Math 751 Lecture note Week 11

December 21, 2015

Excision Theorem

(a) Let $Z \subset A \subset X$ with $\overline{Z} \subset int(A)$.
The inclusion map $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisims $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n. i.e., $H_n(X - Z, A - Z) \cong H_n(X, A)$ for all n.

(b) Let $A, B \subset X$ with $X = int(A) \cup int(B)$.
The inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces $H_n(B, A \cap B) \cong H_n(X, A)$ for all n.

Remark (a) and (b) are equivanlent by taking $B = X - Z$ (or $Z = X - B$). Then we have $A \cap B = A - Z$ and $\overline{Z} \subset int(A)$ iff $X = int(A) \cup int(B)$

Sketch of the proof of the Excision Theorem :

For a topological space X, let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of subspaces of X with $X = \bigcup_{i \in I} int(U_i)$. Consider $C_n^{\mathcal{U}}(X) := \{\sum_{i=1}^{m} n_i \sigma_i \mid im(\sigma_i) \subset U_j$ for some $j \in I\}$, a subcomplex of $C_n(X)$. Then $C_\bullet^{\mathcal{U}}(X)$ forms a Chain complex with boundary map ∂. Denote the homology group of this chain complex by $H_\bullet^{\mathcal{U}}(X)$. We have the following proposition.

Proposition $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.
This is because the inclusion map $i : C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ is a homotopy equivalence. i.e., $\exists \rho : C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$ such that $\rho \circ i = Id_{C_\bullet(X)}$ and $i \circ \rho = Id_{C_\bullet^{\mathcal{U}(X)}}$.

Now, for the proof of the part (b), consider the cover $\mathcal{U} = \{A, B\}$and denote $C_n(A + B) := C_n^{\mathcal{U}}(X)$. The inclusion $C_n(A + B)/C_n(A) \hookrightarrow C_n(X)/C_n(A) = C_n(X, A)$ induces isomorphism on homology.
Moreover, $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \to C_n(A + B)/C_n(A)$ also induces isomorphism on homology. In consequence, we get $H_n(X, A) \cong H_n(B, A \cap B)$.