Math 751 Week 5~6 Notes

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10/07/2015~10/14/2015

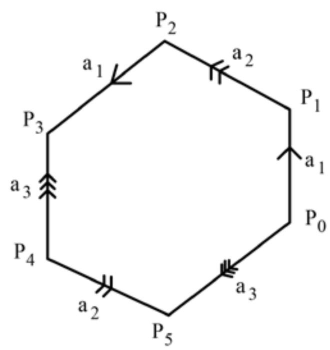
Review:Yuqing Li

Classification of surfaces

Definition An n-dimensional manifold with no boundary is a T_2 topological space X such that every $x \in X$ has a neighborhood U_x homeomorphic to the open ball $B^n \subset R^n$

Definition A surface is a two dimensional connected manifold with no boundary. (A surface is a compact topological space)

Let P a polygonal region in the plane with vertices $P_0, P_1, \cdots, P_{m-1}$ and labeled oriented edges.



Starting at P_0 and going along the perimeter of P in counter clockwise direction gives us a labeling scheme

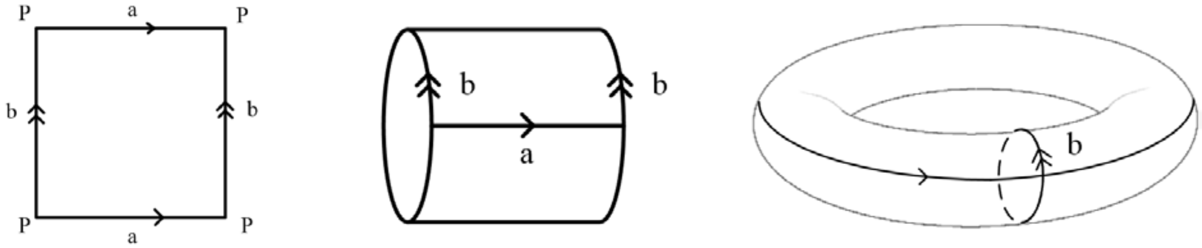
$$a_1 a_2 a_1 a_3^{-1} a_2 a_3^{-1}$$

From P and the labelling scheme we get an identification space as follows:

- Points in the interior of P are identified to themselves
- Two edges of the same label are identified by an orientation preserving linear homeomorphism

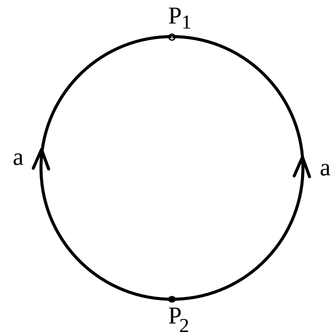
Examples:

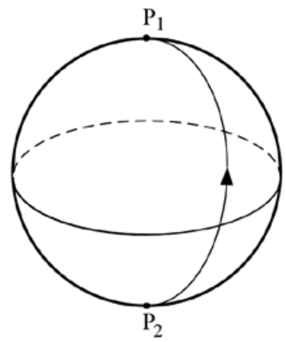
1. The torus $T^2 \cong S^1 \times S^1$

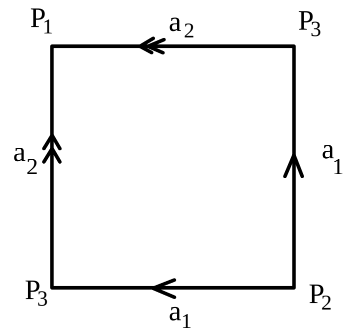


$aba^{-1}b^{-1}$

2. The sphere S^2



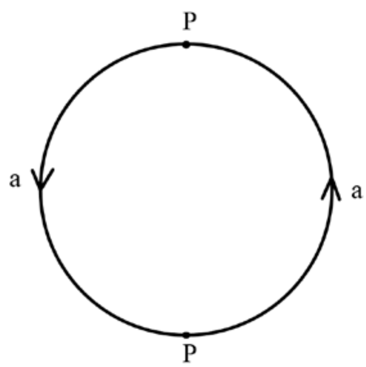


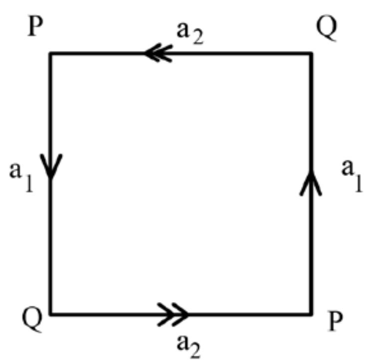


aa^{-1}

$a_1 a_2 a_2^{-1} a_1^{-1}$

3. The real projective plane RP^2

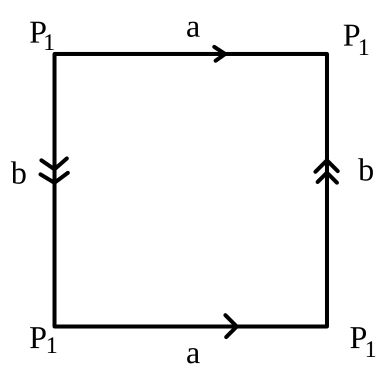




Cannot be embedded in R^2

aa

4. The Klein bottle K

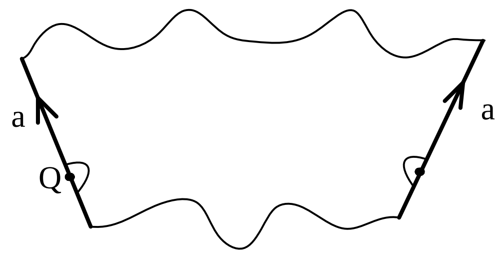


Cannot be embedded in R^3

$aba^{-1}b$

Definition Regular labelling scheme : even number of edges which are identified in pairs.

