FREE GROUPS

Definition: Let G be a group and let $S = \{x_j \mid j \in J$ be a set of elements of G. We say that S generates the group G if every element of G can be written as a finite product of elements of S (i.e. given $g \in G$ there exists $i_1, ..., i_n \in J$ not necessarily distinct and $e_1, ..., e_n \in \mathbb{Z}$ such that $g = i_1^{e_1} ... i_n^{e_n}$. This expression is not necessarily unique.)

Given a set X, we want to define a group $F(X)$ such that
· S generates $F(X)$, and
· $F(X)$ is the freest group containing X (i.e. there are no nontrivial relations between the elements of X).
Definition: The set of words in X is the set

$$W(X) = \{x_1^{e_1} ... x_n^{e_n} | x_i \in X, e_i = \pm 1, n \in \mathbb{N}\}$$

If $w \in W(X)$, we call w a word in X and 1 is called the empty word. We can endow $W(X)$ with a binary operation of concatenation.

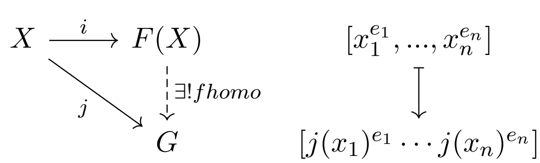
Definition: Let w and w' be two words in X. We say that w can be elementarily reduced to w' (or vice versa) and denote by $w \sim_e w'$ if the word w (resp. w') contains a subword xx^{-1} or $x^{-1}x$ and the word w^{-1} (resp. w) is obtained from it by deleting this subword.

Definition: Let w and w' be two words in X. We say that $w \sim w'$ if and only if there is a unique finite sequence of words $w_1, ..., w_n$ in X such that $w \sim_e w_1 \sim_e ... \sim_e w_n \sim_e w'$.
Obviously $\sim$ is an equivalence relation.

Definition: We define $F(X)$ to be the set of equivalence classes of words in X; i.e. $F(X) := W(X)/\sim$.
Note that the relation $\sim$ is consistent with concatenation of words; i.e. if $w_1, w_1', w_2, w_2' \in X$ and $w_1 \sim w_1'$ $w_2 \sim w_2'$, then $w_1 w_2 \sim w_1, w_1'$.

Theorem: $F(X)$ together with the operation induced by concatenation of words in $W(X)$ is called the free group on X.

Proposition: (Universal mapping property) Let X be a set and G be a group.
Let $i : X \to F(X) : x \mapsto [x]$ and let $j : X \to G$, then there is a unique group homomorphism $f : F(X) \to G$ such that $f \circ i = j$ where $f([x_1^{e_1} ... x_n^{e_n}]) = [j(x_1)^{e_1} ... j(x_n)^{e_n}]$.



Example: Let $X = \{x\}$. Then $F(X) = \{x^n \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}$. Let $G = < a \mid a^n = 1 >$ be the group of order n and let $j : X \to G$ be the map $j(x) = a$. By the UMP, we get an epimorphism $f : F(X) \twoheadrightarrow f([x]) = n$ and we get $G = F(X)/ker(f) \simeq \mathbb{Z}/n\mathbb{Z}$.
More generally, if G is a group generated by a set X, we have an epimorphism

$$f : F(X) \twoheadrightarrow G$$

and therefore $G \simeq F(X)/ker(f)$ which gives us a presentation of G by generators and relations; i.e. $< x \in X \mid r \in ker(f) >$.

FREE PRODUCT

Let H and K be two groups. We want to define a new group from them, the free group $H * K$. Consider the set of the words

$$W(H,K) = \{g_1 g_2 ... g_n \mid g_i \in H \text{ or } g_i \in K\} \cup \{1\}$$

together with the operation of concatenation.

Definition: Let w and w' be two words in $W(H,K)$. We say that w is an elementary reduction of w' (or vice versa) if w contains a subword of the form ab with $a, b \in H$ or $a, b \in K$ and w' is obtained from w by
· substituting ab by a single element in H (or K) which is the product a, b if $a \neq b^{-1}$;
· erasing the subword ab if $a = b^{-1}$.

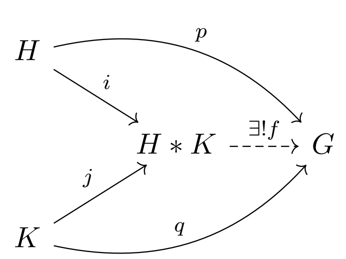
Definition: We say that two words w and w' in $W(H,K)$ are equivalent if and only if there exists a finite sequence $w_1, w_2, ..., w_n$ in $W(H,K)$ such that
$w \sim_e w_1 \sim_e w_2 \sim_e ... \sim_e w_n \sim_e w'$.
Obviously $\sim$ is an equivalence relation.

Definition: We define $H * K$ as the equivalence classes of words in $W(H,K)$; i.e. $H * K := W(H,K)/\sim$

Remark: Any equivalence class contains a unique reduced word satisfying $h_1 k_1 h_2 k_2 ... h_r k_r$
·$h_i \in H$ for all $i = 1, ..., r$
·$k_i \in K$ for all $i = 1, ..., r$
·$h_i \neq 1$ for all $i = 2, ..., r$
·$k_i \neq 1$ for all $i = 1, ..., r-1$
Equivalence relation is consistent with concatenation.

Theorem: $H * K$ is a group, called the free group of H and K.

