on an algebraic curve X of rank n and degree d. Let us discuss this particular case in more detail here now.

Theorem 2.67. *The moduli stack $\mathscr{B}un_X^{n,d}$ of vector bundles of rank n and degree d on a smooth projective irreducible algebraic curve X of genus $g \geq 2$ is an Artin algebraic stack which is smooth and locally of finite type.*

Proof. Let $\mathcal{E}$ resp. $\mathcal{E}'$ be a family of vector bundles of rank n and degree d over X parametrized by the S-scheme U resp. U'. Now let $u : U \to \mathscr{B}un_X^{n,d}$ and $u' : U' \to \mathscr{B}un_X^{n,d}$ be the associated morphisms via representability.

We know that the sheaf $\underline{\mathrm{Isom}}(U \times U', pr_1^* u, pr_2^* u')$ is the open subscheme of invertible morphisms of the fiber bundle $\mathrm{Hom}(pr_1^*\mathcal{E}, pr_2^*\mathcal{E}')$ over $U \times U'$ and the morphism $\mathrm{Hom}(pr_1^*\mathcal{E}, pr_2^*\mathcal{E}') \to U \times U'$ is affine. Therefore the diagonal morphism Δ is representable (by a scheme) and quasi-compact and also separated.

Let us now describe the construction of an atlas for the moduli stack $\mathscr{B}un_X^{n,d}$. Let $P_{n,d}$ be the polynomial given by

$$P_{n,d}(x) = nx + d + n(1-g).$$

For every integer m let $P(m) = P_{n,d}(m)$ and consider the Quot scheme $\mathrm{Quot}(\mathcal{O}_X^{P(m)}, P(x+m))$ parametrizing quotient sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X^{P(m)}$ with prescribed Hilbert polynomial $P_{n,d}$. In general, a Quot scheme $\mathrm{Quot}(\mathcal{F}, P)$ is a fine moduli space for the moduli functor

$$\mathcal{Q}uot : (Sch/S)^{op} \to (Sets)$$

of the moduli problem of classifying quotient sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ with prescribed Hilbert polynomial P and there exists a universal family of such quotient sheaves over the Quot-scheme $\mathrm{Quot}(\mathcal{F}, P)$ (see [HL97], 2.2).

For every integer m we define an open subscheme

$$R_m \hookrightarrow \mathrm{Quot}(\mathcal{O}_X^{P(m)}, P(x+m))$$

by requiring that

(i) the quotient sheaves $\mathcal{O}_X^{P(m)} \to \mathcal{F} \to 0$ parametrized by R_m are vector bundles, i.e. $\mathcal{F}$ is a locally free $\mathcal{O}_X$-sheaf.

(ii) for every U-point of R_m defined by the family $\mathcal{O}_{X\times U}^{P(m)} \to \mathcal{F} \to 0$ we have that $R^1(pr_2)_*\mathcal{F} = 0$ and $(pr_2)_* : \mathcal{O}_{X\times U}^{P(m)} \xrightarrow{\cong} (pr_2)_*\mathcal{F}$ is an isomorphism.

Induced by the universal family over $\mathrm{Quot}(\mathcal{O}_X^{P(m)}, P(x+m))$ we get now a universal family $\mathcal{E}^{univ}$ of vector bundles over X of rank n and degree d parametrized by the subscheme R_m. Therefore we get a morphism of stacks

$$r_m : R_m \to \mathscr{B}un_X^{n,d}.$$

From (ii) it follows that if a point of R_m is represented by a quotient sheaf of the form

$$0 \to \mathcal{G} \to \mathcal{O}_{X\times U}^{P(m)} \to \mathcal{F} \to 0$$

then $H^1(\mathcal{F}\otimes\mathcal{G}^\vee) = 0$, which implies that r_m is a smooth morphism.

For every family of vector bundles $\mathcal{E}$ of rank n and degree d on X we can find an integer m such that the scheme R_m has a geometric point whose corresponding quotient in $\mathrm{Quot}(\mathcal{O}_X^{P(m)}, P(x+m))$ is $\mathcal{E}$. Taking infinite disjoint unions of all morphisms r_m we get a surjective smooth morphism

$$r : \coprod_m R_m \to \mathscr{B}un_X^{n,d}$$

and therefore an atlas for the the stack $\mathscr{B}un_X^{n,d}$. From the nature of the schemes R_m it follows also that $\mathscr{B}un_X^{n,d}$ is locally of finite type.

It remains to show that $\mathscr{B}un_X^{n,d}$ is a smooth stack. For this we apply the lifting criterion for smoothness. We have to prove: Let A be a local Artin algebra with ideal $I \subset A$ and $I^2 = (0)$. If $\bar{\mathcal{E}}$ is a vector bundle on $\mathrm{Spec}(A/I)\times X$, then it can be lifted to a vector bundle $\mathcal{E}$ on $\mathrm{Spec}(A)\times X$.

Let $\{\bar{U}_i\}$ an open affine covering of $\mathrm{Spec}(A/I) \times X$ on which $\bar{\mathcal{E}}$ is the trivial vector bundle. Furthermore let $\{U_i\}_{i \in I}$ be the corresponding covering of $\mathrm{Spec}(A) \times X$ and $\bar{\phi}_{ij} \in \Gamma(\bar{U}_{ij}, GL_n)$ be the transition functions, which can be lifted to $\phi_{ij} \in \Gamma(U_{ij}, GL_n)$. In this way we get a cocycle $\phi_{ij} \circ \phi_{jk} \circ \phi_{ik}^{-1} = 1 + \gamma_{ijk}$ with

$$\gamma_{ijk} \in \Gamma(U_{ijk}, I\mathcal{O}^{n^2}_{\mathrm{Spec}(A)\times X}) = \Gamma(U_{ijk}, I\mathrm{End}(\mathcal{E})).$$

We have a 2-cocycle condition for γ_{ijk} on the intersection U_{ijkl} corresponding to an element $(\gamma_{ijk}) \in H^2(\mathrm{Spec}(A) \times X, I\mathrm{End}(\mathcal{E}))$. But for dimension reason as X is a curve we have:

$$H^2(\mathrm{Spec}(A) \times X, I\mathrm{End}(\mathcal{E})) = 0$$

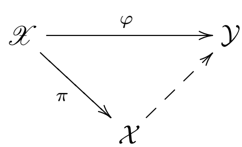
and therefore as the obstruction vanishes, we can change the transition functions $\beta_{ij} \in \Gamma(U_{ij}, GL_n)$ such that they define a vector bundle $\mathcal{E}$ on $\mathrm{Spec}(A) \times X$. Now from the lifting criterion for smoothness it follows that the stack $\mathscr{B}un_X^{n,d}$ is smooth. □

The moduli stack $\mathscr{B}un_X^n$ of all vector bundles of rank n on a smooth projective irreducible curve X is the disjoint union of the moduli stacks $\mathscr{B}un_X^{n,d}$, which are the connected components. [Fal95].

Let us finally discuss briefly the relation between moduli stacks and moduli spaces. We need to define first what it means for an algebraic stack to have a coarse moduli space.

Definition 2.68. *Let $\mathscr{X}$ be an algebraic stack over the category (Sch/S) of S-schemes. A* coarse moduli space *for $\mathscr{X}$ is an algebraic space $\mathcal{X}$ together with a morphism $\pi : \mathscr{X} \to \mathcal{X}$ such that:*

1. *All morphisms $\mathscr{X} \to \mathcal{Y}$, where $\mathcal{Y}$ is an algebraic space factor uniquely through $\mathcal{X}$, i.e. there exists a unique morphism $\mathcal{X} \to \mathcal{Y}$ such that the diagram*



   is commutative.

2. *For every algebraically closed field k the map of sets*

$$[\mathscr{X}(\mathrm{Spec}(k))] \to \mathcal{X}(\mathrm{Spec}(k))$$

is a bijection, where $[\mathscr{X}(\mathrm{Spec}(k))]$ is the set of isomorphism classes of the groupoid $\mathscr{X}(\mathrm{Spec}(k))$

Let us compare this general definition with the classical one for schemes [HL97], Def. 2.2.1.

Definition 2.69. *Let (Sch/S) be the category of S-schemes. A scheme M is a* coarse moduli space *for a functor*

$$\mathcal{M} : (Sch/S)^{op} \to (Sets)$$

if the following holds:

1. *M* corepresents *the functor $\mathcal{M}$, i.e. there is a natural transformation of functors $\phi : \mathcal{M} \to \mathrm{Hom}_{(Sch/S)}(?, M)$ such that given another scheme N and a natural transformation $\psi : \mathcal{M} \to \mathrm{Hom}_{(Sch/S)}(?, N)$ there exists a unique natural transformation $\eta : \mathrm{Hom}_{(Sch/S)}(?, M) \to \mathrm{Hom}_{(Sch/S)}(?, N)$ with $\psi = \eta \circ \phi$, i.e. we have a commutative diagram of the form*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad\psi\quad} & \\ \downarrow{\scriptstyle\phi} & \searrow & \\ \mathrm{Hom}_{(Sch/S)}(?, M) & \overset{\eta}{\dashrightarrow} & \mathrm{Hom}_{(Sch/S)}(?, N) \end{array}$$

2. *For every algebraically closed field k the map of sets*

$$\phi(k) : \mathcal{M}(\mathrm{Spec}(k)) \to \mathrm{Hom}_{(Sch/S)}(\mathrm{Spec}(k), M)$$

is a bijection.

Let us also recall here for completeness, even though it is normally a vacuous term, what it means to have a fine moduli space.

Definition 2.70. *Let (Sch/S) be the category of S-schemes. A scheme M is a* fine moduli space *for a functor*

$$\mathcal{M} : (Sch/S)^{op} \to (Sets)$$

if M represents *the functor $\mathcal{M}$, i.e. there is a natural isomorphism between functors*

$$\eta : \mathcal{M}(?) \cong \mathrm{Hom}_{(Sch/S)}(?, M).$$

A fine or a coarse moduli space M is always given up to unique isomorphism.

Example 2.71. (Coarse moduli space for vector bundles over an algebraic curve) There is a morphism of stacks

$$F : \mathscr{B}un_X^{st,n} \to \mathrm{Bun}_X^{st,n}$$

where $\mathscr{B}un_X^{st,n}$ is the moduli stack of stable vector bundles of rank n on an algebraic curve X and $\mathrm{Bun}_X^{st,n}$ is its coarse moduli space. $\mathrm{Bun}_X^{st,n}$ is a scheme constructed by GIT methods using the machinery of Quot schemes. We refer to [MFK94] or [Est97] for the constructions of GIT quotients and applications to moduli problems. In fact, $\mathrm{Bun}_X^{st,n}$ is a coarse moduli space for the moduli functor

$$\mathcal{M}_X^{st,n} : (Sch/k)^{op} \to (Sets)$$

associated to the moduli problem of classifying stable vector bundles of rank n on X. It follows from the GIT constructions that the moduli stack of stable vector bundles of rank n on X is given as a quotient stack $\mathscr{B}un_X^{st,n} = [R^{st,n}/GL_N]$ and the coarse moduli space as a quotient $\mathrm{Bun}_X^{st,n} = R^{st,n}/PGL_N$ for some scheme $R^{st,n}$. The scheme $R^{st,n}$ is an atlas for the moduli stack and its quotient by the PGL_N-action gives the coarse moduli space. The quotient morphism $\pi : R^{st,n} \to R^{st,n}/PGL_N$ is a principal PGL_N-bundle. Similarly, we have also a morphism of stacks

$$F : \mathscr{B}un_X^{ss,n} \to \mathrm{Bun}_X^{ss,n}$$

where $\mathscr{B}un_X^{ss,n}$ is the moduli stack of semistable vector bundles of rank n on an algebraic curve X and $\mathrm{Bun}_X^{ss,n}$ is again its coarse moduli space.

Let us just state the following proposition to summarize the discussion on moduli stacks versus moduli spaces of vector bundles.

Proposition 2.72. *There is a commutative diagram of algebraic stacks of the form*

$$\begin{array}{ccc} [R^{st,n}/GL_N] & \xrightarrow{\quad q \quad} & [R^{st,n}/PGL_N] \\ {\scriptstyle\cong}\downarrow{\scriptstyle g} & & {\scriptstyle h}\downarrow{\scriptstyle\cong} \\ \mathscr{B}un_X^{st,n} & \xrightarrow{\quad F \quad} & \mathrm{Bun}_X^{st,n} \end{array}$$

where g and h are isomorphism of stacks. There is a similar diagram in the semistable situation, but the right vertical morphism is not an isomorphism of stacks anymore.

Proof. This is [Góm01], Prop. 3.3. □

For a more detailed discussion of the relation between moduli stacks and coarse moduli spaces of semistable and stable vector bundles on a scheme we refer to [Góm01], Sec. 3. Similar statements can also be made for the moduli stacks $\mathscr{M}_g$ and $\widetilde{\mathscr{M}_g}$ of algebraic curves.

Chapter 3

Cohomology of algebraic stacks

3.1 Sheaf cohomology of algebraic stacks

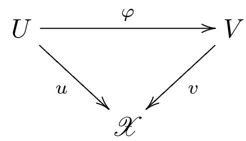
In this section we will define sheaf cohomology for algebraic stacks and discuss some of its properties. For a systematic and general treatment of how to define cohomology of algebraic stacks we refer to [LMB00], [LO08a], [LO08b],[Ols07], [Beh93]. [Beh03] and [Vis89]. An introduction to this material can also be found in [NS05] and [Hei09]. We like to recommend also the article [Hei05] for a similar discussion of cohomology of differentiable stacks.

We will give here only a working definition for sheaf cohomology of algebraic stacks, which has all the properties we will need to determine the l-adic cohomology of the moduli stack of vector bundles of fixed rank and degree on an algebraic curve. First we need to define the smooth site of an algebraic stack, where our sheaves will be living on (see [LMB00], [Beh93], [NS05], [Hei98]).

