1
The Fundamental Group

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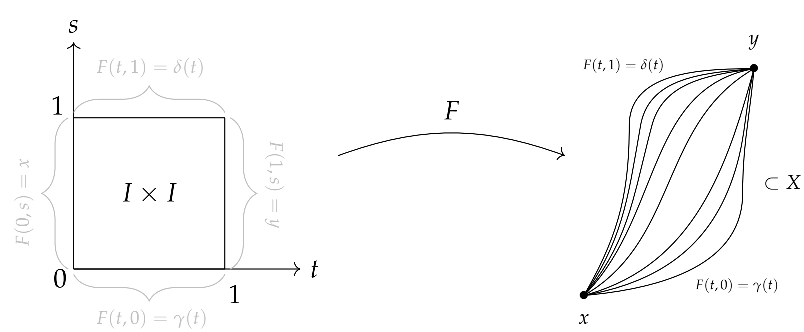
Fundamental Group

Let X be a topological space and let $x, y \in X$. Define by $P(X, x, y)$ the set $\{\gamma : [0,1] \to X : \gamma(0) = x,\ \gamma(1) = y$ and γ is continuous$\}$. In the case, $x = y$, we denote $P(X, x, x)$ by $\Omega(X, x)$. Next we define an equivalence relation.

1.1 Definition (Homotopic paths)**.** Two paths $\gamma, \delta \in P(X, x, y)$ are called **homotopic** (denoted $\gamma \sim \delta$) if there is a continuous map $F : [0,1] \times [0,1] \to X$ such that

$$F(t,0) = \gamma(t),\ \forall t \in [0,1], \qquad F(0,s) = x,\ \forall s \in [0,1],$$
$$F(t,1) = \delta(t),\ \forall t \in [0,1], \qquad F(1,s) = y,\ \forall s \in [0,1].$$



1.2 Lemma. *The homotopy relation $\sim$ is an equivalence relation.*

Proof. • **Reflexivity** $\gamma \sim \gamma$ with $F(t,s) = \gamma(t),\ \forall s \in [0,1]$.

- **Symmetry** If $\gamma \overset{F}{\sim} \delta$ then $\delta \overset{F'}{\sim} \gamma$ with $F'(t,s) = F(t, 1-s)$.
- **Transivity** If $\gamma \overset{F}{\sim} \delta$ and $\delta \overset{G}{\sim} \varphi$ then $\gamma \overset{H}{\sim} \varphi$ with:

$$H(t,s) = \begin{cases} F(t,2s) & 0 \leq s \leq \frac{1}{2} \\ G(t,2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

1.3 Remark. We have:

$$\begin{aligned} H(t,0) &= F(t,0) = \gamma(t) \\ H(t,1) &= G(t,1) = \varphi(t) \\ H(0,s) &= \begin{cases} F(0,2s) = x & 0 \leq s \leq \frac{1}{2} \\ G(0,2s-1) = x & \frac{1}{2} \leq s \leq 1 \end{cases} \\ H(1,s) &= y \end{aligned}$$

Also $H(t, 1/2) = F(t,1) = G(t,0) = \delta(t)$. Hence H is well defined.

Question: Is H continuous?

1.4 Remark. If $X = A \cup B$ with A, B both closed (or open) and if $f : X \to Y$ is a map with continuous restrictions $f|_A$ and $f|_B$ then f is continuous.

Hence H is the desired map, proving transivity.