Proposition: (Universal Mapping Property) Let H, K and G be groups. Let $i : H \to H * K$, $j : K \to H * K$ be inclusion maps and $p : H \to G$, $q : K \to G$ be homomorpshims, then there is a unique group homomorphism $f : H * K \to G$ such that $f \circ i = p$ and $f \circ j = q$.



Corollary 2.1 $H * K$ is the unique group satisfying the UMP up to isomorphism.

Examples:
1. Let $X = \{x_1, ..., x_n\}$ be a set of elements and define $F_i := F(x_i) \simeq \mathbb{Z}$ for $i = 1, ..., n$. Then

$$F(X) \simeq F_1 * F_2 * ... * F_n \simeq \mathbb{Z} * ... * \mathbb{Z} =: \mathbb{Z}^{*n}$$

2. If $H =< h \mid r_h >$ and $K =< k \mid r_k >$ are presentations by generators and relations of the groups H and K then

$$H * K =< h, k \mid r_h, r_k >$$

3. $\mathbb{Z}_2 * \mathbb{Z}_2 =< a, b \mid a^2, b^2 > \simeq < x, y \mid x^2, xyx^{-1} = y^{-1} >= \mathbb{Z} \rtimes \mathbb{Z}_2$
Define $\omega : \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2 : x \mapsto$ the length of x *mod* 2. Then $\ker(\omega) =< ab >$ and it is a normal subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$ that is isomorphic to $\mathbb{Z}$.

Remark: Each factor H_α of a free product $*_{\alpha \in A} H_\alpha$ is identified by a subgroup of $*_{\alpha \in A} H_\alpha$; the subgroups formed by the empty word 1 and one letter words $h \in H_\alpha$. We have that

$$\{1\} = \bigcap_{\alpha \in A} H_\alpha \text{ and } (H_\alpha \setminus \{1\}) \cap (H_\beta \setminus \{1\}) = \emptyset \text{ if } \alpha, \beta \in \text{ A and } \alpha \neq \beta.$$

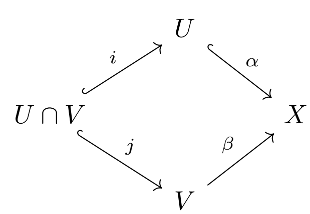
1 Fundamental Theorem of Algebra

Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be complex polynomial. Then $f(z) = 0$ has at least a root in $\mathbb{C}$.

Proof. If $a_n = 0$, then $z = 0$ is a root.
Assume $a_n \neq 0$. Let us show that the only complex polynomials with roots are the constant ones.
We will use a combination of two homotopies.
Let $C_r = \{z \in \mathbb{C} : \|z\| = r\}$. $F_r : I \times C_r \to \mathbb{C}^*$ a homotopy of maps between
· $f : C_r \to \mathbb{C}^*$ and
· $p_n : C_r \to \mathbb{C}^*$ given by $p_n(z) = z^n$ for $r \gg 0$.
The homotopy F_r is defined by $F_r(t, z) = z^n + t(a_1 z^{n-1} + \cdots + a_{n-1}z + a_n)$.
Note that for $r > (|a_1| + \cdots + |a_n|)$ we have that
$z^n > (|a_1| + \cdots + |a_n|)z^{n-1} > |a_1 z^{n-1} + \cdots + a_{n-1}z + a_n|$.
· Consider the homotopy of maps
$G_r : C_r \times I \to \mathbb{C}^*$ defined by $(z, s) \mapsto (sz)$. Now $G_r(z, 0) = a_n$ constant map and $G_r(z, 1) = f(z)$.
Let $\delta : I \to \mathbb{C}^*$ be a path from $\delta(0) = r^n \in \mathbb{C}^*$ to $\delta(1) = a_n \in \mathbb{C}^*$. δ satisfies that $\delta \circ (a_n)_* = (p_n)_*$. □

2 Seifert van Kampen Theorem

Theorem 2.1 (van Kampen). *Let X be a path connected space and $X = U \cup V$, where U and V are open sets whose intersection is non empty and path-connected. Let $x_0 \in U \cap V$ and consider the inclusion maps:*



Then,

$$\pi_1(X, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0)/N$$

where '$*$' denotes the free product and $N = N(i_*(\zeta)\overline{j_*(\zeta)} \mid \zeta \in \pi_1(U \cap V, x_0))$ is the normal subgroup of the free product $\pi_1(U, x_0) * \pi_1(V, x_0)$ generated by elements of the form $i_*(\zeta)\overline{j_*(\zeta)}$ with $\zeta \in \pi_1(U \cap V, x_0)$.

The above theorem is useful for computing the fundamental groups of various spaces.
Note: van Kampen's theorem can be generalized to the case when $X = \bigcup_\alpha A_\alpha$, where the A_α's are open sets satisfying certain conditions. The details of this will be given later.

Corollary 2.1. *Let X, U and V be as above. If $U \cap V$ is simply connected, then $\pi_1(X, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0)$.*

Proof. The proof follows directly by the definition of being simply connected (i.e. π_1 is trivial) □

Corollary 2.2. *The union of two simply connected spaces is simply connected provided their intersection is non empty and path connected.*

Note: The hypothesis that $U \cap V$ be path connected is essential for the theorem to hold. For instance, let $X = S^1$ and let P and Q be two distinct points on it. Let $U = X \setminus \{P\}$ and $V = X \setminus \{Q\}$. Then $U \cap V$ is not path connected. Clearly, $X = U \cup V$ fails to be simply connected even though U and V are.