Definition 3.1. *Let $\mathscr{X}$ be an algebraic stack. The* smooth site $\mathscr{X}_{sm}$ *on $\mathscr{X}$ is defined as the following category:*

1. *The objects are given as pairs (U, u), where U is a scheme and $u : U \to \mathscr{X}$ is a smooth morphisms.*

2. *The morphisms are given as pairs* $(\varphi, \alpha) : (U, u) \to (V, v)$ *where* $\varphi : U \to V$ *is a morphism of schemes and* $\alpha : u \Rightarrow v \circ \varphi$ *is a* 2*-isomorphism, i.e. we have a commutative diagram of the form*



together with a 2*-isomorphism* $\alpha : u \Rightarrow v \circ \varphi$.

3. *The coverings are given by the smooth coverings of the schemes, i.e. coverings of an object* (U, u) *are families of morphisms*

$$\{(\varphi_i, \alpha_i) : (U_i, u_i) \to (U, u)\}_{i \in I}$$

such that the morphism

$$\coprod_{i \in I} \varphi_i : \coprod_{i \in I} U_i \to U$$

is smooth and surjective.

The Grothendieck topology on the smooth site $\mathscr{X}_{sm}$ here is basically induced by the smooth topology on the category (Sch/S) of S-schemes. We can now define the notion of a sheaf on the smooth site $\mathscr{X}_{sm}$.

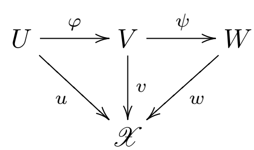
Definition 3.2. *Let* $\mathscr{X}$ *be an algebraic stack. A* sheaf $\mathcal{F}$ *on the smooth site* $\mathscr{X}_{sm}$ *is given by the following data:*

1. *For each object* (U, u) *of* $\mathscr{X}_{sm}$*, where* U *is a scheme and* $u : U \to \mathscr{X}$ *a smooth morphism, a sheaf* $\mathcal{F}_{U,u}$ *on* U.

2. *For each morphism* $(\varphi, \alpha) : (U, u) \to (V, v)$ *of* $\mathscr{X}_{sm}$ *a morphism of sheaves*

$$\theta_{\varphi,\alpha} : \varphi^* \mathcal{F}_{V,v} \to \mathcal{F}_{U,u}$$

satisfying the cocycle condition for composable morphisms, i.e. for each commutative diagram of the form



together with 2-isomorphisms $\alpha : u \Rightarrow v \circ \varphi$ and $\beta : v \Rightarrow w \circ \psi$ we have that

$$\theta_{\varphi,\alpha} \circ \psi^* \theta_{\psi,\beta} = \theta_{\psi\circ\varphi,\varphi_*\beta\circ\alpha}$$

A sheaf $\mathcal{F}$ is called quasi-coherent (resp. coherent, resp. of finite type, resp. of finite presentation, resp. locally free) *if the sheaf $\mathcal{F}_{U,u}$ is quasi-coherent (resp. coherent, resp. of finite type, resp. of finite presentation, resp. locally free) for every morphism $u : U \to \mathscr{X}$, where U is a scheme.*

A sheaf $\mathcal{F}$ is called cartesian *if all morphisms $\theta_{\varphi,\alpha}$ are isomorphisms.*

A morphism of sheaves *$h : \mathcal{F} \to \mathcal{F}'$ on $\mathscr{X}_{sm}$ is a collection of morphisms of sheaves $h_{U,u} : \mathcal{F}_{U,u} \to \mathcal{F}'_{U,u}$ for all objects (U, u) of $\mathscr{X}_{sm}$ which are compatible with the morphisms $\theta_{\varphi,\alpha}$. The category* $\mathrm{Shv}(\mathscr{X})$ *of sheaves of sets on the smooth site $\mathscr{X}_{sm}$ is called the* smooth topos *of $\mathscr{X}$.*

It is normally enough to consider affine schemes in the definition of a sheaf on an algebraic stack as any sheaf on a scheme can be obtained by glueing along affine covers [LMB00].

The category of sheaves on an algebraic stack $\mathscr{X}$ can be described equivalently as the category of sheaves on some atlas $x : X \to \mathscr{X}$ together with descent data [LMB00], Chap. 12.

In general, it is not enough to consider just cartesian sheaves as for example certain abelian categories of cartesian sheaves might not have enough injective objects, and so have a bad homological behaviour. But it can be shown that the category of cartesian sheaves is a thick subcategory of the category of all sheaves, i.e. a full subcategory closed under kernels, quotients and extensions [LMB00].

We record especially here the case of a vector bundle on an algebraic stack.

Definition 3.3. *A* vector bundle *on an algebraic stack $\mathscr{X}$ is a coherent sheaf $\mathcal{E}$ on $\mathscr{X}_{sm}$ such that all coherent sheaves $\mathcal{E}_{U,u}$ are locally free for every morphism $u : U \to \mathscr{X}$, where U is a scheme.*

Let us look now at some important example of sheaves on algebraic stacks.

Example 3.4. (Structure sheaf of an algebraic stack) Let $\mathscr{X}$ be an algebraic stack. The *structure sheaf* $\mathcal{O}_{\mathscr{X}}$ on $\mathscr{X}$ is defined by assembling the structure sheaves $\mathcal{O}_U$ of the schemes U for every smooth morphism $u : U \to \mathscr{X}$, i.e. by setting $(\mathcal{O}_{\mathscr{X}})_{U,u} = \mathcal{O}_U$. In this way we get a ringed site $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ on the algebraic stack $\mathscr{X}$ and we can define sheaves of $\mathcal{O}_{\mathscr{X}}$-modules, sheaves of quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules and, if $\mathscr{X}$ is locally noetherian, also sheaves of coherent $\mathcal{O}_{\mathscr{X}}$-modules [LMB00], Chap. 13 & 15.

Example 3.5. (Constant sheaf $\mathbb{Z}/n\mathbb{Z}$) Let $\mathscr{X}$ be an algebraic stack. Let $n \geq 1$ be a positive integer. The *constant sheaf* $(\mathbb{Z}/n\mathbb{Z})_{\mathscr{X}}$ is given by assembling the constant sheaves $(\mathbb{Z}/n\mathbb{Z})_{U,u} = (\mathbb{Z}/n\mathbb{Z})_U = \mathbb{Z}/n\mathbb{Z}$. It turns out that this is actually a cartesian sheaf on $\mathscr{X}$ [LMB00], 12.7.1 (ii).

Example 3.6. (Sheaf of relative differentials) Let $F : \mathscr{X} \to \mathscr{Y}$ be a representable morphism of algebraic stacks. We can define the *sheaf of relative differentials* $\Omega_{\mathscr{X}/\mathscr{Y}}$.

Let $u : U \to \mathscr{Y}$ be any smooth covering morphism with U a scheme, i.e. an atlas of $\mathscr{Y}$. We have the following cartesian diagram

$$\begin{array}{ccc} U \times_{\mathscr{Y}} \mathscr{X} & \xrightarrow{pr_2} & \mathscr{X} \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle F} \\ U & \xrightarrow{u} & \mathscr{Y} \end{array}$$

The fiber product $U \times_{\mathscr{Y}} \mathscr{X}$ is a scheme and we have the sheaf of relative differentials $\Omega_{U \times_{\mathscr{Y}} \mathscr{X}/U}$. The sheaf of relative differentials $\Omega_{\mathscr{X}/\mathscr{Y}}$ is now defined via descent of the shaf $\Omega_{U \times_{\mathscr{Y}} \mathscr{X}/U}$ on the covering morphism $pr_2 : U \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{X}$. This sheaf is well-defined as it does not depend on the chosen atlas $u : U \to \mathscr{Y}$. [Hei98], [Góm01], Ex. 2.48.

Example 3.7. (Universal vector bundle $\mathcal{E}^{univ}$ on $X \times \mathscr{B}un_X^{n,d}$) Let $\mathscr{B}un_X^{n,d}$ be the moduli stack of rank n and degree d vector bundles on a smooth projective irreducible algebraic curve X of genus $g \geq 2$. There exists a *universal vector bundle* $\mathcal{E}^{univ}$ on the algebraic stack $X \times \mathscr{B}un_X^{n,d}$, because via representability any morphism $U \to \mathscr{B}un_X^{n,d}$, where U is a scheme defines a family of vector bundles of rank n and degree d on the scheme X parametrized by U and the cocycle conditions can easily be checked for vector bundles. Similar, we get universal vector bundles for the moduli stacks $\mathscr{B}un_X^{ss,n,d}$ (resp. $\mathscr{B}un_X^{st,n,d}$) of semistable (resp. stable) vector bundles.

Example 3.8. (Equivariant sheaves) Let (Sch/S) be the category of S-schemes and X be a noetherian S-scheme. Let G be an affine smooth group S-scheme with a right action $\rho : X \times G \to X$ and consider the quotient stack $[X/G]$. Then any cartesian sheaf $\mathcal{F}$ on $[X/G]$ is the same as an G-equivariant sheaf on X.

Example 3.9. (Vector bundle on moduli stack of algebraic curves) Let $\mathscr{M}_g$ be the moduli stack of algebraic curves of genus $g \geq 2$. For any family $\pi : C \to U$ consider the dualizing sheaf $\omega_{C/U}^{\otimes 3}$. The direct image sheaf $\pi_*(\omega_{C/U}^{\otimes 3})$ is a locally free sheaf of rank $5g-5$. We define a sheaf on $\mathscr{M}_g$ by letting $\mathcal{E}_{U,u} = \pi_*(\omega_{C/U}^{\otimes 3})$ for any atlas $u : U \to \mathscr{M}_g$. This gives a vector bundle $\mathcal{E}$ of rank $5g-5$ on the moduli stack $\mathscr{M}_g$ (see also [Vis89], Ex. 7.20 (iii)). The vector bundle $\mathcal{E}$ defines then via representability a morphism of algebraic stacks

$$\mathscr{M}_g \to \mathscr{B}GL_{5g-5}$$

where $\mathscr{B}GL_{5g-5}$ is the classifying stack of vector bundles of rank $5g-5$. It turns out, that this is actually a representable morphism, which is surjective and smooth.

Having defined sheaves on algebraic stacks we can now also define inverse image and direct image functors f^*, f_* of quasicoherent sheaves etc. We refer to [LMB00], [Ols07], [LO08a] and [LO08b] for a systematic treatment. Let us just mention that because such functors f^*, f_* always commute with flat base change we get such functors automatically for any representable morphism of stacks $F : \mathscr{X} \to \mathscr{Y}$.

We will now define sheaf cohomology of an algebraic stack $\mathscr{X}$ with respect to a sheaf of abelian groups $\mathcal{F}$ on $\mathscr{X}_{sm}$. We will give a common man's definition of cohomology here which will be enough for our purposes to illustrate the use of cohomology. We refer to the work of Behrend [Beh93], [Beh03] and Laszlo and Olsson [LO08a] and

[LO08b] for a general and systematic approach in the style of SGA.

Let $\mathscr{X}$ be an algebraic stack and choose an atlas $u : U \to \mathscr{X}$ of $\mathscr{X}$. For cartesian sheaves $\mathcal{F}$ on $\mathscr{X}$ we define the global sections as the equalizer

$$\Gamma(\mathscr{X}, \mathcal{F}) := \mathrm{Ker}(\Gamma(U, \mathcal{F}) \rightrightarrows \Gamma(U \times_{\mathscr{X}} U, \mathcal{F})).$$

It is not hard to see that this definition does not depend on the choice of the atlas $u : U \to \mathscr{X}$ of $\mathscr{X}$ by first checking it on a covering and then on refinements.

For general sheaves we proceed as follows [LMB00], 12.5.3.

Definition 3.10. *Let $\mathscr{X}$ be an algebraic stack and $\mathcal{F}$ a quasi-coherent sheaf on $\mathscr{X}_{sm}$. The set of* global sections *is defined as*

$$\Gamma(\mathscr{X}, \mathcal{F}) := \{(s_{U,u}) : s_{U,u} \in H^0(U, \mathcal{F}_{U,u}), \theta_{\varphi,\alpha} s_{U,u} = s_{V,v}\}.$$

The functor

$$\Gamma(\mathscr{X}, ?) : \mathrm{Shv}(\mathscr{X}) \to (Sets)$$

is called the global section functor*.*

We can rephrase this by saying that the global sections are given as the limit

$$\Gamma(\mathscr{X}, \mathcal{F}) = \varprojlim \Gamma(U, \mathcal{F}_{U,u})$$

where the limit is taken over all atlases $u : U \to \mathscr{X}$ with transition functions given by the restriction maps $\theta_{\varphi,\alpha}$. Again, it is not hard to show that for cartesian sheaves the two notions of global sections coincide.

The category $\mathfrak{Mod}(\mathscr{X})$ of sheaves of $\mathcal{O}_{\mathscr{X}}$-modules and the category $\mathfrak{Ab}(\mathscr{X})$ of sheaves of abelian groups on the site $\mathscr{X}_{sm}$ have enough injective objects and so we can do homological algebra in them and especially can proceed now to define sheaf cohomology of algebraic stacks.

Definition 3.11. *The* i-th smooth cohomology group *of the algebraic stack $\mathscr{X}$ with respect to a sheaf $\mathcal{F}$ of abelian groups on the smooth site $\mathscr{X}_{sm}$ is defined as*

$$H^i_{sm}(\mathscr{X}, \mathcal{F}) := R^i\Gamma(\mathscr{X}, \mathcal{F})$$

where the cohomology functor

$$H^i_{sm}(\mathscr{X}, ?) = R^i\Gamma(\mathscr{X}, ?) : \mathfrak{Ab}(\mathscr{X}) \to \mathfrak{Ab}$$

is the i-th right derived functor of the global section functor $\Gamma(\mathscr{X}, ?)$ *with respect to* $\mathscr{X}_{sm}$.

For cartesian sheaves we can give a simplicial interpretation of the sheaf cohomology of an algebraic stack $\mathscr{X}$ [LMB00], 12.4. Let $x : X \to \mathscr{X}$ be an atlas. As the diagonal morphism of an algebraic stack is representable, we obtain by taking iterated fiber products of the atlas with itself

$$\begin{array}{ccc} X \times_{\mathscr{X}} X & \longrightarrow & X \\ \downarrow & & \downarrow x \\ X & \xrightarrow{x} & \mathscr{X} \end{array}$$

a simplicial scheme $X_\bullet = \{X_i\}_{i \geq 0}$ over $\mathscr{X}$ with layers

$$X_i = X \times_{\mathscr{X}} X \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} X$$

given by the $(i+1)$-fold iterated fiber product of the atlas with itself.