□

1.5 Definition (Fundamental Group)**.** The **fundamental group** of X with base point x (denoted by $\pi_1(X,x)$) is the quotient of $\Omega(X,x)$ with respect to the homotopy equivalence relation on $\Omega(X,x)$:

$$\pi_1(X,x) = \Omega(X,x)/\sim.$$

1.6 Definition (Concatenation of paths in X)**.** We define **concatenation operator** as the map $* : P(X,x,y) \times P(X,y,z) \to P(X,x,z)$ given by $(\gamma, \delta) \mapsto \gamma * \delta$ where $\gamma * \delta : [0,1] \to X$ is defined by

$$\gamma * \delta(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

1.7 Remark. One can also define for fixed $s \in [0,1]$,

$$\gamma \underset{s}{*} \delta(t) = \begin{cases} \gamma(t/s) & 0 \leq t \leq s \\ \delta\left(\frac{t-s}{1-s}\right) & s \leq t \leq 1 \end{cases}.$$

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1.8 Lemma. *Concatenation of paths is consistent with the homotopy relation (i.e. if $\gamma \sim \gamma'$ and $\delta \sim \delta'$ then $\gamma * \delta \sim \gamma' * \delta'$).*

Proof. Let us say $\gamma \overset{F}{\sim} \gamma'$ and $\delta \overset{G}{\sim} \delta'$. Consider: $H : [0,1] \times [0,1] \to X$ given by

$$H(t,s) = \begin{cases} F(2t,s) & \text{for } 0 \leq t \leq \frac{1}{2} \\ G(2t-1,s) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Claim: $\gamma * \delta \overset{H}{\sim} \gamma' * \delta'$. Proof is left as an exercise. □

1.9 Corollary. *The operation, $*$, of concatenation of paths induces a binary law on the set $\pi_1(X,x)$ by $[\gamma] \cdot [\delta] = [\gamma * \delta]$.*

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1.10 Theorem (Fundamental Group). *$(\pi_1(X,x), *)$ is a group.*

Proof. We need to show that the group axioms hold for $(\pi_1(X,x), *)$. $\pi_1(X,x)$ is clearly closed under concatenation.

- **Associativity** We will show that for $\gamma, \delta, \varphi \in \Omega(X,x)$, we have $(\gamma * \delta) * \varphi = \gamma * (\delta * \varphi)$.
  Hint: Recall from Remark 1.7 $\gamma \underset{s}{*} \delta \sim \gamma \underset{s'}{*} \delta$ for any $s, s' \in (0,1)$ and since concatenation is consistent with the homotopy equivalence relation, we have:

  $$(\gamma * \delta) * \varphi \sim (\gamma * \delta) \underset{3/4}{*} \varphi \sim (\gamma \underset{2/3}{*} \delta) \underset{3/4}{*} \varphi = \gamma * (\delta * \varphi), \tag{1.1}$$

  thus proving associativity of the concatenation operator. Note that the equality in eq. (1.1) can be checked by simply expanding the definition of concatenation. We refer the reader to [1] page 27, for a visual demonstration of the idea.

- **Identity Element** We contend the constant map:

  $$e_x : [0,1] \to X \text{ given by } e_x(t) = x$$

  is the left and right identity elements of $\pi_1(X,x)$. Let $\gamma \in \Omega(X,x)$. We show that $\gamma \overset{G}{\sim} e_x * \gamma$ where $G : [0,1] \times [0,1] \to X$ is defined by:

  $$(t,s) \mapsto \begin{cases} x & 0 \leq t \leq \frac{1}{2}s \\ \gamma\left(\frac{2t-s}{2-s}\right) & \frac{1}{2}s \leq t \leq 1 \end{cases}$$

[1] Allen Hatcher. *Algebraic topology*. Cambridge ; New York : Cambridge University Press, 2002

We have:

$$G(t,0) = \gamma(t)$$
$$G(t,1) = \begin{cases} x & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

It is easy to check that $G(0,s) = G(1,s) = x$, $\forall x \in [0,1]$. The continuity of G follows from Remark 1.4.

A similar proof shows that $\gamma \sim \gamma * e_x$. Hence e_x is the identity element.

