Math 751 Week 3 Notes

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Last time, we showed that the fundamental group of a path connected space is independent of our choice of base point. The precise statement is as follows:

Theorem. *Let X be a path connected space, and $x, y \in X$. Let $\alpha : [0,1] \to X$ be a continuous path with $\alpha(0) = x$, $\alpha(1) = y$. Then*

$$\begin{aligned} \alpha_\# : \pi_1(X,x) &\to \pi_1(X,y) \\ [\gamma] &\mapsto [\alpha^{-1} * \gamma * \alpha] \end{aligned}$$

is an isomorphism.

Remark. The path connectedness assumption might be redundant.

1 Induced Homomorphism

If $f : X \to Y$ is a continuous map, with $f(x) = y$, then what is the relation between $\pi_1(X,x)$ and $\pi_1(Y,y)$? Given a continuous loop $\gamma : [0,1] \to X$ with fixed base point x, $f \circ \gamma : [0,1] \to Y$ is a continuous loop in Y with fixed base point y. Thus, we have a map

$$\begin{aligned} f_\# : \Omega(X,x) &\to \Omega(Y,y) \\ \gamma &\mapsto f \circ \gamma. \end{aligned}$$

Moreover, if two paths $\gamma, \gamma' \in \Omega(X,x)$ are homotopic, then $f_\#(\gamma)$ and $f_\#(\gamma')$ are also homotopic. (Indeed, suppose $\gamma \overset{H}{\sim} \gamma'$. Then $f_\#(\gamma) \overset{f\circ H}{\sim} f_\#(\gamma')$.) It follows that the map

$$\begin{aligned} f_* : \pi_1(X,x) &\to \pi_1(Y,y) \\ [\gamma] &\mapsto [f_\#(\gamma)] \end{aligned}$$

is well-defined.
Also, if $\gamma, \delta \in \Omega(X,x)$, then

$$\begin{aligned} f_\#(\gamma * \delta)(t) &= \begin{cases} f(\gamma(2t)) & 0 \le t \le \frac{1}{2} \\ f(\delta(2t-1)) & \frac{1}{2} \le t \le 1 \end{cases} \\ &= (f_\#(\gamma) * f_\#(\delta))(t). \end{aligned}$$

Taking equivalence classes gives

$$\begin{aligned} f_*(\gamma * \delta) &\stackrel{\text{def}}{=} [f_\#(\gamma * \delta)] \\ &= [f_\#(\gamma) * f_\#(\delta)] \\ &= [f_\#(\gamma)] \cdot [f_\#(\delta)] \\ &\stackrel{\text{def}}{=} f_*(\gamma) \cdot f_*(\delta). \end{aligned}$$

Therefore, $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is a group homomorphism.

The above results can be summarized as follows:

Theorem. *A continuous map $f : X \to Y$ induces a group homomorphism*

$$\begin{aligned} f_* : \pi(X, x) &\to \pi(Y, f(x)) \\ [\gamma] &\mapsto [f \circ \gamma]. \end{aligned}$$

Here are some basic properties of the induced homomorphism:

Proposition. *Let X, Y, Z be topological spaces, and $f : X \to Y$, $g : Y \to Z$ be continuous maps.*

(i) $(g \circ f)_* = g_* \circ f_*$;

(ii) $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}$.

2 Functoriality

Definition. A category $\mathcal{C}$ consists of the following things:

(i) A class of objects, denoted by $\mathrm{Obj}(\mathcal{C})$;

(ii) A set of morphisms $\mathbf{Hom}(A, B)$ for every pair objects A, B in $\mathrm{Obj}(\mathcal{C})$;

(iii) An identity morphism $\mathrm{id}_A \in \mathbf{Hom}(A, A)$ for every object A;

(iv) A composition rule $\mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) \to \mathbf{Hom}(A, C)$ for every triple A, B, C of objects, such that $(f \circ g) \circ h = f \circ (g \circ h)$ and $\mathrm{id} \circ f = f = f \circ \mathrm{id}$.

Example. Sets is a category. The objects are sets, and the morphisms are maps between sets.

Example. Grp is a category. The objects are groups, and the morphisms are group homomorphisms.

Example. Top is a category. The objects are topological spaces, and the morphisms are continuous maps.

Example. Pointed topological spaces (pairs (X, x), where $x \in X$) form a category. The morphisms in $\mathbf{Hom}((X, x), (Y, y))$ are continuous maps sending x to y.

Definition. A functor $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule that associates an object A of $\mathcal{C}$ with an object $\mathbf{F}(A)$ of $\mathcal{D}$, and a morphism $f \in \mathbf{Hom}(A, B)$ with a morphism $\mathbf{F}(f) \in \mathbf{Hom}(\mathbf{F}(A), \mathbf{F}(B))$, such that $\mathbf{F}(\mathrm{id}) = \mathrm{id}$ and $\mathbf{F}(f \circ g) = \mathbf{F}(f) \circ \mathbf{F}(g)$.

If you want a thorough introduction to category theory, I recommend *Categories for the Working Mathematician* by Mac Lane.

We can now restate our results from the previous section in the language of category theory:

Theorem. *The fundamental group π_1 is a functor from the category of pointed topological spaces to the category of groups.*

3 Invariance

In this section, we shall see that homeomorphic topological spaces "share the same fundamental group." We will also work towards establishing a similar result when the condition "homeomorphism" is replaced by "homotopy equivalence."

