Notes from Week 3 of Topology 751
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9/21/15-9/25/15

Monday, Sept. 21st

Definition 1. *A topological space X is called* <u>*contractible*</u> *if the identity map $Id_x : X \to X$ is homotopic to a constant map (i.e. the space can be contracted to a point).*

Some examples: $\{x\}$, $\mathbb{R}^n$, and D^n are contractible, but S^n for $n \geq 1$ are not contractible.
Note that if X is path-connected and contractible, then $\pi_1(X)$ is trivial.

Definition 2. *A path-connected topological space is called* <u>*simply connected*</u> *if $\pi_x(X)$ is trivial.*

Note that if X is contractible then X is simply connected, but the reverse is not necessarily true. As a counterexample, take $X = S^n$ where $n \geq 2$. This space is simply connected (it will be proven later in class that $\pi_1(S^n)$ is trivial if $n \geq 2$), but is not contractible.

Proposition 1. *Let X be a topological space. Then X is simply connected if and only if there is a unique homotopy class of paths connecting any pair of points in X.*

Proof. To prove the forward direction ($\Rightarrow$), let $x, y \in X$ and let $\alpha, \beta \in \Omega(X, x, y)$. Then since X is simply connected

$$\alpha \sim \alpha * e_y \sim \alpha * (\overline{\beta} * \beta) \sim (\alpha * \overline{\beta}) * \ \beta \underset{\substack{\uparrow \\ \text{Since } \pi_1(X) \\ \text{is trivial}}}{\sim} e_x \ * \beta \sim \beta.$$

For the other direction ($\Leftarrow$), let $\alpha, \beta \in \Omega(X, x)$. Then by assumption $\alpha \sim \beta$. so $\pi_1(X, x)$ is trivial.

□

Definition 3. *A subspace $A \subset X$ is called a* <u>*retract*</u> *of X is there is a continuous map $r : X \to A$ such that $r|_A = Id_A$, i.e. $r(a) = a$ whenever $a \in A$, or equivalently if $i : A \to X$ is the inclusion map and $r \circ i = Id_A$. Such an r is called a retraction of X to A.*

Definition 4. *A subspace $A \subset X$ is called a* <u>*(weak) deformation reatract*</u> *if A is a retract of X under the map $r : A \to X$ and $i \circ r \sim Id_X$.*

A subspace $A \subset X$ is called a <u>*strong deformation retract*</u> *if A is a weak deformation retract and additionally $i \circ r \underset{\substack{\text{relative} \\ \text{to } A}}{\sim} Id_X$, i.e. there exists a homotopy $H : I \times X \to X$ such that $H(t, a) = a$ for all $t \in I$ and for all $a \in A$.*

Exercise: Prove that if X is contractible, then X deformation retracts to any of its points but does not strong deformation retract to any of its points. An example of this is shown in Hatcher Exercise 0.6 with the "rational comb."

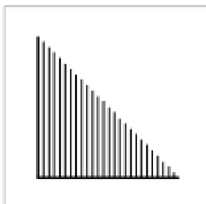


Figure 1: A Diagram of X

Proposition 2 (The Rational Comb Problem)**.** *Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ along with $\{r\} \times [0, 1-r]$ for each rational r in $[0,1]$. Then*

$$X = [0,1] \times \{0\} \cup \left(\bigcup_{r \in \mathbb{Q} \cap [0,1]} \{r\} \times [0, 1-r] \right).$$

It can be shown that if p is a point on the baseline of X, (i.e. $p \in [0,1] \times \{0\}$) then p is a strong deformation retract of X, but if p is not on that baseline then it is not. If we let Y be an infinite union of such "rational combs" then for all $y \in Y$, y is not a strong deformation retract of Y.

Figure 2: A diagram of Y, an infinite union of rational combs X

Another example of deformation retracts on spaces follows. First we will define the notion of the binary set operation known as "join", denoted by $\vee$. Essentially this is a formalization of gluing two sets together at exactly one point.