A simplicial scheme $X_\bullet$ over $\mathscr{X}$ can simply be interpreted as a functor

$$X_\bullet : \Delta^{op} \to (Sch/\mathscr{X})$$

where Δ^{op} is the category with objects finite sets $[n] = \{0, 1, \ldots n\}$ and morphisms order preserving maps and $(Sch/\mathscr{X})$ is the category of schemes over the algebraic stack $\mathscr{X}$, i.e. the category of schemes X together with morphisms $x : X \to \mathscr{X}$.

Now let $\mathcal{F}$ be a sheaf on $\mathscr{X}$. This defines a sheaf $\mathcal{F}_\bullet$ on the simplicial scheme $X_\bullet$, i.e. a sheaf $\mathcal{F}_i$ on all schemes X_i together with morphisms for all simplicial maps $f : [m] \to [n]$ of the form f^* :

$X_\bullet(f)^*\mathcal{F}_n \to \mathcal{F}_m$. We call a sheaf on a simplicial scheme *cartesian* if all morphisms f^* are isomorphisms. If we start with a cartesian sheaf $\mathcal{F}$ on $\mathscr{X}$, we get a cartesian sheaf $\mathcal{F}_\bullet$ on the simplicial scheme $X_\bullet$. In this way we get a functor $\mathrm{Shv}(\mathscr{X}) \to \mathrm{Shv}(X_\bullet)$.

Conversely, for any smooth morphism $u : U \to \mathscr{X}$ a sheaf $\mathcal{F}_\bullet$ on the simplicial scheme $X_\bullet$ gives a sheaf on the covering $U \times_{\mathscr{X}} X \to U$ via taking global sections and by assembling them to a sheaf on $\mathscr{X}$. Again starting with a cartesian sheaf $\mathcal{F}_\bullet$ on $X_\bullet$ gives a cartesian sheaf on $\mathscr{X}$.

We can define cohomology of sheaves of abelian groups on simplicial schemes generalizing the classical homological approach for sheaf cohomology on schemes [Fri82].

The relation between the cohomology groups of an algebraic stack $\mathscr{X}$ and the cohomology groups of an atlas $x : X \to \mathscr{X}$ is given by a descent spectral sequence [Fri82]:

Theorem 3.12. *Let $\mathscr{X}$ be an algebraic stack and $\mathcal{F}$ be a cartesian sheaf of abelian groups on $\mathscr{X}$. Let $x : X \to \mathscr{X}$ be an atlas and $\mathcal{F}_\bullet$ the induced sheaf on the simplicial scheme $X_\bullet$ over $\mathscr{X}$. Then there is a convergent spectral sequence*

$$E_1^{p,q} \cong H^p_{sm}(X_q, \mathcal{F}_q) \Rightarrow H^{p+q}_{sm}(\mathscr{X}, \mathcal{F}).$$

which is functorial with respect to morphisms $F : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks.

Proof. For a proof see [Del74b], [Fri82]. □

In [Beh93] the general framework of topoi is used to give a rigorous definition of sheaf cohomology on the smooth site $\mathscr{X}_{sm}$ of an algebraic stack $\mathscr{X}$. The resulting cohomology groups agree with the ones defined here in an ad hoc way. We are mainly interested in the special case of l-adic cohomology employing the cartesian constant sheaf $(\mathbb{Z}/l^n\mathbb{Z})_{\mathscr{X}}$, where l is a prime number different from the characteristic p of the ground field $\mathbb{F}_q$.

Example 3.13. (l-adic smooth cohomology) Let $\mathscr{X}$ be an algebraic stack defined over the field $\mathbb{F}_q$ of characteristic p. Via base change we get an associated algebraic stack $\overline{\mathscr{X}}$ over the algebraic closure $\overline{\mathbb{F}}_q$ by setting

$$\overline{\mathscr{X}} = \mathscr{X} \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q).$$

Let l be a prime number different from p. The *l-adic smooth cohomology* of the algebraic stack $\overline{\mathscr{X}}$ is defined as

$$H^*_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) = \lim_{\leftarrow} H^*_{sm}(\overline{\mathscr{X}}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

It is possible to conclude directly that Künneth decomposition, Gysin sequences and Leray spectral sequences also exist for smooth l-adic cohomology of algebraic stacks by using the simplicial scheme associated to an atlas X and the descent spectral sequence.

In the special case of smooth schemes, the smooth l-adic cohomology groups agree with the étale l-adic cohomology groups [Mil80].

Let us just record here the following fundamental properties of l-adic smooth cohomology of algebraic stacks:

Theorem 3.14. *We have the following properties:*

(1.) (Künneth decomposition) Let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic stacks. There is a natural isomorphism of graded $\mathbb{Q}_l$-algebras

$$H^*_{sm}(\overline{\mathscr{X}} \times \overline{\mathscr{Y}}, \mathbb{Q}_l) \cong H^*_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) \otimes H^*_{sm}(\overline{\mathscr{Y}}, \mathbb{Q}_l).$$

(2.) (Gysin sequence) Let $\mathscr{Z} \hookrightarrow \mathscr{X}$ be a closed embedding of algebraic stacks of codimension c. There is a long exact sequence

$$\cdots \to H^{i-2c}_{sm}(\overline{\mathscr{Z}}, \mathbb{Q}_l(c)) \to H^i_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) \to H^i_{sm}(\overline{\mathscr{X} \setminus \mathscr{Z}}, \mathbb{Q}_l) \to \cdots$$

In particular, $H^i_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) \overset{\cong}{\to} H^i_{sm}(\overline{\mathscr{X} \setminus \mathscr{Z}}, \mathbb{Q}_l)$ is an isomorphism in the range $i < 2c - 1$.

Proof. The first part follows from the descent spectral sequence and the analogous results for simplicial schemes [Del77], [Fri82]. (See also [NS05] for a brief discussion). The second part is [Beh93], Prop. 2.1.2 and Cor. 2.1.3. □

A general l-adic formalism of derived categories of l-adic sheaves for algebraic stacks was systematically developed by Behrend [Beh03] and used to prove a general version of a Lefschetz trace formula for algebraic stacks.

Example 3.15. (Cohomology of the classifying stack $\mathscr{B}\mathbb{G}_m$) Let $\mathbb{G}_m$ be the multiplicative group over $\mathrm{Spec}(\mathbb{F}_q)$. The quotient morphism $\mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ is a principal $\mathbb{G}_m$-bundle and we have a cartesian diagram of the form

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q) \\ \downarrow & & \downarrow \\ \mathbb{P}^{n-1} & \xrightarrow{\ \pi\ } & \mathscr{B}\mathbb{G}_m \end{array}$$

The fiber of the morphism π is $\mathbb{A}^n - \{0\}$ and we can employ the Leray spectral sequence

$$E_2^{p,q} \cong H^p_{sm}(\overline{\mathbb{P}}^{n-1}, R^q\pi_*\mathbb{Q}_l) \Rightarrow H^*_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l)$$

and because $R^0\pi_*\mathbb{Q}_l \cong \mathbb{Q}_l$ and $R^q\pi_*\mathbb{Q}_l = 0$ if $q \leq 2n-1$ it follows for $q \leq 2n-1$ that

$$H^q_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong H^q_{sm}(\overline{\mathbb{P}}^{n-1}, \mathbb{Q}_l)$$

and therefore

$$H^*_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1]$$

where c_1 is a generator of degree 2 given as the Chern class of the universal bundle $\mathcal{E}^{univ}$ on the classifying stack $\mathscr{B}\mathbb{G}_m$.

We will later need a bit of the theory of gerbes on algebraic stacks. We refer to [Lie07], [Hei09] for the technical details.

Definition 3.16. *A morphism $F : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks is a* gerbe *over $\mathscr{Y}$ if the following holds:*

1. *F is locally surjective, i.e. for any morphism $U \to \mathscr{Y}$ from a scheme, there exists a covering $U' \to U$ such that the morphism $U' \to \mathscr{Y}$ can be lifted to a morphism $U' \to \mathscr{X}$, i.e.*

$$\begin{array}{ccc} & & \mathscr{X} \\ & \nearrow & \downarrow F \\ U' & \longrightarrow & \mathscr{Y} \end{array}$$

2. *All objects in a fiber of F are locally isomorphic, i.e. if $u_1, u_2 : U \to \mathscr{X}$ are objects of $\mathscr{X}(U)$ such that $F(u_1) \cong F(u_2)$, then there exists a covering $U' \to U$ such that $u_1|_{U'} \cong u_2|_{U'}$.*

A gerbe $F : \mathscr{X} \to \mathscr{Y}$ is a $\mathbb{G}_m$-gerbe *if for all morphisms $u : U \to \mathscr{X}$ the relative automorphism group $\mathrm{Aut}_{\mathscr{Y}}(u)$ is canonically isomorphic to $\mathbb{G}_m(U)$.*

We can think of a $\mathbb{G}_m$-gerbe over a scheme Y as a $\mathscr{B}\mathbb{G}_m$-bundle over Y, i.e. a bundle over Y with fiber $\mathscr{B}\mathbb{G}_m$.

Example 3.17. As mentioned before, there is a morphism of stacks

$$F : \mathscr{B}un_X^{st,n} \to \mathrm{Bun}^{st,n}$$

where $\mathscr{B}un_X^{st,n}$ is the moduli stack of stable vector bundles of rank n on an algebraic curve X with coarse moduli space $\mathrm{Bun}_X^{st,n}$. The morphism F has the following property: For any morphism $U \to \mathrm{Bun}_X^{st,n}$ of schemes there exists an étale covering $U' \to U$ such that the morphism $U' \to \mathrm{Bun}_X^{st,n}$ lifts to a morphism $U' \to R^{st,n}$ and so it lifts to the moduli stack $\mathscr{B}un_X^{st,n} = [R^{st,n}/GL_N]$.

Therefore F is a gerbe and because all automorphisms of stable bundles are given by scalars the fiber of F is isomorphic to $\mathscr{B}\mathbb{G}_m$, i.e. F is actually a $\mathbb{G}_m$-gerbe.

In general, a morphism of quotient stacks of the form

$$F : [R/GL_N] \to [R/PGL_N]$$

is a $\mathbb{G}_m$-gerbe. This is useful in order to compare "stacky" quotients with GIT quotients.

The following proposition gives a criterion for triviality of a $\mathbb{G}_m$-gerbe on a stack $\mathscr{Y}$.

Proposition 3.18. *Let $F : \mathscr{X} \to \mathscr{Y}$ be a $\mathbb{G}_m$-gerbe. Then the following are equivalent:*

1. *The $\mathbb{G}_m$-gerbe F is trivial, i.e. we have a splitting of algebraic stacks*
$$\mathscr{X} \cong \mathscr{Y} \times \mathscr{B}\mathbb{G}_m.$$

2. The morphism F has a section.

Proof. This is [Hei09], Lemma 3.10. □

Example 3.19. There is also a morphism of stacks

$$F : \mathscr{B}un_X^{st,n,d} \to \mathrm{Bun}_X^{st,n,d}$$

where $\mathscr{B}un_X^{st,n,d}$ is the moduli stack of stable vector bundles of rank n and degree d on X and $\mathrm{Bun}_X^{st,n,d}$ its coarse moduli space, given again as a scheme via GIT methods. A section of the morphism F is a vector bundle on $X \times \mathrm{Bun}_X^{st,n,d}$ such that the fiber over every geometric point of $\mathrm{Bun}_X^{st,n,d}$ lies in the isomorphism class of stable bundles defined by this geometric point. Such a vector bundle is also called a *Poincaré family*.

There are many interesting $\mathbb{G}_m$-gerbes over an algebraic stack and we will make use of the splitting criteria for the triviality of a $\mathbb{G}_m$-gerbe when we determine the cohomology of the moduli stack $\mathscr{B}un_X^{n,d}$.

Chapter 4

Moduli stacks of vector bundles I

4.1 Cohomology of the moduli stack

In this section we will determine the l-adic cohomology algebra of the moduli stack $\mathscr{B}un_X^{n,d}$ of vector bundles of rang n and degree d on a smooth projective irreducible algebraic curve X over the field $\mathbb{F}_q$.

Let us recall the l-adic cohomology algebra of the moduli stack

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l) = \varprojlim H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

The main ingredients used to produce the cohomology algebra will be the l-adic cohomology of the classifying stack $\mathscr{B}GL_n$ of all rank n vector bundles and the l-adic cohomology of the algebraic curve X.

The l-adic cohomology of the algebraic curve X is completely determined from the Weil Conjectures.

Theorem 4.1 (Weil, Deligne)**.** *Let X be a smooth projective curve of genus g over $\mathbb{F}_q$ and $\overline{X} = X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ the associated curve over the algebraic closure $\overline{\mathbb{F}}_q$. Then we have*

$$H^0_{et}(\overline{X};\mathbb{Q}_l) = \mathbb{Q}_l \cdot 1$$
$$H^1_{et}(\overline{X};\mathbb{Q}_l) = \bigoplus_{i=1}^{2g} \mathbb{Q}_l \cdot \alpha_i$$
$$H^2_{et}(\overline{X};\mathbb{Q}_l) = \mathbb{Q}_l \cdot [\overline{X}]$$
$$H^i_{et}(\overline{X};\mathbb{Q}_l) = 0, \text{ if } i \geq 3$$

where $[\overline{X}]$ is the fundamental class and the α_i are eigenclasses under the action of the geometric Frobenius morphism

$$\overline{F}^*_X : H^*_{et}(\overline{X}, \mathbb{Q}_l) \to H^*_{et}(\overline{X}, \mathbb{Q}_l)$$

given as

$$\overline{F}^*_X(1) = 1$$
$$\overline{F}^*_X([\overline{X}]) = q[\overline{X}]$$
$$\overline{F}^*_X(\alpha_i) = \lambda_i \alpha_i \ (i = 1, 2, \dots 2g)$$

where $\lambda_i \in \overline{\mathbb{Q}}_l$ is an algebraic integer with $|\lambda_i| = q^{1/2}$ for any embedding of λ_i in $\mathbb{C}$.

Proof. This is the étale l-adic cohomology analogue of a similar result in the complex analytic case of a Riemann surface. For a proof see for example [Mil80] or [FK88]. □

We could have also used here smooth l-adic cohomology, but as mentioned before, it will give the same result as étale l-adic cohomology as on smooth schemes both cohomology theories agree [Mil80].

