MATH 751 LECTURE NOTES

MAYER-VIETORIS SEQUENCES

Recall that we can utilize a long exact sequence of homology groups for a pair (X, A) to compute homology. We now introduce another type of long exact sequence that is useful in computation.

Theorem. *Assume $A, B \subset X$ with $X = \operatorname{int} A \cup \operatorname{int} B$. Then we have the following long exact sequence of homology groups*

$$\cdots \xrightarrow{\phi} H_n(A \cap B) \xrightarrow{\psi} H_n(A) \oplus H_n(B) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

Proof. We denote $C_n(A+B)$ the subgroup of $C_n(X)$ consisting of chains that are sums of chains in A and chains in B. The boundary map comes from the restriction of $\partial : C_n(X) \to C_{n-1}(X)$. Note that the restriction takes $C_n(A+B)$ to $C_{n-1}(A+B)$ so we get a chain complex $(C_\bullet(A+B), \partial_\bullet)$ with inclusions inducing an isomorphism $H_n(A+B) \cong H_n(X)$.
Consider the following SES

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{\phi} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\psi} C_\bullet(A+B) \to 0$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$. Exactness is due to the following facts:

1. $\psi_n : C_n(A) \oplus C_n(B) \longrightarrow C_n(A+B)$ by definition of $C_\bullet(A+B)$.
2. $\phi_n : C_n(A \cap B) \longrightarrow C_\bullet(A) \oplus C_\bullet(B)$ is injective by definition.
3. $(\psi \circ \phi)(x) = \psi(x, -x) = x - x = 0$ for all $x \in C_n(A \cap B)$. Thus $\operatorname{Im} \phi_n \subset \ker \psi_n$
4. If $(x, y) \in \ker \psi_n$, then $x = -y \in C_n(A \cap B)$ and $\phi_n(x) = (x, y)$ and $(x, y) \in \operatorname{Im} \phi_n$. Thus $\ker \psi_n \subset \operatorname{Im} \phi_n$

We apply the long exact sequence theorem to complete the proof. □

Remark 0.1. It is also possible to define a reduced Mayer-Vietoris LES of reduced homology groups.

Example 0.2 (Homology of S^n)**.** Let $X = S^n$ with $A = S^n - \{s\}$ and $B = S^n - \{n\}$ where s is the southern most point and n is the northern most point. Then $A \cap B \cong \mathbb{R}^n$ and $A \cap B \cong S^{n-1}$. By Mayer-Vietoris, we get the LES

$$\cdots \longrightarrow \tilde{H}_n(\mathbb{R}^n) \oplus \tilde{H}_n(\mathbb{R}^n) \longrightarrow \tilde{H}_n(S^n) \longrightarrow \tilde{H}_{n-1}(S^{n-1}) \longrightarrow \tilde{H}_{n-1}(\mathbb{R}^n) \oplus \tilde{H}_{n-1}(\mathbb{R}^n) \longrightarrow \cdots$$

But $\tilde{H}_n(\mathbb{R}^n) \oplus \tilde{H}_n(\mathbb{R}^n) = 0$ since $\mathbb{R}^n$ is contractible. It follows that

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

Example 0.3 (Homology of the Klein Bottle)**.** We decompose the Klein bottle K as the union of two mobius bands M_1 and M_2 glued together by a homeomorphism between their boundary circles. We know $M_1 \sim M_2 \sim S^1$ (i.e. same homotopy type) and $M_1 \cap M_2 \cong \partial M_1 = \partial M_2 \cong S^1$. Note that actually we need to take a neighborhood around each M_1 and M_2 that deformation retracts back to the mobius band so that the union of their interiors form the Klein bottle. Since M_1, M_2, and $M_1 \cap M_2$ are all homotopy equivalent to circles, the interesting part of the Mayer-Vietoris sequence is the segment

$$0 \longrightarrow \tilde{H}_2(K) \xrightarrow{\partial} \tilde{H}_1(M_1 \cap M_2) \xrightarrow{\phi} \tilde{H}_1(M_1) \oplus \tilde{H}_1(M_2) \xrightarrow{\psi} \tilde{H}_1(K) \to 0$$

Notice that $\tilde{H}_2(K) \cong \operatorname{Im} \partial \cong \ker \phi \cong 0$. Also

$$\tilde{H}_1(K) \cong \operatorname{Im} \psi \cong \frac{\tilde{H}_1(M_1) \oplus \tilde{H}_1(M_2)}{\ker \psi} \cong \frac{\tilde{H}_1(M_1) \oplus \tilde{H}_1(M_2)}{\operatorname{Im} \phi} \cong \frac{(1,0)\mathbb{Z} \oplus (1,-1)\mathbb{Z}}{(2,-2)\mathbb{Z}} \cong \mathbb{Z} \oplus \mathbb{Z}_2$$