Math 751 Week 10 Notes

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Definition A short sequence of chain complex:

$$0 \longrightarrow A_. \xrightarrow{i} B_. \xrightarrow{j} C_. \longrightarrow 0$$

is said to be exact if

$$0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \longrightarrow 0$$

is a short exact sequence for each n, i.e, (1) i is injective, (2) $Im(i) = Ker(j)$, (3) j is surjective.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We will show that when we pass to homology groups, this short exact sequence of chain complexes stretches out into a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A_.) \xrightarrow{i_*} H_n(B_.) \xrightarrow{j_*} H_n(C_.) \xrightarrow{\partial} H_{n-1}(A_.) \xrightarrow{i_*} H_{n-1}(B_.) \longrightarrow \cdots$$

where $H_n(A_.)$ denotes the homology group $Ker\partial/Im\partial$ at A_n in the chain complex $A_.$, and $H_n(B_.)$ and $H_n(C_.)$ are defined similarly.

$$A_{n-1} \xrightarrow{i} B_{n-1} \xleftarrow[\partial]{} B_n \xrightarrow{j} C_n$$

The commutativity of the squares in the short exact sequence of chain complexes means that i and j are chain maps. These therefore induce maps i_* and j_* on homology. To define the boundary map $\partial : H_n(C_.) \longrightarrow H_{n-1}(A_.)$, let $c \in C_n$ be a cycle. Since j is onto, $c = j(b)$ for some $b \in B_n$. The element $\partial b \in B_{n-1}$ is in $Kerj$ since $j(\partial b) = \partial j(b) = \partial c = 0$. So $\partial b = i(a)$ for some $a \in A_{n-1}$ since $Kerj = Imi$. Note that $\partial a = 0$ since $i(\partial a) = \partial i(a) = \partial \circ \partial b = 0$ and i is injective. We define $\partial : H_n(C_.) \longrightarrow H_{n-1}(A_.)$ by sending the homology class of c to the homology class of a, $\partial[c] = [a]$. This is well-defined since:

- The element a is uniquely determined by ∂b since i is injective.
- A different choice b' for b would have $j(b') = j(b)$, so $b' - b$ is in $Kerj = Imi$. Thus $b' - b = i(a')$ for some a', hence $b' = b + i(a')$. The effect of replacing b by $b + i(a')$ is to change a to the homologous element $a + \partial a'$ since $i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b + i(a'))$.

- A different choice of c within its homology class would have the form $c+\partial c^{'}$. Since $c^{'} = j(b^{'})$ for some $b^{'}$, we then have $c+\partial c^{'} = c+\partial j(b^{'}) = c+j(\partial b^{'}) = j(b+\partial b^{'})$, so b is replaced by $b+\partial b^{'}$, which leaves ∂b and therefore also a unchanged.

The map $\partial : H_n(C_.) \longrightarrow H_{n-1}(A_.)$ is a homomorphism since if $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a]$ via elements b_1 and b_2 as above, then $j(b_1+b_2) = j(b_1)+j(b_2) = c_1+c_2$ and $i(a_1+a_2) = i(a_1)+i(a_2) = \partial b_1+\partial b_2 = \partial(b_1+b_2)$, so $\partial([c_1]+[c_2]) = [a_1]+[a_2]$.

Exactness at $H_n(B.)$:

- Im $i_* \subset$ Kerj_*. This is immediate since $ji = 0$ implies j_*i_*=0.
- Ker $j_*\subset$ Im i_*. A homology class in Ker j_* is represented by a cycle b$\in B_n$ with $j(b)$ a boundary, so $j(b) = \partial c$ for some $c' \in C_{n+1}$. Since j is surjective, $c' = j(b')$ for some $b' \in B_{n+1}$. We have $j(b-\partial b')$ $= j(b)$ - $j(\partial b')$ = $j(b)$ - ∂ j(b') = 0 since $\partial j(b' = \partial c' = j(b)$. So b - $\partial b' = i(a)$ for some a $\in A_n$. This a is a cycle since $i(\partial a) = \partial i(a) = \partial(b-\partial b') = \partial b = 0$ and i is injective. Thus $i_*[a] = [b-\partial b'] = [b]$, showing that i_* maps onto Ker j_*.

Exactness at $H_n(C.)$:

- Im $j_* \subset$ Ker∂. We have $\partial j_* = 0$ since in this case $\partial b = 0$ in the definition of ∂.
- Ker $\partial \subset$ Im j_*. In the notation used in the definition of ∂, if c represents a homology class in Ker∂, then $a = \partial a'$ for some $a' \in A_n$. The element $b - i(a')$ is a cycle since $\partial(b-i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a)$=0. And $j(b-i(a')) = j(b) - ji(a') = j(b) = c$, so j_* maps $[b-i(a')]$ to $[c]$.

Exactness at $H_n(A.)$:

- Im$\partial \subset$ Keri_*. Here $i_*\partial = 0$ since $i_*\partial$ takes $[c]$ to $[\partial b] = 0$.
- Ker$i_* \subset Im\partial$. Given a cycle $a \in A_n - 1$ such that $i(a) = \partial b$ for some b $\in B_n$, then $j(b)$ is a cycle since $\partial j(b)$= $j(\partial b) = ji(a) = 0$, and ∂ takes $[j(b)]$ to $[a]$.

Applications

1) Now we define relative topology. Given a space X and a subspace $A \subset X$, let $C_n(X,A)$ be the quotient group $C_n(X)/C_n(A)$. Thus chains in A are trivial in $C_n(X,A)$. Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X,A) \to C_{n-1}(X,A)$. Letting n vary, we have a sequence of boundary maps

$$\cdots \longrightarrow C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \longrightarrow \cdots$$

The relation $\partial^2 = 0$ holds for these boundary maps since it holds before passing to quotient groups. So we have a chain complex, and the homology groups Ker ∂/Im∂ of this chain complex are by definition the relative homology groups $H_n(X,A)$. By the definition of relative topology, we have the following short exact sequence:

$$0 \longrightarrow \mathbb{C}_.(A) \xrightarrow{i_*} \mathbb{C}_.(X) \xrightarrow{p_*} \mathbb{C}_.(X,A) \longrightarrow 0$$

Therefore, by the previous argument we have the long exact sequence

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

Remark This also works for augmented chain complex

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_.(A) \arrow[r] \arrow[d, "\epsilon"] & \mathbb{C}_.(X) \arrow[r] \arrow[d, "\epsilon"] & \mathbb{C}_.(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and this gives

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X,A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

Exercise:

- If $A = x_0 \subset X$ for some pint $x_0 \in X$, then $\widetilde{H}_n(X) \cong H_n(X, x_0)$ for all n.
- If $A = \partial D^n \subset X = D^n$, then

$$H_i(D^n, \partial D^n) \underset{\cong}{\xrightarrow{\partial}} \widetilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & otherwise. \end{cases}$$

2) For $A \subset B \subset X$, we have short exact sequence

$$0 \longrightarrow \mathbb{C}_.(A,B) \xrightarrow{i_*} \mathbb{C}_.(X,B) \xrightarrow{p_*} \mathbb{C}_.(X,A) \longrightarrow 0$$

and hence the corresponding long exact sequence

$$\cdots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A,B) \longrightarrow \cdots$$