The other ingredient in the determination of the l-adic cohomology algebra of $\mathscr{B}un^{n,d}_X$ will be the l-adic cohomology of the classifying stack $\mathscr{B}GL_n$ of all rank n vector bundles. Let $\overline{\mathscr{B}GL}_n = \mathscr{B}GL_n \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ be the associated classifying stack over the algebraic closure $\overline{\mathbb{F}}_q$. We also have a geometric Frobenius morphism

$$\overline{F}^*_{\mathscr{B}GL_n} : H^*_{sm}(\overline{\mathscr{B}GL}_n, \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathscr{B}GL}_n, \mathbb{Q}_l).$$

The l-adic cohomology algebra of $\mathscr{B}GL_n$ and the action of the Frobenius morphism $\overline{F}_{\mathscr{B}GL_n}$ is completely determined by the following theorem [Beh93].

Theorem 4.2. *There is an isomorphism of graded $\mathbb{Q}_l$-algebras*

$$H^*_{sm}(\overline{\mathscr{B}GL}_n, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \dots c_n]$$

and the geometric Frobenius morphism $\overline{F}^*_{\mathscr{B}GL_n}$ acts as follows

$$\overline{F}^*_{\mathscr{B}GL_n}(c_i) = q^i c_i \ \ (i \geq 1).$$

where the c_i are the Chern classes of the universal vector bundle $\tilde{\mathcal{E}}^{univ}$ of rank n over the classifying stack $\mathscr{B}GL_n$.

Proof. This is [Beh93] Thm. 2.3.2. This is an analogue of a similar result in algebraic topology for singular cohomology of the classifying space BGL_n. The associated simplicial scheme of the classifying stack $\mathscr{B}GL_n$ has the same homotpy type as the classifying space. □

We will embark now on the determination of the l-adic cohomology algebra of the moduli stack $\mathscr{B}un_X^{n,d}$. Let $\mathcal{E}^{univ}$ be the universal vector bundle of rank n and degree d over the algebraic stack $\overline{X} \times \overline{\mathscr{B}un}_X^{n,d}$. Via representability it gives a morphism of stacks

$$u : \overline{X} \times \overline{\mathscr{B}un}_X^{n,d} \to \overline{\mathscr{B}GL}_n.$$

The universal vector bundle $\mathcal{E}^{univ}$ has Chern classes given as

$$c_i(\mathcal{E}^{univ}) = u^*(c_i) \in H^{2i}_{sm}(\overline{X} \times \overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l).$$

Fixing a basis $1 \in H^0_{sm}(\overline{X}, \mathbb{Q}_l)$, $\alpha_j \in H^1_{sm}(\overline{X}, \mathbb{Q}_l)$ with $j = 1, \dots 2g$ and $[\overline{X}] \in H^2_{sm}(\overline{X}, \mathbb{Q}_l)$ we get the following Künneth decomposition of Chern classes:

$$c_i(\mathcal{E}^{univ}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [\overline{X}] \otimes b_{i-1}.$$

where the classes $c_i \in H^{2i}_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l)$, $a_i^{(j)} \in H^{2i-1}_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l)$ and $b_{i-1} \in H^{2(i-1)}_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l)$ are the so-called *Atiyah-Bott classes.*

We can now state and prove the main theorem of this chapter about the l-adic cohomology algebra of the moduli stack $\mathscr{B}un_X^{n,d}$.

Historically, first the l-adic Betti numbers of the coarse moduli space of stable bundles of coprime rank n and degree d on an algebraic curve over a finite field were determined by Harder and Narasimhan [HN75] using arithmetic techniques and the Weil Conjectures. Later Atiyah and Bott [AB83] calculated the Betti numbers in the complex analytic case for the coarse moduli space of stable bundles of coprime rank n and degree d on a compact Riemann surface using Yang-Mills gauge theory and equivariant Morse theory. A new algebro-geometric determination of the Betti numbers with the aim to compare and understand the two different calculations was given by Bifet, Ghione and Letizia [BGL94].

More recently, Heinloth [Hei98] determined the l-adic cohomology for the whole moduli stack $\mathscr{B}un_X^{n,d}$ and another determination of this cohomology algebra together with actions of the various Frobenius actions was given in [NS05], [NS05]. The proof we will present here is a mixture of the proof outlined in [Hei09], which is a variant of the proof in [Hei98], and the one in [NS05] and [NS].

Theorem 4.3. *Let X be a smooth projective irreducible algebraic curve of genus $g \geq 2$ over the field $\mathbb{F}_q$ and $\mathscr{B}un_X^{n,d}$ be the moduli stack of vector bundles of rank n and degree d on X. There is an isomorphism of graded $\mathbb{Q}_l$-algebras*

$$\begin{aligned} H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l) \cong\ & \mathbb{Q}_l[c_1, \ldots, c_n] \otimes \mathbb{Q}_l[b_1, \ldots, b_{n-1}] \\ & \otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}, \ldots, a_n^{(1)}, \ldots, a_n^{(2g)}). \end{aligned}$$

Proof. The proof has two steps. As a first step, we will show that the cohomology algebra of the moduli stack $\overline{\mathscr{B}un}_X^{n,d}$ contains the graded $\mathbb{Q}_l$-algebra of the right hand side, i.e. the Atiyah-Bott classes generate a free subalgebra of the cohomology algebra. This uses induction over the rank of vector bundles and reduction to closed substacks of vector bundles being direct sums of line bundles. In the second step we will calculate the Poincaré series of the stack $\overline{\mathscr{B}un}_X^{n,d}$ by "stackifying" the approach of [BGL94], where it is basically shown that the moduli stack $\overline{\mathscr{B}un}_X^{n,d}$ is quasi-isomorphic to a certain ind-scheme $\overline{Div}^{n,d}$ representing a moduli functor of effective divisors on X. This ind-scheme has the same Poincaré series as the $\mathbb{Q}_l$-algebra on the right hand side and therefore we will get an isomorphism of the two algebras.

First step. We do induction over the rank n. Let $n = 1$ and consider the moduli stack $\mathscr{B}un_X^{1,d}$, which classifies line bundles of degree d on X. A coarse moduli space for the stack $\mathscr{B}un_X^{1,d}$ is simply given by the Picard scheme Pic_X^d on X. There exists a Poincaré family on the product $X \times Pic_X^d$ (see [Ram73] or [DN89], Thm. G) and so we get a section of the $\mathbb{G}_m$-gerbe $\mathscr{B}un_X^{1,d} \to Pic_X^d$ and a splitting of algebraic stacks

$$\overline{\mathscr{B}un}_X^{1,d} \cong \overline{Pic}_X^d \times \overline{\mathscr{B}\mathbb{G}_m}.$$

The cohomology of the Picard scheme Pic_X^d is ismorphic to the Jacobian $Jac(X)$ of the algebraic curve X. But $Jac(X)$ is an abelian variety and therefore we have:

$$H^*_{sm}(\overline{Jac(X)}, \mathbb{Q}_l) \cong \Lambda_{\mathbb{Q}_l}(H^1_{et}(\overline{X}, \mathbb{Q}_l)) \cong \Lambda_{\mathbb{Q}_l}(\alpha_1, \ldots, \alpha_{2g})$$

and using the Künneth decomposition of l-adic cohomology we get:

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{1,d}, \mathbb{Q}_l) \cong H^*_{sm}(\overline{Pic}_X^d, \mathbb{Q}_l) \otimes H^*_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l).$$

And this therefore gives already the desired isomorphism of $\mathbb{Q}_l$-algebras in the case of rank one vector bundles:

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{1,d}, \mathbb{Q}_l) \cong \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}) \otimes \mathbb{Q}_l[c_1].$$

Now let $n > 1$ and consider the moduli stack $\mathscr{B}un_X^{n,d}$. Let

$$d = \sum_{i=1}^{n} d_i, \ d_i \in \mathbb{Z}; \ \underline{d} = (d_1, \ldots, d_n)$$

be an arbitrary partition of the degree d. We can consider the following morphism of algebraic stacks

$$\oplus_{\underline{d}} : \prod_{i=1}^{n} \mathscr{B}un_X^{1,d_i} \to \mathscr{B}un_X^{n,d}, \ (\mathcal{L}_i) \mapsto \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n.$$

and its induced homomorphism $(\oplus_{\underline{d}})^*$ in l-adic cohomology. Chern classes of direct sums of line bundles can be expressed as elementary symmetric polynomials σ_i in the Chern classes of the line bundles involved in the direct sum. Now for every $k = 1, \ldots, n$ let $\mathcal{L}_{d_k}^{univ}$ be the universal line bundle on the stack $\overline{X} \times \overline{\mathscr{B}un}_X^{1,d_k}$. Fixing a basis, we have the following Künneth decomposition of the Chern class $c_1(\mathcal{L}_{d_k}^{univ})$:

$$c_1(\mathcal{L}_{d_k}^{univ}) = 1 \otimes C_k + \sum_{j=1}^{2g} \alpha_j \otimes A_k^{(j)} + [\overline{X}] \otimes d_k.$$

Now we can describe the effect of the induced map $(\oplus_{\underline{d}})^*$ in l-adic cohomology on the Chern classes of the universal vector bundle $\mathcal{E}^{univ}$ on the stack $\overline{X} \times \overline{\mathscr{B}un}_X^{n,d}$.

$$\begin{aligned}(\oplus_{\underline{d}})^*(c_i(\mathcal{E}^{univ})) &= \sigma_i(c_1(\mathcal{L}_{d_1}^{univ}, \ldots, c_1(\mathcal{L}_{d_n}^{univ})) \\ &= \sigma_i(C_i, \ldots, C_n) + \sum_{r,s} \alpha_r \otimes \partial_s \sigma_i(C_1, \ldots, C_n) A_s^{(r)} \\ &+ \sum_{r+u=2g+1,s,t} [\overline{X}] \otimes \partial_s \partial_t \sigma_i(C_1, \ldots, C_n) A_s^{(r)} A_t^{(u)} \\ &+ \sum_{r,s} [\overline{X}] \otimes \partial_s \sigma_i(C_1, \ldots, C_n) d_s.\end{aligned}$$

We take the product of all homomorphisms $(\oplus_{\underline{d}})^*$ for all partitions $\underline{d} = (d_1, \ldots, d_n)$ with $\sum_i d_i = d$ and get a commutative diagram:

$$\begin{array}{ccc} H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}; \mathbb{Q}_l) & \longrightarrow & \prod_{\underline{d}} \mathbb{Q}_l[C_1, \ldots C_n] \otimes \bigotimes_{i,j}^{n,2g} \Lambda_{\mathbb{Q}_l}(A_i^{(j)}) \\ \uparrow \alpha & & \uparrow \psi \\ \mathbb{Q}_l[c_i] \otimes \Lambda_{\mathbb{Q}_l}(a_i^{(j)}) \otimes \mathbb{Q}_l[b_i] & \xhookrightarrow{\varphi} & \mathbb{Q}_l[C_i] \otimes \Lambda_{\mathbb{Q}_l}(A_i^{(j)}) \otimes \frac{\mathbb{Q}_l[D_1, \ldots D_n]}{(\sum_s D_s - d)} \end{array}$$

The algebra homomorphism φ is given on generators by

$$c_i \mapsto \sigma_i(C_1, \ldots C_n), \; a_i^{(j)} \mapsto \sum_s \partial_s \sigma_i(C_1, \ldots C_n) A_s^{(j)}$$

$$b_i \mapsto \sum_{r+u=2g+1,s,t} \partial_s \partial_t \sigma_i(C_1, \ldots, C_n) A_s^{(r)} A_t^{(u)} + \sum_{r,s} \partial_s \sigma_i(C_1, \ldots, C_n) D_s.$$

It follows that φ is a monomorphism, because the elementary symmetric polynomials σ_i and their derivatives $\partial_s \sigma_i$ are linearly independent. The linear independence of the derivatives basically follows from the fact that the map $\mathbb{A}^n \to \mathbb{A}^n/\Sigma_n$ is generically a Galois covering, i.e. an étale covering with the symmetric group Σ_n as Galois group.

The algebra homomorphism ψ is given by simultaneous evaluation of the variables D_s, i.e. $D_s \mapsto (d_1, \ldots, d_s, \ldots d_n)$ simultaneously at all integers d_s. Therefore ψ is also a monomorphism. Because both ϕ and ψ are monomorphisms it finally follows that the homomorphism α must be a monomorphism for all i, j and therefore that the Atiyah-Bott classes generate a free subalgebra of the cohomology algebra of the moduli stack.

Second step. We will now show that the cohomology algebra of the moduli stack is isomorphic to the algebra on the right hand side. In order to do so we determine the Poincaré series of both algebras and show that they are equal.

The Poincaré series of the graded $\mathbb{Q}_l$-algebra

$$Alg^*_{\mathbb{Q}_l} := \mathbb{Q}_l[c_1, \ldots, c_n] \otimes \mathbb{Q}_l[b_1, \ldots, b_{n-1}] \otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \ldots, a_1^{(2g)}, \ldots, a_n^{(1)}, \ldots, a_n^{(2g)}).$$

can be read off immediately. We have

$$P(Alg^*_{\mathbb{Q}_l}, t) = \sum_{k=0}^{\infty} \dim_{\mathbb{Q}_l}(Alg^k_{\mathbb{Q}_l}) t^k = \frac{\prod_{i=1}^n (1 + t^{2i-1})^{2g}}{\prod_{i=1}^n (1 - t^{2i}) \prod_{i=2}^n (1 - t^{2i-2})}$$

In order to calculate the Poincaré series of the cohomology algebra of the moduli stack, we approximate the moduli stack via a certain ind-scheme of effective divisors on X. For this we will "stackify" the approach of [BGL94].