Definition 5. *Let X, Y be sets with $x_0 \in X$ and $y_0 \in Y$. First define an equivalence on $X \sqcup Y$ that $x \sim y$ if $x = y$ and $x_0 \sim y_0$. Then we define $X \vee Y = X \sqcup Y / \sim$.*

Now that we have this notation, define $J = S^1_1 \vee S^1_2$. Then define $r : J \to S^1_1$ by

$$r(x) = \begin{cases} x & \text{if } x \in S^1_1 \\ x_0 & \text{if } x \in S^1_2. \end{cases}$$

It is not hard to show that r is a retract.

Lemma 1. *S^n is a deformation of $\mathbb{R}^{n+1} \setminus \{0\}$.*

Proof. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and a map $r : X \to S^n$ by $r(x) = \frac{x}{|x|}$. Then $r \circ i = r|_{S^n} = Id_{S^n}$ and r is continuous, implying r is a retract. Notice that the map $H : I \times X \to X$ by $H(t, x) = \frac{x}{(1-t)+t(|x|)}$ is a homotopy from Id_X to $i \circ r$. We see this since $H(0, x) = x$ and $H(1, x) = \frac{x}{|x|} = i\left(\frac{x}{|x|}\right) = i(r(x))$. □

In the proof of the following theorem we will need to use the fact that convex sets are contractible. This is perhaps easiest to visualize in the example of a disc in $\mathbb{R}^2$. Since every point can be connected to the center by a straight line, we can define a deformation retract to the center pointwise by moving each point along the line between it and the center of the disc. It is not hard to show that this is a continuous map, and we can use a similar argument to prove that any convex set is contractible.

Proposition 3. *S^n is simply connected for $n \geq 2$.*

Proof. Let $\gamma : [0, 1] \to S^n$ be a continuous map with $\gamma(0) = \gamma(1) = x \in S^n$. It is enough to show that there is a continuous loop $\eta : [0, 1] \to S^n$ with $\eta(0) = \eta(1) = x$, $\eta \sim \gamma$, and such that $\eta([0, 1]) \neq S^n$. In such case, we can take $y \in S^n$ such that $y \notin \eta([0, 1])$ (in particular $y \neq x$) and we have

$$[0, 1] \xrightarrow{\eta} S^n \setminus \{y\} \simeq \mathbb{R}^n$$

Where $S^n \setminus \{y\}$ and $\mathbb{R}^n$ are homeomorphic by stereographic projection. Since $\mathbb{R}^n$ contractible, we can conclude that $\gamma \sim \eta \sim e_x$ and that $\pi_1(S^n, x)$ is trivial.

But first we must show that such a η exists. Consider a small open ball $B \subset S^n$ about y for some $y \neq x$ such that $x \notin B$. Then $\gamma^{-1}(B)$ is an open set in $[0, 1]$. Note that since $x \notin B$, we know the endpoints $0, 1 \notin \gamma^{-1}(B)$, so $\gamma^{-1}(B) \neq [0, 1]$. Since $\gamma^{-1}(B)$ is open, it consists of a disjoint union of open intervals, i.e.

$$\gamma^{-1}(B) = \bigsqcup_{i \in I} (a_i, b_i).$$

Next, since $\gamma^{-1}(y)$ is a compact set contained in $\gamma^{-1}(B)$, there is a finite subcollection $\{(a_i, b_i)\}_{i=1}^N$ such that $\gamma^{-1}(y) \subset \bigsqcup_{i=1}^{N} (a_i, b_i)$.

Note that since these intervals are disjoint and cover $\gamma^{-1}(B)$, we may conclude that $\gamma(a_i), \gamma(b_i) \in \partial B$ for each i. As proof, assume not. Then $\gamma(a_i) \in Int(B)$, which implies that there exists $\epsilon > 0$ such that $\gamma((a_i - \epsilon, a_i + \epsilon)) \subset Int(B)$. But this means $(a_i - \epsilon, a_i + \epsilon) \subset \gamma^{-1}(B) = \bigsqcup_{i \neq j} (a_i, b_i)$. So there must be

$j \in \{1, \ldots, N\}$, $j \neq i$ such that $(a_i - \epsilon, a_i + \epsilon) \subset (a_j, b_j)$. All this implies that $a_i \in (a_j, b_j)$, contradicting the disjointedness assumed in $\gamma^{-1}(B)$.