Theorem. *$\pi_1(X, x)$ is homeomorphism-invariant. In other words, if $f : X \to Y$ is a homeomorphism with $f(x) = y$, then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.*

Proof. Let $g : Y \to X$ be the inverse of f. By functoriality of π_1, we have

$$f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y,y)},$$
$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x)}.$$

Thus, $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism (whose inverse is g_*.) □

Remark. The homeomorphism assumption is overkill. In fact, the fundamental group is invariant under homotopy equivalence (so long as our spaces are path connected.)

Definition. Let $f, g : X \to Y$ be continuous maps. We say that f and g are homotopic if there is a continuous map $H : [0, 1] \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$.

Definition. Let $A \subset X$ be a subspace, and assume that $f = g$ on A. We say that f and g are homotopic relative to A if there is a homotopy H as before, which satisfies the additional condition $H(t, a) = f(a) = g(a)$ for all $t \in [0, 1]$ and all $a \in A$.

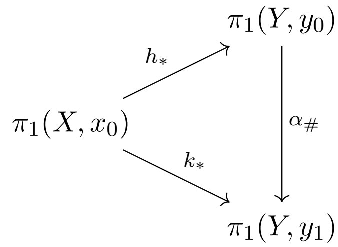
Example. A path is a continuous map $\gamma : [0, 1] \to X$. Our usual notion of path homotopy $\gamma \sim \gamma'$ is equivalent to homotopy relative to $\{0, 1\} \subset [0, 1]$.

Definition. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. We say that f and g are homotopic equivalences if $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. In this case, X and Y are said to have the same homotopy type.

Theorem. *Let X, Y be path connected spaces. If $f : X \to Y$ is a homotopy equivalence. Then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.*

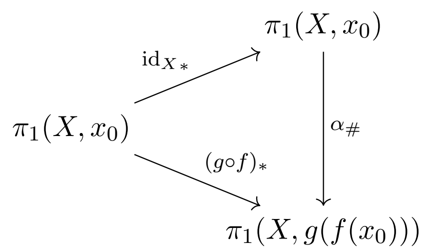
To prove this theorem, we first give a technical lemma:

Lemma. *Let $h, k : X \to Y$ be two continuous maps. Let $x_0 \in X$, and denote $y_0 = h(x_0)$, $y_1 = k(x_0)$. If $h \sim k$, then there is a path $\alpha : [0,1] \to Y$ with $\alpha(0) = y_0$, $\alpha(1) = y_1$ such that the following diagram commutes:*



The proof is very involved, so we shall omit it. (For reference, see Munkres, Lemma 58.4.)

Proof of Theorem. Fix $x_0 \in X$ and let $g : Y \to X$ be so that $g \circ f \sim \mathrm{id}_X$. By the previous lemma, there is a path $\alpha : [0,1] \to X$ with $\alpha(0) = x_0$, $\alpha(1) = g(f(x_0))$, such that the diagram below commutes:



By functoriality, $g_* \circ f_* = \alpha_\# \circ \mathrm{id}_{\pi_1(X,x_0)}$. But we know that $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, g(f(x_0)))$ is an isomorphism. So f_* is mono (the categorical way of saying that f_* is injective.), and g_* is epi (which means surjective.) Reversing the roles of f and g, one can show that f_* is mono, and g_* is epi. Therefore, f_*, g_* are isomorphisms. □

4 Deformation Spaces

Definition. A topological space X is called contractible if the identity map $\mathrm{id}_X : X \to X$ is homotopic to the constant map.

Example. $\{x\}$, $\mathbb{R}^n$, D^n are contractible.

Example. S^n $(n \geq 1)$ is not contractible.

One can show that a contractible space is always path connected. Also, if X is contractible, then $\pi_1(X)$ is trivial.

Remark. If X is contractible, then it is simply connected. The converse is false: take S^n, $n \geq 2$ for example.

Proposition. *X is simply connected if and only if there is a unique homotopy class of paths connecting every pair of points in X.*

Proof. ($\Rightarrow$) Let $x, y \in X$. Since X is path connected, there is a path $\alpha : [0,1] \to X$ with $\alpha(0) = x$, $\alpha(1) = y$. Also assume that there is a second path $\beta : [0,1] \to X$ with $\beta(0) = x$, $\beta(1) = y$. But now

$$\alpha \sim \alpha * e_y \sim \alpha * \overline{\beta} * \beta \sim e_x * \beta \sim \beta.$$

($\Leftarrow$) Since there is a class of paths connecting every pair of points, X is path connected. If we consider the pair (x, x), then $\pi_1(X, x)$ is trivial by assumption. $\square$

Definition. A subspace $A \subset X$ is called a retraction of X if there is a continuous map $r : X \to A$ with $r_{|A} = \mathrm{id}_A$. (Equivalently, if we denote the inclusion $A \hookrightarrow X$ by i, then $r \circ i = \mathrm{id}_A$.

Definition. A subspace $A \subset X$ is called a (weak) deformation retract if A is a retraction of X (under $r : X \to A$) and $i \circ r \sim \mathrm{id}_X$.

Definition. A subspace $A \subset X$ is called a strong deformation retract if A is a (weak) definition retract and $\mathrm{id} \circ r \overset{\mathrm{rel}\, A}{\sim} \mathrm{id}_X$.

Exercise. If X is contractible, then it deformation retracts to any of its points. However, it does not strongly deformation retracts to any of its points.