Let Λ be a partially ordered set of effective divisors on the algebraic curve X. For any fixed divisor $D \in \Lambda$ we have a moduli functor

$$\mathcal{D}iv^{n,d}(D) : (Sch/\mathbb{F}_q)^{op} \to (Sets)$$

where $\mathcal{D}iv^{n,d}(D)(U)$ is given as the set of (n,d)-divisors, i.e. equivalence classes of inclusions $\mathcal{E} \hookrightarrow \mathcal{O}_{X\times U}(D)^{\oplus n}$ with $\mathcal{E}$ a family of rank n and degree d vector bundles on X parametrized by U [BGL94]. The moduli functor $\mathcal{D}iv^{n,d}(D)$ is representable by a Quot scheme $\mathrm{Div}^{n,d}(D)$ given as

$$\mathrm{Div}^{n,d}(D) := \mathrm{Quot}(\mathcal{O}_X(D)^{\oplus n}, s)$$

which parametrizes the torsion sheaf quotients of $\mathcal{O}_X(D)^{\oplus n}$ of degree $s = n \cdot \deg(D) - d$ [Gro95b], [BGL94]. These are smooth projective schemes and for every pair $D, D' \in \Lambda$ of effective divisors with $D \leq D'$ we have a closed immersion of Quot schemes

$$\mathrm{Div}^{n,d}(D) \hookrightarrow \mathrm{Div}^{n,d}(D').$$

We get a directed system of schemes $\{\mathrm{Div}^{n,d}(D), D\}$ and therefore get an ind-scheme $\mathrm{Div}^{n,d}$ by taking the direct limit

$$\mathrm{Div}^{n,d} := \varinjlim \mathrm{Div}^{n,d}(D).$$

Let $\underline{s} = (s_1, \ldots, s_n)$ be a partition of the integer $s = n \cdot \deg(D) - d$ by non-negative integers. The automorphism group of the vector bundle $\mathcal{O}_X(D)$ is given by $\mathbb{G}_m$. Therefore we obtain an action of the split maximal torus $T \subset GL_n$ of diagonal matrices on the sum $\mathcal{O}_X(D)^{\oplus n}$ on the scheme $\mathrm{Div}^{n,d}(D)$. The components of the fixed-point sets of this torus action correspond to line bundles of the form

$$\mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_n).$$

Taking cokernels of inclusions of bundles of this form into $\mathcal{O}_X(D)^{\oplus n}$ we get torsion sheaves on X. We can identify the fixed-point sets with products $H^{\underline{s}}$ of Hilbert schemes of points on X given as

$$H^{\underline{s}} = \mathrm{Hilb}(s_1, X) \times \cdots \times \mathrm{Hilb}(s_n, X)$$

and canonically embedded in $\mathrm{Div}^{n,d}(D)$ [Bif89], [BGL94]. Since X is 1-dimensional, these Hilbert schemes $\mathrm{Hilb}(s_k, X)$ of points are just given as symmetric powers $X^{(s_k)}$ of the algebraic curve X. From general results of fixed points of algebraic group actions [BB73], [BB74] and deformation theory it follows from this [BGL94], Prop. 4.2 that the l-adic cohomology of the ind-scheme $\mathrm{Div}^{n,d}$ stabilizes, i.e. if $D \leq D'$, then the homomorphism

$$H^i_{et}(\overline{\mathrm{Div}}^{n,d}(D'), \mathbb{Q}_l) \to H^i_{et}(\overline{\mathrm{Div}}^{n,d}(D), \mathbb{Q}_l)$$

induced by the closed immersion $\mathrm{Div}^{n,d}(D) \hookrightarrow \mathrm{Div}^{n,d}(D')$ is an isomorphism in the range $0 \leq i \leq n \cdot \deg(D) - d$, i.e. the cohomology is given as the inverse limit

$$H^*_{et}(\overline{\mathrm{Div}}^{n,d}, \mathbb{Q}_l) \cong \lim_{\leftarrow} H^*_{et}(\overline{\mathrm{Div}}^{n,d}(D), \mathbb{Q}_l).$$

And we also get that

$$H^*_{et}(\overline{\mathrm{Div}}^{n,d}(D), \mathbb{Q}_l) \cong \bigoplus_{\underline{s}} H^*_{et}(\overline{X^{(s_1)}} \times \cdots \times \overline{X^{(s_n)}}, \mathbb{Q}_l).$$

It follows also that the l-adic cohomology of $\mathrm{Div}^{n,d}(D)$ for every divisor is torsion free and the Poincaré series therefore can be calculated via the Poincaré series of the symmetric powers $X^{(s_k)}$ [Bif89], [BGL94]. But those Poincaré series can be calculated directly [Mac62] and from this it follows finally that (see again [BGL94], Prop. 4.2.)

$$\begin{aligned} P(H^*_{et}(\overline{\mathrm{Div}}^{n,d}, \mathbb{Q}_l), t) &= \frac{\prod_{i=1}^n (1+t^{2i-1})^{2g}}{\prod_{i=1}^n (1-t^{2i}) \prod_{i=2}^n (1-t^{2i-2})} \\ &= P(Alg^*_{\mathbb{Q}_l}, t) \end{aligned}$$

For every divisor $D \in \Lambda$ there is a canonical morphism of algebraic stacks

$$\mathrm{Div}^{n,d}(D) \to \mathscr{B}un^{n,d}_X$$

as geometric points in $\mathrm{Div}^{n,d}(D)$ are families of vector bundles of rank n and degree d on X parametrized by U. Therefore we get a morphism from the ind-scheme to the moduli stack

$$\mathrm{Div}^{n,d} \to \mathscr{B}un^{n,d}_X$$

inducing a homorphism in l-adic cohomology

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathrm{Div}}^{n,d}, \mathbb{Q}_l).$$

Using the Shatz stratification of both the ind-scheme $\mathrm{Div}^{n,d}$ and the moduli stack $\mathscr{B}un_X^{n,d}$, it follows from [Dhi06], Thm 4.6 that this homomorphism is in fact an isomorphism. Therefore the Poincaré series of the moduli stack is also given as

$$P(H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l), t) = P(Alg^*_{\mathbb{Q}_l}, t)$$

From both steps we can now conclude that we have an isomorphism of graded $\mathbb{Q}_l$-algebras

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l) \cong Alg^*_{\mathbb{Q}_l}$$

which finishes the proof of the theorem. □

As a benefit of the proof of the above theorem we get immediately a complete calculation of the l-adic cohomology of the ind-scheme $\mathrm{Div}^{n,d}$ extending the calculations of Betti numbers in [BGL94].

Corollary 4.4. *Let X be a smooth projective and irreducible algebraic curve of genus g over the field $\mathbb{F}_q$ and $\mathrm{Div}^{n,d}$ be the ind-scheme of (n, d)-divisors on X. There is an isomorphism of graded $\mathbb{Q}_l$-algebras*

$$\begin{aligned} H^*_{et}(\overline{\mathrm{Div}}^{n,d}, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \dots, c_n] \otimes \mathbb{Q}_l[b_1, \dots, b_{n-1}] \\ \otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \dots, a_1^{(2g)}, \dots, a_n^{(1)}, \dots, a_n^{(2g)}). \end{aligned}$$

Recently Heinloth and Schmitt [HS] determined also the l-adic cohomology of the moduli stack $\mathscr{B}un_{G,X}$ of principal G-bundles on a smooth projective algebraic curve X in arbitrary characteristic, where G is a reductive algebraic group. Again, they showed that the cohomology algebra is freely generated by the corresponding Atiyah-Bott characteristic classes. This reformulates and extends to algebraic stacks the fundamental results of Atiyah and Bott [AB83] in the complex analytic case on the cohomology of the coarse moduli space of principal G-bundles on a Riemann surface. In the complex analytic case a calculation of the cohomology of the moduli stack of principal G-bundles was already obtained by Teleman [Tel98], where in fact the homotopy type of this moduli stack is determined. This suggests another way of determining the l-adic cohomology algebra of the moduli stack of vector bundles and more generally the moduli stack of principal G-bundles using the machinery of étale homotopy theory [AM69], [Fri82] along similar lines as in the case of moduli stacks of algebraic curves with symmetries [FN03].

In [ADK08] Asok, Doran and Kirwan recently determined the motivic cohomology à la Voevodsky of the coarse moduli space of stable bundles of rank n and degree d on a smooth projective curve over an algebraically closed field and thereby unified and extended the different classical approaches to the natural setting of motivic cohomology. In fact, more generally they are able to calculate the equivariant motivic cohomology of GIT quotients. It would be interesting to extend this approach to the whole moduli stack and calculate its motivic cohomology or even its motivic homotopy type. This would need a good definition of motivic cohomology for algebraic stacks, which in the case of quotient stacks should correspond to equivariant motivic cohomology. Some other interesting results on motives of moduli stacks of vector bundles and principal G-bundled on curves were also recently obtained by Behrend and Dhillon [BD07].

Chapter 5

Moduli stacks of vector bundles II

5.1 A primer on the classical Weil Conjectures

In this section we will give a brief introduction into the fascinating circle of ideas behind the Weil Conjectures. The Weil Conjectures bring to light a deep connection between the arithmetic and topology of a smooth complex vartiety. The main aim here is to motivate the discussion on Frobenius morphisms and analogues of the Weil Conjectures for the moduli stack of vector bundles of algebraic curves in the last section of this chapter.

As a guide and reference for the discussion of the classical Weil Conjectures here we refer to the introduction by Dieudonné in [FK88] and a similar overview in [KW06].

Let $X \subseteq \mathbb{CP}^n$ be an m-dimensional smooth complex projective variety defined over an algebraic number ring R, for example let X be defined over the ring $R = \mathbb{Z}$ of integers. In other words, X can be defined as the zero locus of homogeneous polynomials with coefficients in the ring R. Let $\mathfrak{m}$ be a maximal ideal in R. Then $R/\mathfrak{m} \cong \mathbb{F}_q$ is a finite field of characteristic $\mathrm{char}(R/\mathfrak{m}) = p$ with $q = p^s$ elements, where s is a positive integer. For example, if $R = \mathbb{Z}$, we can take $\mathfrak{m} = p\mathbb{Z}$ for a prime number p and get the finite field $\mathbb{F}_p$.

We define a new projective variety $X_\mathfrak{m}$ over the field $\mathbb{F}_q$

$$X_\mathfrak{m} \subseteq \mathbb{F}_q\mathbb{P}^N = \frac{\mathbb{F}_q^{N+1}\backslash\{(0,\dots,0)\}}{\mathbb{F}_q\backslash\{0\}}$$

by reducing the equations defining X with coefficients in the ring R modulo $\mathfrak{m}$. Let $\overline{X}_\mathfrak{m}$ be the associated projective variety over the algebraic closure $\overline{\mathbb{F}}_q$ defined by the same equations as for $X_\mathfrak{m}$ but viewed over $\overline{\mathbb{F}}_q$.

We like to count the number $N_r = \#X(\mathbb{F}_{q^r})$ of $\mathbb{F}_{q^r}$-rational points of $\overline{X}_\mathfrak{m}$. The number $N_r = \#X(\mathbb{F}_{q^r})$ is basically given as the number of points in $\overline{X}_\mathfrak{m}$ of the form $(x_0 : \ldots : x_k)$ such that $x_j \in \mathbb{F}_{q^r}$ for all $j = 0,\dots,k$. It is clear from the definition that N_r is a finite number. We can define a generating function for the different numbers $N_1, N_2, N_3, \dots$ by:

Definition 5.1. *Let X be an m-dimensional smooth complex projective variety defined over an algebraic number ring R. The* zeta function *of the associated variety $\overline{X}_\mathfrak{m}$ is defined as*

$$\begin{aligned} Z(t) = Z(X_\mathfrak{m}, t) &= \exp(\sum_{r\geq 1} N_r \frac{t^r}{r}) \\ &= 1 + \sum_{r\geq 1} N_r \frac{t^r}{r} + \frac{1}{2!}(\sum_{r\geq 1} N_r \frac{t^r}{r})^2 + \ldots \in \mathbb{Q}[[t]] \end{aligned}$$

Let us look at a simple example, the complex projective space.

Example 5.2. Let $X = \mathbb{CP}^m$ be the m-dimensional complex projective space defined over the number ring $R = \mathbb{Z}$. Furthermore let $\mathfrak{m} = p\mathbb{Z}$, so $R/\mathfrak{m} = \mathbb{F}_p$ and $q = p$. Then we get for the number of $\mathbb{F}_{p^r}$-rational points via direct calculation

$$N_r = 1 + p^r + p^{2r} + \ldots p^{mr}$$

and therefore the zeta function is simply given as

$$\begin{aligned} Z(t) &= \exp(\sum_{r\geq 1}(1 + p^r + p^{2r} + \ldots p^{mr})\frac{t^r}{r}) \\ &= \frac{1}{(1-t)(1-pt)(1-p^2t)\cdots(1-p^mt)} \end{aligned}$$

André Weil suggested that there is a deep relation between the arithmetic and the topology of the complex projective variety X. He conjectured that the numbers N_r of $\mathbb{F}_{q^r}$-rational points of the complex projective variety X are related to the Betti numbers $\dim H_j(X, \mathbb{C})$ of X. The classical Weil Conjectures can be formulated as follows [Wei49].

Theorem 5.3 (Weil Conjectures)**.** *Let $Z(t) = Z(X,t)$ be the zeta function of an m-dimensional smooth complex projective variety X over an algebraic number ring R. Then*

$$Z(t) = \frac{P_1(t)P_3(t)\cdots P_{2m-1}(t)}{P_0(t)P_2(t)\cdots P_{2m}(t)}$$
$$= \prod_j \Big(\prod_{1\le i\le \dim H_j(X,\mathbb{C})} (1-\alpha_{ji}t)\Big)^{(-1)^{j+1}}$$

with $P_0(t) = 1-t$, $P_{2m} = 1 - q^m t$ and for $1 \le j \le 2m-1$

$$P_j(t) = \prod_{1\le i\le \dim H_j(X,\mathbb{C})} (1-\alpha_{ji}t)$$

where the α_{ji} are algebraic integers with $|\alpha_{ji}| = q^{\frac{j}{2}}$, i.e. the zeta function $Z(t)$ determines uniquely the polynomials $P_j(t)$ and hence the Betti numbers $\dim H_j(X, \mathbb{C}) = \deg P_j(t)$. *Let*

$$\chi = \chi(X) = \sum_j (-1)^j \dim H_j(X,\mathbb{C})$$

be the Euler characteristic *of the variety X. Then we have the following functional equation*

$$Z(\frac{1}{q^m t}) = \pm q^{\frac{\chi}{2}} t^{\chi} Z(t)$$

Proof. This is of course a very special case of the general Weil conjectures proved by Deligne [Del74a], [Del80] and the proof uses the full machinery of étale cohomology. For an overview of the proof and background material, especially on étale cohomology we refer also to [FK88] and [Mil80]. □

Weil was able to prove some special cases of these conjectures and realized that the general case would follow if one would be able to construct a suitable cohomology theory for algebraic varieties in positive characteristic, which would play an analogous role as singular cohomology for complex varieties in algebraic topology.