Thus we have that for each $i = 1, \ldots, N$ we can define $\gamma_i = \gamma|_{[a_i,b_i]} : [a_i, b_i] \to \overline{B}$ which satisfies that $\gamma(a_i), \gamma(b_i) \in \partial B$. For $i = 1, \ldots, N$ let g_i be the arc along ∂B from $\gamma(a_i)$ to $\gamma(b_i)$. Because $\overline{B}$ is a convex set and thus by the example above contractible, we can show $\gamma_i \sim g_i$ for all $i = 1, \ldots, N$. Define the path $\eta : [0, 1] \to S^n$ in the following way:

$$\eta(t) = \begin{cases} \gamma(t) & \text{if } t \notin (a_i, b_i) \text{ for all } i = 1, \ldots, N \\ g_i(t) & \text{if } t \in [a_i, b_i] \text{ for all} i = 1, \ldots, N \end{cases}$$

where $[0, 1] = [0, a_1] \cup (a_1, b_1) \cup [b_1, a_2] \cup \ldots \cup (a_N, b_N) \cup [b_N, 1]$. So $\eta \sim \gamma$ and $y \in \eta([0, 1])$. Then the rest of the proof follows.

□

Proposition 4. $\mathbb{R}^2 \not\simeq \mathbb{R}^n$ *for* $n \neq 2$.

Proof. If $n = 0$, then $\mathbb{R}^0$ is finite whereas $\mathbb{R}^2$ is not. Thus, $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^0$.

If $n = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism, then we have that $\mathbb{R}^2 \setminus \{0\} \simeq \mathbb{R} \setminus \{f(0)\}$, but this is impossible since $\mathbb{R}^2 \setminus \{0\}$ is connected while $\mathbb{R} \setminus \{f(0)\}$ is not.

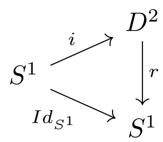
If $n \geq 3$, assume $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. Then $\mathbb{R}^2 \setminus \{0\} \simeq \mathbb{R}^n \setminus \{f(0)\}$ and therefore $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(\mathbb{R}^n \setminus \{f(0)\})$. However, from Lemma 1 we know that

$$\pi_1(\mathbb{R}^n \setminus \{f(0)\}) \cong \pi_1(S^{n-1}) \begin{cases} = 0 & \text{if } n = 2 \\ \neq 0 & \text{if } n \geq 3. \end{cases}$$

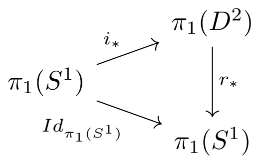
□

Theorem 1 (Brouwer's Fixed Point Theorem)**.** *Any continuous map* $f : D^2 \to D^2$ *has a fixed point.*

Proof. Assume $f(x) \neq x$ for all $x \in D^2$. Then define the map $r : D^2 \to S^1$ which sends a point $x \in D^2$ to the point $r(x) \in \partial D^2$ such that $r(x)$ lies on the line formed by x and $f(x)$ and such that x lies in between $f(x)$ and $r(x)$. We can check that r is continuous. Note that if $x \in \partial D^2 = S^1$ then $r(x) = x$. Thus r is a retraction of D^2 into S^1. Thus the following diagram commutes:



These induce the homomorphisms $i_* : \pi_1(S^1) \to \pi_1(D^2)$, $r_* : \pi_1(D^2) \to \pi_1(S^1)$, and $Id_{\pi_1(S^1)} : S^1 \to S^1$, and since $r \circ i = Id_{S^1}$, the following diagram commutes :



However, since $\pi_1(D^2) = 0$ and $r_* \circ i_*$ maps through $\pi_1(D^2)$, it is impossible for these to commute.

□

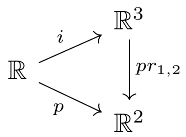
Wednesday, Sept. 23rd

We will speak today about the fundamental group of S^1. We claim that $\pi_1(S^1) \cong (\mathbb{Z}, +)$. But before we prove this, we will need to construct a foundation.