- **Inverse Elements** Let $\gamma : [0,1] \to X$ be a closed continuous path with $\gamma(0) = \gamma(1) = x$. Consider $\bar{\gamma} : [0,1] \to X$ given by $\bar{\gamma}(t) = \gamma(1-t)$. Clearly $\bar{\gamma}$ is continuous. $\bar{\gamma}(0) = \bar{\gamma}(1) = x$. We contend that $e_x \overset{F}{\sim} \gamma * \bar{\gamma}$ where $F : [0,1] \times [0,1] \to X$ is given by:

$$F(t,s) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2}s \\ \gamma(s) & \frac{1}{2}s \le t \le 1 - \frac{1}{2}s \\ \gamma(2-2t) & 1 - \frac{1}{2}s \le t \le 1 \end{cases}$$

It is easy to see that F is well defined. Continuity of F follows from Remark 1.4. We have:

$$F(t,0) = \gamma(0) = x = e_x,$$
$$F(t,1) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \gamma(2-2t) & \frac{1}{2} \le t \le 1 \end{cases}$$
$$= \gamma * \bar{\gamma}$$

□

1.11 Example. Fundamental groups of certain spaces:

1. If $X = \{x\}$ (point space), $\Omega(X,x) = \{e_x\}$ and $\pi_1(X,x) = \langle [e_x] \rangle$.
2. If X is a convex subset of $\mathbb{R}^n$ and $x \in X$ then $\pi_1(X,x) = \langle [e_x] \rangle$ because if $\gamma \in \Omega(X,x)$ then $H(t,s) = se_x(t) + (1-s)\gamma(t)$ is a continuous map with $H(t,0) = \gamma(t)$ and $H(t,1) = e_x(t) = x$. So $\gamma \overset{H}{\sim} e_x$.
3. For $n \ge 2$, we will see $\pi_1(S^n, x) = \langle [e_x] \rangle$.
4. We will see that $\pi_1(S^1, x) \cong \langle [\gamma] \rangle \cong \mathbb{Z}$ where $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$.

Base Point Independence: Question: How does $\pi_1(X,x)$ change under base point change? Assume that X is path connected. Take two distinct points $x, y \in X$ where $x \neq y$.

Choose a continuous path $\alpha : [0,1] \to X$ with $\alpha(0) = x$, $\alpha(1) = y$. If $\gamma \in \Omega(X,x)$ then $\bar{\alpha} * \gamma * \alpha \in \Omega(X,y)$.

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1.12 Remark. If $\gamma \overset{H}{\sim} \gamma'$ then $\bar{\alpha} * \gamma * \alpha \overset{H'}{\sim} \bar{\alpha} * \gamma' * \alpha$ where H' is a reparametrization of H. Hence this operation is compatible with homotopies, and we get the map:

$$\alpha_\# : \pi_1(X, x) \to \pi_1(X, y)$$

1.13 Theorem. *If X is path-connected and α is a path from y to x, $\alpha_\#$ is an isomorphism.*

Proof. First we will show that $\alpha_\#$ is a homomorphism. Take $[\gamma], [\eta] \in \pi_1(X, x)$, we have $\alpha_\# ([\gamma] \cdot [\eta]) = \alpha_\# ([\gamma * \eta]) = [\bar{\alpha} * \gamma * \eta * \delta] = [\bar{\alpha} * \gamma * e_y * \eta * \alpha] = [\bar{\alpha} * \gamma * \alpha * \bar{\alpha} * \eta * \alpha] = [\bar{\alpha} * \gamma * \alpha] \cdot [\bar{\alpha} * \eta * \alpha] = \alpha_\#([\gamma]) \cdot \alpha_\#([\eta])]$.

Secondly, $\alpha_\#$ is an isomorphism with inverse map $\bar{\alpha}_\#$ since $\forall \gamma \in \pi_1(X, x)$:

$$\begin{aligned}
(\alpha_\# \circ \bar{\alpha}_\#)([\gamma]) &= \alpha_\# ([\alpha * \gamma * \bar{\alpha}]) \\
&= [\bar{\alpha} * \alpha * \gamma * \bar{\alpha} * \alpha] \\
&= [e_x * \gamma * e_x] \\
&= [\gamma].
\end{aligned}$$

Similarly $\forall \gamma \in \pi_1(X, y)$, we have $(\bar{\alpha}_\# \circ \alpha_\#)([\gamma]) = [\gamma]$. $\square$