The Weil Conjectures can also be viewed as an analogue of the Riemann Hypothesis for algebraic curves. Let us discuss this here also briefly.

Let X again be an m-dimensional smooth complex projective variety defined over an algebraic number ring R and $\mathfrak{p}$ be a prime divisor of X, i.e. an equivalence class of points of $\overline{X}_{\mathfrak{m}}$ modulo conjugation over $\mathbb{F}_q$. We define the *norm* as

$$\mathrm{Norm}(\mathfrak{p}) = q^{\deg(\mathfrak{p})}$$

where the *degree* of the divisor $\mathfrak{p}$ is given as the number of points in the equivalence class of $\mathfrak{p}$. Because $\mathbb{F}_{q^i} \subseteq \mathbb{F}_{q^j}$ if and only if $i|j$ it follows immediately that

$$N_r = \#X(\mathbb{F}_{q^r}) = \sum_{\deg(\mathfrak{p})|r} \deg(\mathfrak{p}).$$

Substituting now $t = q^{-s}$ in the zeta function $Z(t) = Z(X, t)$ of the variety X we get

$$\begin{aligned} Z(q^{-s}) &= \exp(\sum_{r\geq 1} N_r \frac{q^{-rs}}{r}) \\ &= \exp(\sum_{r\geq 1} \sum_{\deg(\mathfrak{p})|r} \frac{\deg(\mathfrak{p})\mathrm{Norm}(\mathfrak{p})^{-rs/\deg(\mathfrak{p})}}{r}) \\ &= \exp(\sum_{\mathfrak{p}} \sum_{i} \frac{\mathrm{Norm}(\mathfrak{p})^{-si}}{i}) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathrm{Norm}(\mathfrak{p})^{-s}} \end{aligned}$$

In this way the zeta function of X looks very much like the classical Riemann zeta function given as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p\,\mathrm{prime}} \frac{1}{1 - p^{-s}}$$

The Riemann Hypothesis says that if $\zeta(s) = 0$, then $Re(s) = \frac{1}{2}$. We can now see the Weil Conjectures as an analogue of the Riemann Hypothesis by looking at the particular case of an algebraic curve. Let X be a complex smooth projective algebraic curve, i.e. $\dim_{\mathbb{C}}(X) = 1$. Then we get as a special case the following expression for the zeta function of X

$$
\begin{aligned}
Z(t) &= \prod_j \Big(\prod_{1 \le i \le \dim H_j(X,\mathbb{C})} (1 - \alpha_{ji} t) \Big)^{(-1)^{j+1}} \\
&= \Big(\prod_{1 \le i \le \dim H_1(X,\mathbb{C})} (1 - \alpha_{1i}) \Big) (1-t)^{-1} (1-qt)^{-1}
\end{aligned}
$$

So $|\alpha_{1i}| = q^{\frac{1}{2}}$ implies that if $Z(t) = 0$, then $|t| = q^{\frac{1}{2}}$ or equivalently, that if $Z(q^{-s}) = 0$, then $Re(s) = \frac{1}{2}$, which is the analogue of the Riemann Hypothesis.

The question now is to understand how the numbers α_{ij} are related to the complex variety X. We have a special automorphism of the variety $\overline{X}$ over $\mathbb{F}_q$, the *Frobenius morphism.*

Definition 5.4. *Let X be an m-dimensional smooth complex projective variety defined over an algebraic number ring R and $\mathfrak{m}$ be a maximal ideal in R. Let $\overline{X}_{\mathfrak{m}}$ be the associated projective variety over the algebraic closure $\overline{\mathbb{F}}_q$. The* Frobenius morphism *f is defined as the morphism*

$$f : \overline{X}_{\mathfrak{m}} \to \overline{X}_{\mathfrak{m}}, \; f(x_0 : \ldots : x_k) = (x_0^q : \ldots : x_k^q).$$

It is obvious that f is well-defined, because basically polynomial equations defining $\overline{X}_{\mathfrak{m}}$ have coefficients in $\mathbb{F}_q$ and if $p(X_0, \ldots, X_k) \in \mathbb{F}_q[X_0, \ldots, X_k]$ then it follows

$$p(X_0^q, \ldots, X_k^q) = p(X_0, \ldots, X_k)^q.$$

Therefore we can iterate the Frobenius morphism f for any $r \geq 1$ and we get that a point x of $\overline{X}_{\mathfrak{m}}$ is a fixed point of f^r, $r \geq 1$ if and only if all coordinates of the point x are in $\mathbb{F}_{q^r}$.

Let $L(f^r)$ be the number of fixed points of the iterated Frobenius morphism f^r. It follows immediately from the definition that we have

$$N_r = \#X(\mathbb{F}_{q^r}) = L(f^r).$$

The number $L(f^r)$ can be calculated algebraically via a Lefschetz trace formula for the l-adic étale cohomology of the variety $\overline{X}/\mathfrak{m}$.

Theorem 5.5 (Grothendieck, Deligne)**.** *Let X be an m-dimensional smooth complex projective variety defined over an algebraic number ring R and $\mathfrak{m}$ be a maximal ideal in R. Let $\overline{X}_{\mathfrak{m}}$ be the associated projective variety over the algebraic closure $\overline{\mathbb{F}}_q$ and $f : \overline{X}_{\mathfrak{m}} \to \overline{X}_{\mathfrak{m}}$ be the Frobenius morphism. Then the number of fixed points of the iterated Frobenius morphism f^r can be calculated via the Lefschetz trace formula*

$$L(f^r) = \sum_{0 \leq j \leq 2m} (-1)^j Tr((f^r)^* : H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l) \to H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l))$$

where l is a prime with $l \neq p$.

The cohomology used in the trace formula is the *l-adic étale cohomology* of the projective variety X

$$H^*_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l) := \lim_{\leftarrow} H^*_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Z}/l^n\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

It gives the "right" cohomology theory for algebraic varieties over finite fields analogous to the rational singular cohomology $H^*(X, \mathbb{Q})$ of topological spaces. This cohomology theory was introduced by Grothendieck, Artin and Deligne in SGA4 and gives the necessary cohomological framework to prove the Weil Conjectures. We refer to [Tam94] and [Mil80] for an introduction to étale cohomology and to SGA $4\frac{1}{2}$ [Del77] for a general account.

The Lefschetz trace formula and a bit of linear algebra now gives the following reinterpretation of the zeta function:

$$\begin{aligned}
Z(t) &= \exp(\sum_{r\geq 1} N(r)\frac{t^r}{r}) = \exp(\sum_{r \geq 1} L(f^r)\frac{t^r}{r}) \\
&= \exp(\sum_{r\geq 1}\sum_{0\leq j\leq 2m} (-1)^j Tr((f^r)^* : H^j_{et}(\overline{X}_{\mathfrak{m}}\mathbb{Q}_l) \to H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l))\frac{t^r}{r}) \\
&= \prod_{j=0}^{2m} \det(1 - tf^* : H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l) \to H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l))^{(-1)^{j+1}} \\
&= \frac{P_1(t)P_3(t)\cdots P_{2m-1}(t)}{P_0(t)P_2(t)\cdots P_{2m}(t)}
\end{aligned}$$

and with α_{ji} being the eigenvalues of the induced Frobenius morphism f^* in l-adic cohomology we get

$$P_j(t) = \det(1 - tf^* : H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l) \to H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l)) = \prod_{1 \le i \le \dim H^j_{et}(\overline{X}_{\mathfrak{m}}, \mathbb{Q}_l)} (1 - \alpha_{ji} t)$$

In the next sections we will indicate how to prove an analogue of the classical Weil Conjectures for the moduli stack of vector bundles of rank n and degree d on a smooth projective algebraic curve X.

5.2 Frobenius morphisms for the moduli stack

We like to classify vector bundles on a given algebraic curve X in characteristic p up to their isomorphisms. In order to do so we need to count the number of isomorphism classes of these vector bundles, i.e. we need to determine the number of $\mathbb{F}_q$-rational points of the moduli stack $\mathscr{B}un_X^{n,d}$. First we will consider the various Frobenius morphisms of the moduli stack $\mathscr{B}un_X^{n,d}$ following [NS05].

Definition 5.6. *Let X be a smooth projective algebraic curve of genus g over the field $\mathbb{F}_q$. The geometric Frobenius morphism of X is defined as the morphism of schemes given by*

$$F_X : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X),\ F_X = (id_X, f \mapsto f^q).$$

We also get a geometric Frobenius on $\overline{X} = X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ via base change

$$\overline{F}_X = F_X \times id_{\mathrm{Spec}(\overline{\mathbb{F}}_q)} : \overline{X} \to \overline{X}.$$

Now pullback along the geometric Frobenius morphism $\overline{F}_X$ of the algebraic curve $\overline{X}$ induces a functor given by

$$\overline{\mathscr{B}un}_X^{n,d}(U) \to \overline{\mathscr{B}un}_X^{n,d}(U),\ \mathcal{E} \mapsto \overline{F}^*(\mathcal{E}) = (\overline{F}_X \times id_U)^*(\mathcal{E})$$

for every object U of the category $(Sch/\mathbb{F}_q)$ of schemes over $\mathbb{F}_q$. It induces an endomorphism of algebraic stacks

$$\varphi : \overline{\mathscr{B}un}_X^{n,d} \to \overline{\mathscr{B}un}_X^{n,d}.$$

We will call this endomorphism the *induced geometric Frobenius morphism* of $\overline{\mathscr{B}un}_X^{n,d}$.

The induced geometric Frobenius morphism on $\overline{\mathscr{B}un}_X^{n,d}$ induces an endormorphism in l-adic cohomology

$$\Phi = \varphi^* : H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}\mathbb{Q}_l) \to H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d};\mathbb{Q}_l).$$

The relation between the geometric Frobenius morphism of the algebraic curve $\overline{X}$ and the induced geometric Frobenius morphism is given by the following functorial property.

Proposition 5.7. *There is a canonical isomorphism*

$$(\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(\mathcal{E}^{univ}) \cong (id_{\overline{X}} \times \varphi)^*(\mathcal{E}^{univ})$$

where $\mathcal{E}^{univ}$ *is the universal vector bundle on* $\overline{X} \times \overline{\mathscr{B}un}_X^{n,d}$.

Proof. Using representability we know that for any algebraic stack $\mathscr{T}$ over $\overline{\mathbb{F}}_q$ a vector bundle $\mathcal{E}$ of rank n and degree d on $\overline{X} \times \mathscr{T}$ is given by a morphism of stacks

$$u : \mathscr{T} \to \overline{\mathscr{B}un}_X^{n,d}$$

such that

$$\mathcal{E} \cong (id_{\overline{X}} \times u)^*(\mathcal{E}^{univ}).$$

We apply this to the vector bundle $(\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(\mathcal{E}^{univ})$ and the proposition follows at once. □

The moduli stack $(\mathscr{B}un_X^{n,d}, \mathcal{O}_{\mathscr{B}un_X^{n,d}})$ is an algebraic stack with structure sheaf. We can define a *genuine geometric Frobenius morphism* by raising sections to the q-the power as in the classical case of the geometric Frobenius morphism for schemes using the atlas of the algebraic stack $\mathscr{B}un_X^{n,d}$.

In this way we get an endormorphism of algebraic stacks

$$F_{\mathscr{B}un_X^{n,d}} : (\mathscr{B}un_X^{n,d}, \mathcal{O}_{\mathscr{B}un_X^{n,d}}) \to (\mathscr{B}un_X^{n,d}, \mathcal{O}_{\mathscr{B}un_X^{n,d}})$$

together with its base change extension

$$\overline{F}_{\mathscr{B}un_X^{n,d}} = F_{\mathscr{B}un_X^{n,d}} \times id_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}.$$

This genuine geometric Frobenius morphism induces again an endomorphism in l-adic cohomology

$$\overline{F}^*_{\mathscr{B}un_X^{n,d}} : H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}; \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}; \mathbb{Q}_l)$$

There is yet another Frobenius morphism acting on the moduli stack $\mathscr{B}un_X^{n,d}$. Let

$$\mathrm{Frob} : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \ a \mapsto a^q$$

be the classical Frobenius morphism given by a generator of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ of the field extension $\overline{\mathbb{F}}_q/\mathbb{F}_q$. This induces an endomorphism of schemes

$$\mathrm{Frob}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)} : \mathrm{Spec}(\overline{\mathbb{F}}_q) \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$$

which again gives rise to an endomorphism of algebraic stacks

$$\psi := id_{\mathscr{B}un_X^{n,d}} \times \mathrm{Frob}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)} : \overline{\mathscr{B}un}_X^{n,d} \to \overline{\mathscr{B}un}_X^{n,d}.$$

We will call this endomorphism ψ of algebraic stacks the *arithmetic Frobenius morphism.*

The arithmetic Frobenius morphism induces again an endomorphism in l-adic cohomology

$$\Psi = \psi^* : H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}; \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}; \mathbb{Q}_l).$$

We will now determine the actions of the various Frobenius morphisms on the l-adic cohomology algebra of the moduli stack $\overline{\mathscr{B}un}_X^{n,d}$ and express its effect on the Chern classes. It turns out that using the language of algebraic stacks these calculations are straightforward and topological in flavour.

For the induced geometric Frobenius morphism φ the action in l-adic cohomology of $\overline{\mathscr{B}un}_X^{n,d}$ is described by the following theorem [NS05].