Consider the following maps:

$$p : \mathbb{R} \to S^1 \subset \mathbb{R}^2 \qquad i : \mathbb{R} \to \mathbb{R}^3 \qquad pr_{1,2} : \mathbb{R}^3 \to \mathbb{R}^2$$
$$s \mapsto (\cos(2\pi s), \sin(2\pi s)) \qquad s \mapsto (\cos(2\pi s), \sin(2\pi s), s) \qquad (x, y, z) \mapsto (x, y)$$

p maps the real line onto the circle, i maps the real line onto a spiral in $\mathbb{R}^3$, and $pr_{1,2}$ projects $\mathbb{R}^3$ onto $\mathbb{R}^2$. Thus, the following diagram commutes:



Note that $p^{-1}((1,0)) = \mathbb{Z} \subset \mathbb{R}$.

Consider the following paths (loops):

$$\begin{aligned} \omega_n : I &\to S^1 & \omega_n(0) = \omega(1) &= (1,0) \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) & \omega_n &\text{ is a loop} \\ \tilde{\omega}_n : I &\to \mathbb{R} & 0 = \tilde{\omega}_n(0) \neq \tilde{\omega}_n(1) &= n \\ s &\mapsto ns & \tilde{\omega}_n &\text{ is not a loop} \end{aligned}$$

Note that $p \circ \tilde{\omega}_n = \omega_n$.

Theorem 2. *The map $\phi : \mathbb{Z} \to \pi_1(S^1, (1,0))$ by $\phi(n) = [\omega_n]$ is an isomorphism.*

Remark 1. *$\phi(n) = [\omega_n] = [p \circ \tilde{\omega}_n]$. Moreover, if $\tilde{f} : I \to \mathbb{R}$ is continuous with $\tilde{f}(0) = 0$ and $\tilde{f}(1) = n$ then we can claim that $[p \circ \tilde{f}] = \phi(n)$. This is because $\mathbb{R}$ is convex and therefore $\tilde{\omega}_n \overset{hom}{\sim} \tilde{f}$ relative to $(0,1)$. See the example from Wednesday on convex sets for further explanation.*

First, let's check that ϕ is a homomorphism. Consider the translations $\tau_m : \mathbb{R} \to \mathbb{R}$ defined by $\tau_m(x) = x + m$, where $m \in \mathbb{Z}$. We have that $\tilde{\omega}_{m+n} = \tilde{\omega}_m * \tau_m(\tilde{\omega}_n)$. Thus

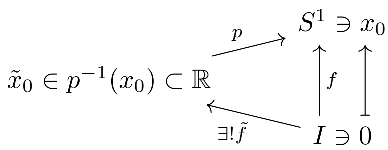
$$\begin{aligned} \phi(m+n) &= [p \circ \tilde{\omega}_{m+n}] \\ &= [p \circ (\tilde{\omega}_m * \tau_m(\tilde{\omega}_n))] \\ &= [(p \circ \tilde{\omega}_m) * (p \circ (\tau_m(\tilde{\omega}_n)))] \\ &= [\omega_m * \omega_n] \\ &= \phi(m) * \phi(n). \end{aligned}$$

So ϕ is a homomorphism.

To prove that ϕ is an isomorphism we need the following two lemmas:

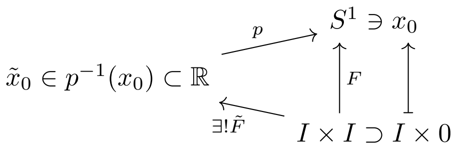
Lemma 2 (Path Lifting Lemma)**.** *For every path $f : I \to S^1$ continuous with $f(0) = x_0 \in S^1$ and for any $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifting $\tilde{f} : I \to \mathbb{R}$ with $p \circ \tilde{f} = f$ and $\tilde{f}(0) = \tilde{x}_0$.*

This lemma can be summarized by the following commuting diagram:



Lemma 3 (Homotopy Lifting Lemma)**.** *For every homotopy $F : I \times I \to S^1$ relative to x_0 (i.e. a homotopy F such that $F(t,0) = x_0$ for all $t \in I$) and for any $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifting $\tilde{F} : I \times I \to \mathbb{R}$ such that $\tilde{F}$ is a homotopy relative to $\tilde{x}_0$ and $p \circ \tilde{F} = F$.*

This lemma can also be summarized by the following commuting diagram:



We will prove these lemmas on Friday.