Theorem 5.8. *The geometric Frobenius φ^* acts on the l-adic cohomology algebra as follows:*

$$\varphi^*(c_i) = c_i \ \ (i \geq 1)$$

$$\varphi^*(a_i^{(j)}) = \lambda_j a_i^{(j)} \ \ (i \geq 1; j = 1, \ldots, 2g)$$
$$\varphi^*(b_i) = q b_i \ \ (i \geq 1)$$

where the $a_i^{(j)}, b_i$ *and* c_i *are the Atiyah-Bott classes generating the cohomology algebra.*

Proof. Using functoriality of Chern classes we get the following equality

$$(\overline{F}_X \times id_{\overline{\mathscr{B}un_X^{n,d}}})^*(c_i(\mathcal{E}^{univ})) = (id_{\overline{X}} \times \varphi)^*(c_i(\mathcal{E}^{univ}))$$

Using Künneth decomposition for the Chern classes $c_i(\mathcal{E}^{univ})$ of the universal vector bundle $\mathcal{E}^{univ}$ we have

$$c_i(\mathcal{E}^{univ}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [\overline{X}] \otimes b_{i-1}.$$

where the classes $c_i \in H_{sm}^{2i}(\overline{\mathscr{B}un_X^{n,d}}; \mathbb{Q}_l)$, $a_i^{(j)} \in H_{sm}^{2i-1}(\overline{\mathscr{B}un_X^{n,d}}; \mathbb{Q}_l)$ and $b_{i-1} \in H_{sm}^{2(i-1)}(\overline{\mathscr{B}un_X^{n,d}}; \mathbb{Q}_l)$ are again the Atiyah-Bott classes.

Evaluating the two expressions from the equality above we get the following two equations

$$(\overline{F}_X \times id_{\overline{\mathscr{B}un_X^{n,d}}})^*(c_i(\mathcal{E}^{univ})) = 1 \otimes c_i + \sum_{j=1}^{2g} \lambda_j \alpha_j \otimes a_i^{(j)} + q[\overline{X}] \otimes b_{i-1}$$

$$(id_{\overline{X}} \times \varphi)^*(c_i(\mathcal{E}^{univ})) = 1 \otimes \varphi^*(c_i) + \sum_{j=1}^{2g} \alpha_j \otimes \varphi^*(a_i^{(j)}) + [\overline{X}] \otimes \varphi^*(b_{i-1}).$$

Comparing coefficients of the right hand sides of the two equations finally gives the desired action of the induced geometric Frobenius morphism φ^* on the l-adic cohomology algebra of $\mathscr{B}un_X^{n,d}$. □

For the genuine geometric Frobenius morphism $\overline{F}$ the action in l-adic cohomology of $\overline{\mathscr{B}un_X^{n,d}}$ is described by the following theorem [NS05].

Theorem 5.9. *The geometric Frobenius* $\overline{F}^*_{\mathscr{B}un_X^{n,d}}$ *acts on the* l*-adic cohomology algebra as follows:*

$$\overline{F}^*_{\mathscr{B}un_X^{n,d}}(c_i) = q^i c_i \ \ (i \geq 1)$$

$$\overline{F}^*_{\mathscr{B}un_X^{n,d}}(a_i^{(j)}) = \lambda_j^{-1} q^i a_i^{(j)} \ \ (i \geq 1; j = 1, \dots, 2g)$$
$$\overline{F}^*_{\mathscr{B}un_X^{n,d}}(b_i) = q^{i-1} b_i \ \ (i \geq 1)$$

where the $a_i^{(j)}, b_i$ *and* c_i *are the Atiyah-Bott classes generating the cohomology algebra.*

Proof. Let $\tilde{\mathcal{E}}^{univ}$ be the universal rank n vector bundle over the classifying stack $\mathscr{B}GL_n$ of rank n vector bundles.

The universal vector bundle $\mathcal{E}^{univ}$ of rank n and degree d over the algebraic stack $X \times \mathscr{B}un_X^{n,d}$ is given via representability by a classifying morphism

$$u : X \times \mathscr{B}un_X^{n,d} \to \mathscr{B}GL_n$$

with $u^*(\tilde{\mathcal{E}}^{univ}) \cong \mathcal{E}^{univ}$. We have the following pullback diagram related to the actions of the geometric Frobenius morphisms

$$\begin{array}{ccc} X \times \mathscr{B}un_X^{n,d} & \xrightarrow{u} & \mathscr{B}GL_n \\ {\scriptstyle F_{X \times \mathscr{B}un_X^{n,d}}} \downarrow & & \downarrow {\scriptstyle F_{\mathscr{B}GL_n}} \\ X \times \mathscr{B}un_X^{n,d} & \xrightarrow{u} & \mathscr{B}GL_n \end{array}$$

which gives the following commutativity law for the geometric Frobenius morphisms:

$$F_{\mathscr{B}GL_n} \circ u = u \circ F_{X \times \mathscr{B}un_X^{n,d}}.$$

Evaluating the geometric Frobenius morphism $\overline{F}_{X \times \mathscr{B}un_X^{n,d}}$ on the pullback of the universal vector bundle gives now:

$$\begin{aligned} (\overline{F}_{X \times \mathscr{B}un_X^{n,d}})^*(c_i(\overline{u}^*(\mathcal{E}^{univ})) &= (\overline{F}_{X \times \mathscr{B}un_X^{n,d}})^*(\overline{u}^*(c_i(\tilde{\mathcal{E}}^{univ}))) \\ &= \overline{u}^*(\overline{F}^*_{\mathscr{B}GL_n}(c_i(\tilde{\mathcal{E}}^{univ}))) \\ &= q^i \overline{u}^*(c_i(\tilde{\mathcal{E}}^{univ})) \\ &= q^i c_i(\overline{u}^*(\tilde{\mathcal{E}}^{univ})) \\ &= q^i c_i(\mathcal{E}^{univ}) \\ &= q^i(1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [\overline{X}] \otimes b_{i-1}). \end{aligned}$$

We have also the following general relation between the geometric Frobenius morphisms evaluated on the Chern classes:

$$\begin{aligned}
&\overline{F}^*_{X\times \mathscr{B}un_X^{n,d}}(c_i(\overline{u}^*(\tilde{\mathcal{E}}^{univ}))) \\
&\qquad = \overline{F}^*_{X\times \mathscr{B}un_X^{n,d}}(c_i(\mathcal{E}^{univ})) \\
&\qquad = (\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(id_{\overline{X}} \times \overline{F}_{\mathscr{B}un_X^{n,d}})^*(c_i(\mathcal{E}^{univ}))
\end{aligned}$$

From this we get now the following expression:

$$\begin{aligned}
&(\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(id_{\overline{X}} \times \overline{F}_{\mathscr{B}un_X^{n,d}})^*(c_i(\mathcal{E}^{univ})) \\
&= (id_{\overline{X}} \times \overline{F}_{\mathscr{B}un_X^{n,d}})^*(\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(c_i(\mathcal{E}^{univ})) \\
&= (id_{\overline{X}} \times \overline{F}_{\mathscr{B}un_X^{n,d}})^*(\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(1\otimes c_i + \\
&\qquad\qquad + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [\overline{X}] \otimes b_{i-1}) \\
&= (id_{\overline{X}} \times \overline{F}_{\mathscr{B}un_X^{n,d}})^*(\overline{F}_X \times id_{\overline{\mathscr{B}un}_X^{n,d}})^*(1\otimes c_i + \\
&\qquad\qquad + \sum_{j=1}^{2g} \lambda_j\alpha_j \otimes a_i^{(j)} + q[\overline{X}] \otimes b_{i-1})
\end{aligned}$$

Comparing coefficients again gives the description of the action of the geometric Frobenius morphism as stated in the theorem. □

Finally we can also analyze the effect of the arithmetic Frobenius morphism Ψ^* on the l-adic cohomology algebra of $\overline{\mathscr{B}un}_X^{n,d}$. The action is completely described by the following theorem [NS05].

Theorem 5.10. *The arithmetic Frobenius Ψ^* acts on the l-adic cohomology algebra as follows:*

$$\begin{aligned}
\psi^*(c_i) &= q^{-i}c_i \ \ (i \geq 1) \\
\psi^*(a_i^{(j)}) &= \lambda_j q^{-i} a_i^{(j)} \ \ (i\geq 1; j = 1,\dots,2g) \\
\psi^*(b_i) &= q^{-i+1}b_i \ \ (i\geq 1)
\end{aligned}$$

where the $a_i^{(j)}, b_i$ and c_i are the Atiyah-Bott classes generating the cohomology algebra.

Proof. The relations follow from the theorem before by observing that the arithmetic Frobenius morphism ψ^* is the inverse of the geometric Frobenius morphism $\overline{F}^*_{\mathscr{B}un_X^{n,d}}$. □

5.3 Weil Conjectures for the moduli stack

In order to determine the number of $\mathbb{F}_q$-rational points of the moduli stack $\mathscr{B}un_X^{n,d}$ we need a Lefschetz trace formula for the arithmetic Frobenius Ψ like in the classical case. In this section we will now discuss the trace formula and analogues of the Weil Conjectures for the moduli stack $\mathscr{B}un_X^{n,d}$.

Definition 5.11. *Let $\mathscr{X}$ be an algebraic stack over the category (Sch/S) of S-schemes and U an object of (Sch/S). Let $[\mathscr{X}(U)]$ be the* set of isomorphism classes *of objects in the groupoid $\mathscr{X}(U)$. If convergent, let*

$$\#\mathscr{X}(U) = \sum_{x\in[\mathscr{X}(U)]} \frac{1}{\#\mathrm{Aut}_{\mathscr{X}(U)}(x)}$$

be the number of U-points of the algebraic stack $\mathscr{X}$.

We can also define the dimension of an algebraic stack in analogy to the dimension of schemes as follows:

Definition 5.12. *Let $\mathscr{X}$ be an algebraic stack and $X/\mathscr{X}$ be an atlas of $\mathscr{X}$ i.e. a representable smooth surjective morphism $x : X \to \mathscr{X}$. The* dimension *of $\mathscr{X}$ is defined as*

$$\dim(\mathscr{X}) = \dim(X) - \mathrm{rel.dim}(X/\mathscr{X})$$

where rel.dim$(X/\mathcal{X})$ *is the dimenson of the fibers of $X \times_{\mathscr{X}} Y \to Y$ for any morphism $Y \to \mathscr{X}$*

The dimension of an algebraic stack $\mathscr{X}$ can be determined in the fundamental examples we have discussed before.

Example 5.13. Let $[X/G]$ be a quotient stack, then we have

$$\dim([X/G]) = \dim(X) - \dim(G)$$

and especially it follows for the special case of the classifying stack $\mathscr{B}G$ that

$$\dim(\mathscr{B}G) = -\dim(G)$$

which shows that the dimension of an algebraic stack can actually be a negative integer.

Example 5.14. Let $\mathscr{B}un_X^{n,d}$ be the moduli stack of vector bundles of rank n and degree d on a smooth projective algebraic curve X, then we have for the dimension

$$\dim(\mathscr{B}un_X^{n,d}) = n^2(g-1).$$

To count the number of $\mathbb{F}_q$-rational points of the moduli stack $\mathscr{B}un_X^{n,d}$ we will need the following Lefschetz trace formula for algebraic stacks [Beh93], [Beh03].

Theorem 5.15 (Lefschetz trace formula)**.** *Let $\mathscr{X}$ be a smooth algebraic stack locally of finite type and Ψ be the arithmetic Frobenius morphism, then we have*

$$q^{\dim(\mathscr{X})} \sum_{p \geq 0} tr(\Psi | H^p_{sm}(\overline{\mathscr{X}}; \mathbb{Q}_l)) = \sum_{x \in [\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#\mathrm{Aut}_{\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)}.$$

Here the expression on the right hand side

$$\sum_{x \in [\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#\mathrm{Aut}_{\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \#\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))$$

is the number $\#\mathscr{X}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of the algebraic stack $\mathscr{X}$, where $\#\mathrm{Aut}_{\mathscr{X}(\mathrm{Spec}(\mathbb{F}_q))}(x)$ is the order of the group of automorphisms of the isomorphism class x.

We cannot give a proof of the Lefschetz trace formula here, but instead look at an interesting example. For a proof and further applications see [Beh93], [Beh03].

Example 5.16. Let $\mathscr{B}\mathbb{G}_m$ be the classifying stack of line bundles. For the dimension of this algebraic stack we have

$$\dim(\mathscr{B}\mathbb{G}_m) = -\dim(\mathbb{G}_m) = -1$$

The number $\#\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$ of $\mathbb{F}_q$-rational points of $\mathscr{B}\mathbb{G}_m$ is given as the number of line bundles (up to isomorphism) over the "point" $\mathrm{Spec}(\mathbb{F}_q)$. But all line bundles over the "point" $\mathrm{Spec}(\mathbb{F}_q)$ are trivial, so there is just one isomorphism class x in $\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))$.

Furthermore we have

$$\#\mathrm{Aut}_{\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x) = \#\mathbb{G}_m(\mathbb{F}_q) = \#\mathbb{F}_q^* = q-1$$

so in other words we have that

$$\#\mathscr{B}\mathbb{G}_m(\mathbb{F}_q) = \sum_{x \in [\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))]} \frac{1}{\#\mathrm{Aut}_{\mathscr{B}\mathbb{G}_m(\mathrm{Spec}(\mathbb{F}_q))}(x)} = \frac{1}{q-1}.$$

The l-adic cohomology of $\mathscr{B}\mathbb{G}_m$ is basically the cohomology of an infinite projective space, as we calculated already, so from the Lefschetz trace formula we get

$$q^{\dim(\mathscr{B}\mathbb{G}_m)} \sum_{i \geq 0} tr(\Psi | H^{2i}_{sm}(\overline{\mathscr{B}\mathbb{G}_m}; \mathbb{Q}_l)) = \frac{1}{q} \sum_{i=0}^{\infty} \frac{1}{q^i}.$$

This simple calculation therefore gives a "stacky" proof for the well-known formula

$$\sum_{i=0}^{\infty} \frac{1}{q^{i+1}} = \frac{1}{q-1}.$$

We now state an analogue of the Weil Conjectures for the moduli stack $\mathscr{B}un_X^{n,d}$ [Hei98], [NS].

Theorem 5.17 (Weil Conjectures for the moduli stack)**.** *Let X be a smooth projective irreducible algebraic curve of genus g over $\mathbb{F}_q$ and α_i the eigenvalues of the geometric Frobenius acting on $H^1_{et}(\overline{X}, \mathbb{Q}_l)$. Then we have*

(1) The number of $\mathbb{F}_q$-rational points of the moduli stack $\mathscr{B}un_X^{n,d}$ is given as

$$\#\mathscr{B}un_X^{n,d}(\mathbb{F}_q) = q^{n^2(g-1)} \frac{\prod_{i=1}^{n} \prod_{j=1}^{2g} (1 - \alpha_j q^{-i})}{\prod_{i=1}^{n} (1 - q^{-i}) \prod_{i=2}^{n} (1 - q^{-i+1})}$$

(2) The zeta function of the moduli stack $\mathscr{B}un_X^{n,d}$

$$Z_{\mathscr{B}un_X^{n,d}}(t) = \exp(\sum_{i=1}^{\infty} \#\mathscr{B}un_X^{n,d}(\mathbb{F}_{q^i}) \frac{t^i}{i})$$

is a meromorphic function with a convergent product expansion

$$Z_{\mathscr{B}un_X^{n,d}}(t) = \prod_{i=1}^{\infty} \det(1 - \Psi q^{\dim(\mathscr{B}un_X^{n,d})} t | H^i_{sm}(\overline{\mathscr{B}un}_X^{n,d}; \mathbb{Q}_l))^{(-1)^{i+1}}.$$

(3) The eigenvalues of the arithmetic Frobenius Ψ have absolute value $q^{i/2}$ and the Poincaré series is given as

$$P_{\mathscr{B}un_X^{n,d}}(t) = \frac{\prod_{i=1}^n (1+t^{2i-1})^{2g}}{\prod_{i=1}^n (1-t^{2i}) \prod_{i=2}^n (1-t^{2i-2})}$$

Proof. We will only outline the ingredients for the proof here. For more details we refer to [Hei98] and [NS].

Part (1). The first part is basically a variation of arguments by Harder and Narasimhan [HN75]. To calculate the number of $\mathbb{F}_q$-rational points we "stackify" the calculation in [HN75] and use the Lefschetz trace formula [Beh93] for the arithmetic Frobenius Ψ acting on the moduli stack of vector bundles of rank n and degree d on X. This takes into account the existence of automorphisms of vector bundles in contrast with the analogous calculation for the coarse moduli space in [HN75].

$$q^{\dim(\mathscr{X})} \sum_{p \geq 0} tr(\Psi | H^p_{sm}(\overline{\mathcal{X}}, \mathbb{Q}_l)) = \sum_{x \in [\mathscr{X}(\mathbb{F}_q)]} \frac{1}{\#\mathrm{Aut}_{\mathscr{X}(\mathbb{F}_q)}(x)}$$

where for $\mathscr{X} = \mathscr{B}un_X^{n,d}$ we have $\dim(\mathscr{B}un_X^{n,d}) = n^2(g-1)$.

Part (2). The product expansion of the zeta function is proved generally by [Beh93] for algebraic stacks $\mathscr{X}$ of finite type using the Lefschetz trace formula. Now let $\mathscr{B}un_X^{n,d,\leq p} \subset \mathscr{B}un_X^{n,d}$ be the open substack where $\mathscr{B}un_X^{n,d,\leq p}(U)$ is the groupoid of families of vector bundles $\mathcal{E}$ over X of rank n and degree d parametrized by U with given Shatz polygon $sh(\mathcal{E}|X \times u) \leq p$ for all closed points u of the scheme U in $(Sch/\mathbb{F}_q)$. Similarly we have the open substack $\mathscr{B}un_X^{n,d,<p} \subset \mathscr{B}un_X^{n,d}$. The complement

$$\mathscr{B}un_X^{n,d,p} \subset \mathscr{B}un_X^{n,d,\leq p} \backslash \mathscr{B}un_X^{n,d,<p}$$

is a closed substack of finite type. The Gysin sequence in l-adic cohomology for the closed embedding

$$\mathscr{B}un_X^{n,d,p} \subset \mathscr{B}un_X^{n,d,\leq p}$$

splits into short exact sequences [Hei98], [NS05]

$$0 \to H^{*-2c}_{sm}(\overline{\mathscr{B}un}^{n,d,p}_X, \mathbb{Q}_l(m)) \to H^*_{sm}(\overline{\mathscr{B}un}^{n,d,\leq p}_X, \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathscr{B}un}^{n,d,<p}_X, \mathbb{Q}_l) \to 0$$

and the Lefschetz trace formula gives for the traces

$$q^{\dim(\mathscr{B}un^{n,d}_X)} tr(\Psi | H^*_{sm}(\overline{\mathscr{B}un}^{n,d}_X, \mathbb{Q}_l)) = = \sum_p q^{\dim(\mathscr{B}un^{n,d,p}_X)} tr(\Psi | H^*_{sm}(\overline{Bun}^{n,d,p}_X \mathbb{Q}_l)).$$

from which the product expansion now follows having reduced it to substacks of finite type.

Part (3.) The Poincaré series for the moduli stack was derived in the last chapter by using the approximation via the ind-scheme $\mathrm{Div}^{n,d}$ and the action of the arithmetic Frobenius on the l-adic cohomology algebra of the moduli stack was determined in the last section. □

It is an interesting question to ask if some kind of analogue of the Weil Conjectures also holds for the geometric Frobenius morphisms. It turns out that a naive Lefschetz trace formula does not hold [NS05]. The geometry of the actions of the geometric Frobenius morphisms on the moduli stack is very mysterious. See [NS05], [NS] for more discussions on the geometry of geometric Frobenius actions on the moduli stack.

Another open problem is to proof the Weil Conjectures for the action of the arithmetic Frobenius morphism on any algebraic stack of finite type over finite fields using the general machinery developed by Deligne [Del74a], [Del80] for his proof of the Weil Conjectures for algebraic varieties over finite fields.

Bibliography

[AB83] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[ADK08] Aravind Asok, Brent Doran, and Frances Kirwan. Yang-Mills theory and Tamagawa numbers: the fascination of unexpected links in mathematics. *Bull. Lond. Math. Soc.*, 40(4):533–567, 2008.

[AM69] M. Artin and B. Mazur. *Etale homotopy.* Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin, 1969.

[Art71] Michael Artin. *Algebraic spaces.* Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 3.

[Art74] M. Artin. Versal deformations and algebraic stacks. *Invent. Math.*, 27:165–189, 1974.

[BB73] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[BB74] A. Bialynicki-Birula. On fixed points of torus actions on projective varieties. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 22:1097–1101, 1974.

[BD07] Kai Behrend and Ajneet Dhillon. On the motivic class of the stack of bundles. *Adv. Math.*, 212(2):617–644, 2007.

[Beh93] Kai A. Behrend. The Lefschetz trace formula for algebraic stacks. *Invent. Math.*, 112(1):127–149, 1993.

[Beh03] Kai A. Behrend. Derived l-adic categories for algebraic stacks. *Mem. Amer. Math. Soc.*, 163(774):viii+93, 2003.

[BGL94] Emili Bifet, Franco Ghione, and Maurizio Letizia. On the Abel-Jacobi map for divisors of higher rank on a curve. *Math. Ann.*, 299(4):641–672, 1994.

[Bif89] Emili Bifet. Sur les points fixes du schéma $\mathrm{Quot}_{\mathcal{O}_X^r/X/k}$ sous l'action du tore $\mathbf{G}^r_{m,k}$. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(9):609–612, 1989.

[Bor44] J. L. Borges. La biblioteca de Babel. In *Ficciones*, 1ra. ed. Editorial Sur, Buenos Aires, 1944.

[Del74a] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[Del74b] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[Del77] P. Deligne. *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 41$\o er$2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[Dhi06] Ajneet Dhillon. On the cohomology of moduli of vector bundles and the Tamagawa number of SL_n. *Canad. J. Math.*, 58(5):1000–1025, 2006.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[DN89] J.-M. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97(1):53–94, 1989.

[Edi00] Dan Edidin. Notes on the construction of the moduli space of curves. In *Recent progress in intersection theory (Bologna, 1997)*, Trends Math., pages 85–113. Birkhäuser Boston, Boston, MA, 2000.

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Est97] Eduardo Esteves. *Construção de espaços de moduli.* 21º Colóquio Brasileiro de Matemática. [21st Brazilian Mathematics Colloquium]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1997.

[Fal95] Gerd Faltings. Vector bundles on curves. Lecture Notes by Michael Stoll, MPI Bonn, 1995.

[Fan01] Barbara Fantechi. Stacks for everybody. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 349–359. Birkhäuser, Basel, 2001.

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[FN03] Paola Frediani and Frank Neumann. Étale homotopy types of moduli stacks of algebraic curves with symmetries. *K-Theory*, 30(4):315–340, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part IV.

[Fri82] Eric M. Friedlander. *Étale homotopy of simplicial schemes*, volume 104 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1982.

[Gat00] Letterio Gatto. *Intersection theory on moduli spaces of curves*, volume 61 of *Monografias de Matemática [Mathematical Monographs]*. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000.

[GD67a] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. I, II, III, IV. *Inst. Hautes Études Sci. Publ. Math.*, (4, 8, 11, 17, 20, 24, 28, 30, 32), 1960-1967.

[GD67b] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. IV. *Inst. Hautes Études Sci. Publ. Math.*, (20, 24, 28, 30, 32), 1964, 1965, 1966, 1967.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[Gir71] Jean Giraud. *Cohomologie non abélienne.* Springer-Verlag, Berlin, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[Góm01] Tomás L. Gómez. Algebraic stacks. *Proc. Indian Acad. Sci. Math. Sci.*, 111(1):1–31, 2001.

[Gro57a] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[Gro57b] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[Gro62] Alexander Grothendieck. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.].* Secrétariat mathématique, Paris, 1962.

[Gro95a] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 190, 299–327. Soc. Math. France, Paris, 1995.

[Gro95b] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[Hak72] Monique Hakim. *Topos annelés et schémas relatifs.* Springer-Verlag, Berlin, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 64.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hei98] Jochen Heinloth. Über den Modulstack der Vektorbündel auf Kurven. Diplomarbeit, Rheinische Friedrich-Wilhelms-Universität Bonn, 1998.

[Hei05] J. Heinloth. Notes on differentiable stacks. In *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005*, pages 1–32. Universitätsdrucke Göttingen, Göttingen, 2005.

[Hei09] Jochen Heinloth. Lectures on the moduli stack of vector bundles on a curve. preprint, 2009.

[HL97] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.

[HN75] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.*, 212:215–248, 1974/75.

[Hol08] Sharon Hollander. A homotopy theory for stacks. *Israel J. Math.*, 163:93–124, 2008.

[HS] Jochen Heinloth and Alexander H. W. Schmitt. The cohomology ring of moduli stacks of principal bundles over curves.

[Knu71] Donald Knutson. *Algebraic spaces.* Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin, 1971.

[KW06] Frances Kirwan and Jonathan Woolf. *An introduction to intersection homology theory*. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2006.

[Lie07] Max Lieblich. Moduli of twisted sheaves. *Duke Math. J.*, 138(1):23–118, 2007.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[LO08a] Yves Laszlo and Martin Olsson. The six operations for sheaves on Artin stacks. I. Finite coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (107):109–168, 2008.

[LO08b] Yves Laszlo and Martin Olsson. The six operations for sheaves on Artin stacks. II. Adic coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (107):169–210, 2008.

[LP97] J. Le Potier. *Lectures on vector bundles*, volume 54 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Translated by A. Maciocia.

[Mac62] I. G. Macdonald. The Poincaré polynomial of a symmetric product. *Proc. Cambridge Philos. Soc.*, 58:563–568, 1962.

[Met03] David Metzler. Topological and smooth stacks. arXiv:math.DG/0306176, 2003.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, third edition, 1994.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MLM94] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[Mum99] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[Nit05] Nitin Nitsure. Construction of Hilbert and Quot schemes. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 105–137. Amer. Math. Soc., Providence, RI, 2005.

[Noo05] Behrang Noohi. Foundations of topological stacks I. arXiv:math.AG/0503247, 2005.

[NS] Frank Neumann and Ulrich Stuhler. Frobenius morphisms and cohomology of moduli stacks of vector bundles. In preparation.

[NS05] Frank Neumann and Ulrich Stuhler. Moduli stacks of vector bundles and Frobenius morphisms. In *Algebra and Number Theory*, pages 126–146. Hindustan Book Agency, Delhi, 2005.

[Ols07] Martin Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007.

[Pro96] Dorette A. Pronk. Etendues and stacks as bicategories of fractions. *Compositio Math.*, 102(3):243–303, 1996.

[Ram73] S. Ramanan. The moduli spaces of vector bundles over an algebraic curve. *Math. Ann.*, 200:69–84, 1973.

[Sat56] I. Satake. On a generalization of the notion of manifold. *Proc. Nat. Acad. Sci. U.S.A.*, 42:359–363, 1956.

[Ser95] Jean-Pierre Serre. Espaces fibrés algébriques (d'après André Weil). In *Séminaire Bourbaki, Vol. 2*, pages Exp. No. 82, 305–311. Soc. Math. France, Paris, 1995.

[sga72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos. (SGA 4).* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[sga03] *Revêtements étales et groupe fondamental. (SGA 1).* Documents Mathématiques (Paris), 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin.

[Sha77a] I. R. Shafarevich. *Basic algebraic geometry.* Springer-Verlag, Berlin, study edition, 1977. Translated from the Russian by K. A. Hirsch, Revised printing of Grundlehren der mathematischen Wissenschaften, Vol. 213, 1974.

[Sha77b] Stephen S. Shatz. The decomposition and specialization of algebraic families of vector bundles. *Compositio Math.*, 35(2):163–187, 1977.

[Sor00] Christoph Sorger. Lectures on moduli of principal G-bundles over algebraic curves. In *School on Algebraic Geometry (Trieste, 1999)*, volume 1 of *ICTP Lect. Notes*, pages 1–57. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000.

[Ste65] Robert Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

[Tam94] Günter Tamme. *Introduction to étale cohomology.* Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.

[Tel98] Constantin Teleman. Borel-Weil-Bott theory on the moduli stack of G-bundles over a curve. *Invent. Math.*, 134(1):1–57, 1998.

[Vis89] Angelo Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.*, 97(3):613–670, 1989.

[Vis05] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.

[VLP85] Jean-Louis Verdier and Joseph Le Potier, editors. *Module des fibrés stables sur les courbes algébriques*, volume 54 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985. Papers from the conference held at the École Normale Supérieure, Paris, 1983.

[Wei49] André Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949.

Index