Proof of Theorem. Let us fix $x_0 \in S^1$ and $\tilde{x}_0 = 0 \in \mathbb{R}$. We will prove the surjectivity of ϕ first, then prove its injectivity.

Pf of Surjectivity: Let $f : I \to S^1$ be continuous with $f(0) = f(1) = x_0$. Then by the Path Lifting Lemma there exists a unique $\tilde{f} : I \to \mathbb{R}$ continuous map such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = 0 \in \mathbb{R}$. Obviously $\tilde{f}(1) \in \mathbb{Z}$ because $p(\tilde{f}(1)) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$. Suppose $\tilde{f}(1) = n \in \mathbb{Z}$. Hence we have that

$$\phi(n) = [p \circ \tilde{f}] = [f].$$

In other words ϕ is surjective.

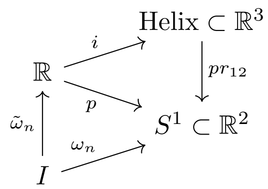
Pf of Injectivity: Assume $\phi(m) = \phi(n)$, where $m, n \in \mathbb{Z}$, $m \neq n$. Then $\phi(m) = \phi(n)$ implies that $\omega_n \sim \omega_m$ (i.e. there exists a homotopy $F : I \times I \to S^1$ such that $F(0,t) = \omega_m(t)$, $F(1,t) = \omega_n(t)$, and $F(s,0) = x_0 = (1,0)$). Then by the Homotopy Lifting Lemma there exists a unique $\tilde{F} : I \times I \to \mathbb{R}$ such that $p \circ \tilde{F} = F$, implying $\tilde{F}(s,0) = \tilde{x}_0$ for all $s \in I$.

By construction we know $p(\tilde{F}(s,1)) = F(s,1) = x_0 \in S^1$. Hence $F(s,1) \in \mathbb{Z}$ for all $s \in I$. Since $\tilde{F}$ is continuous this implies that $\tilde{F}(s,1)$ is constant for all $s \in I$. In particular $\tilde{F}(0,1) = \tilde{F}(1,1)$. Now, we know that $\tilde{F}(0,t)$ and $\tilde{\omega}_m$ are two lifts of ω_m and $\tilde{F}(0,t) = \tilde{\omega}_m(t)$ for all t. In particular, $\tilde{F}(0,1) = \tilde{\omega}_m(1) = m$. We can similarly conclude that $\tilde{F}(1,1) = \tilde{\omega}_n(1) = n$. So $m = n$ as desired.

□

Friday, Sept 25th

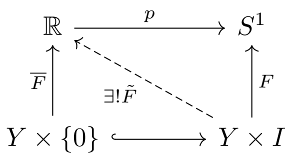
Recall that we constructed this diagram on Wednesday:



We proved using this diagram that the map $\phi : \mathbb{Z} \to \pi_1(S^1)$ defined by $\phi(n) = [\omega_n]$ is an isomorphism. To prove this we used two specific cases of general lemmas, the Path Lifting Lemma and the Homotopy Lifting Lemma. These two lemmas are consequences of a third lemma, which for lack of a better term we will call the Lifting Lemma.

Lemma 4 (Lifting Lemma)**.** *Let Y be a connected space and let a continuous map $F : Y \times I \to S^1$ be given. Let $p : \mathbb{R} \to S^1$ be the map given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$. Assume that there is a continuous map $\overline{F} : Y \times \{0\} \to \mathbb{R}$ which lifts $F|_{Y \times \{0\}} : Y \times 0 \to S^1$, i.e. $p \circ \overline{F} = F|_{Y \times \{0\}}$. Then there is a unique continuous lift $\tilde{F} : Y \times I \to \mathbb{R}$ of F such that $\tilde{F}(y, 0) = \overline{F}(y)$.*

This lemma is summarized in the following commuting diagram:



The idea of the proof of the Lifting Lemma is as follows: first we define a local lift of F on sets of the form $N \times I$ with N being a suitable neighborhood of a given point $y_0 \in Y$. Next we will prove the uniqueness of this lift on $\{y_0\} \times I$. A very important fact in this proof is that there exists an open cover $\left(\bigcup_\alpha\right)_{\alpha \in A}$ of S^1 such that for all U_α

$$p^{-1}(U_\alpha) = \bigsqcup_{\beta \in B} \tilde{U}_\beta$$

where $\tilde{U}_\beta$ are open sets in $\mathbb{R}$ such that $p(\tilde{U}_\beta) = U_\alpha$ and $p|_{\tilde{U}_\beta} : U_\beta \to U_\alpha$ is a homeomorphism.

Proof. Step 1: Given a point $(y_0, t') \in Y \times I$ there exists α' such that $F(y_0, t') \in U_{\alpha'}$. Since F is continuous, there are open neighborhoods $y_0 \in N_{y_0} \subset Y$ and $t' \in (a_{t'}, b_{t'}) \subset I$ of y_0 and t' respectively such that

$$F(N_{t'} \times (a_{t'}, b_{t'})) \subset U_{\alpha_0}.$$

Fixing y_0 and letting t vary in I we get an open cover of $y_0 \times I$ by sets of the form $N_t \times (a_t, b_t)$. Since $y_0 \times I$ is a compact set there exists a finite subcover of $y_0 \times I$. First, this implies a partition of I into $0 = t_0 < t_1 < \ldots < t_m = 1$ such that for each i there exist open neighborhoods $N_i \ni y_0$ and $(a_{t_i}, b_{t_i}) \ni t_i$ such that $F(N_i \times (a_{t_i}, b_{t_i})) \subset U_{\alpha_i}$ for some $\alpha_i \in A$. Second, this implies that there exists an open neighborhood $y_0 \in N = \cap_{i=1}^m N_i$ such that $F(N \times [a_{t_i}, a_{t_{i+1}}]) \subset U_{\alpha_i}$. The homeomorphism $p|_{\tilde{U}_{\alpha_0}} : \tilde{U}_{\alpha_0} \to U_{\alpha_0}$ allows us to define a lifting $\tilde{F}_0 : N \times [0, t_0] \to \mathbb{R}$ of $F|_{N \times [0,t_0]}$ as follows:

$$\tilde{F}(y,t) = \left(\left(p|_{\tilde{U}_{\alpha_0}} \right)^{-1} \circ F \right)(y,t).$$

Now, assume there is a lifting $\tilde{F}_{\circ i} : N \times [0, t_i] \to \mathbb{R}$ of $F|_{N \times [0,t_i]}$. we extend it to a lifting

$$\tilde{F}_{\circ i+1} : N \times [0, t_{i+1}] \to \mathbb{R}$$

by means of the homeomorphism $p|_{\tilde{U}_{\alpha_i}} : \tilde{U}_{\alpha_i} \to U_{\alpha_i}$. So

$$\tilde{F}(y,t) = \left(\left(P|_{\tilde{U}_{\alpha_i}} \right)^{-1} \circ F \right)(y,t) \text{ for } (y,t) \in [t_i, t_{i+1}].$$

Step 2: Choose a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ as before and assume that there are two liftings $\tilde{F}_1, \tilde{F}_2 : I \to \mathbb{R}$ of $F : I \to S^1$ with initial condition $\tilde{F}_1(0) = \tilde{F}_2(0) = x_0$. Since $[0, t_1] \subset U_{\alpha_0}$ and $p|_{\tilde{U}_{\alpha_0}} : \tilde{U}_{\alpha_0} \to U_{\alpha_0}$ is a homeomorphism we have $\tilde{F}_1|_{[0,t_1]} \equiv \tilde{F}_2|_{[0,t_1]}$. By induction we can check that this equality is true by the same argument for each t_i.

□

The Path Lifting Lemma follows from the Lifting Lemma by considering the special case where Y is a point. The derivation of the Homotopy Lifting Lemma is harder to show and we will deal with it on Monday.