Theorem If P is a polygonal region with a regular labelling scheme then the corresponding identification space is a compact connected 2-dimensional manifold



Remark: Note that in this case, any point Q in the sides of P has a neighborhood homeomorphic to the open ball B^2. See figure.

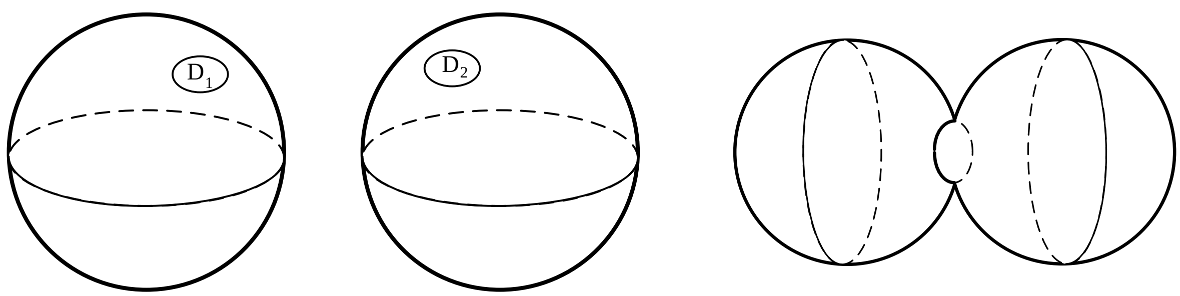
Definition (connected sum) Let M and N be two surfaces. We define the connected sum of M and N, denoted as $M \# N$, as

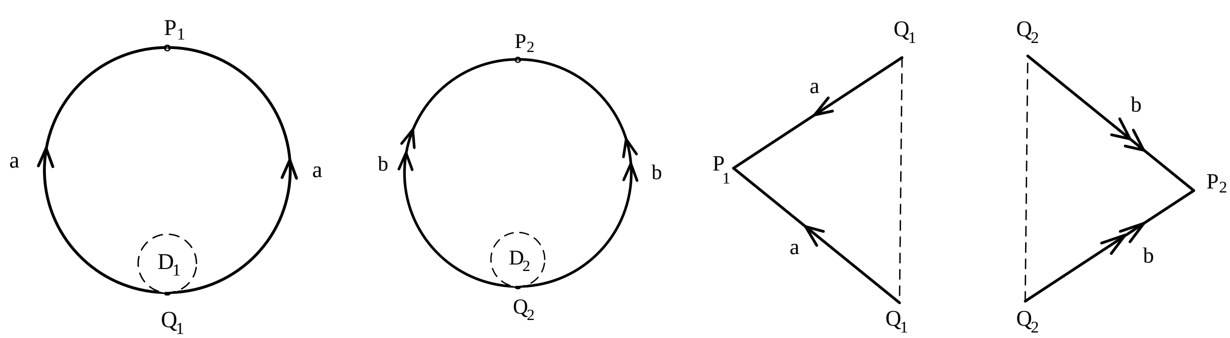
$$(M \setminus D_1) \cup (N \setminus D_2) \big/ (\partial D_1 \sim \partial D_2)$$

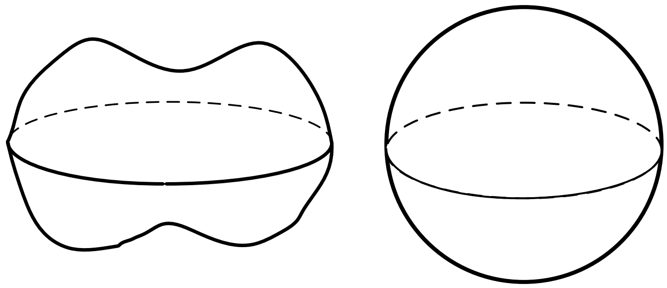
where, D_1 is a disk in M and D_2 is a disk in N and the circles ∂D_1 and ∂D_2 are identified by a homeomorphism.

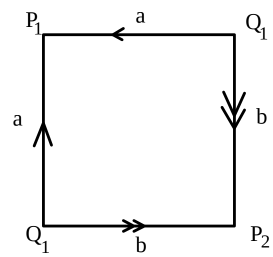
Example

1. $S^2 \# S^2 \simeq S^2$



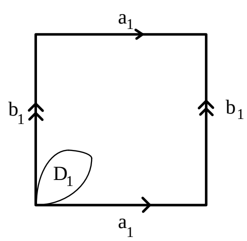


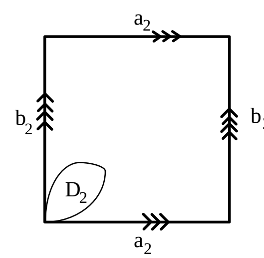


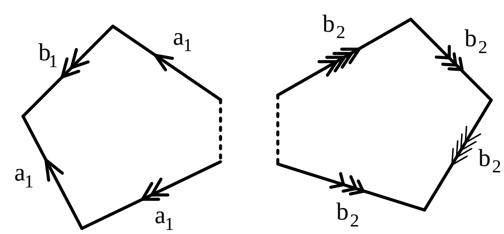


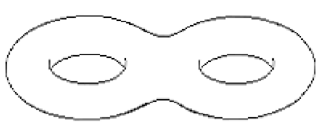
2. $M\#S^2 \cong M$

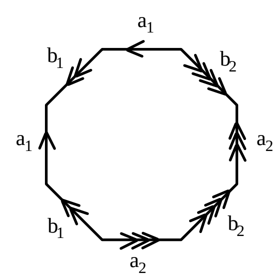
3. $T^2\#T^2 \cong 2T^2$



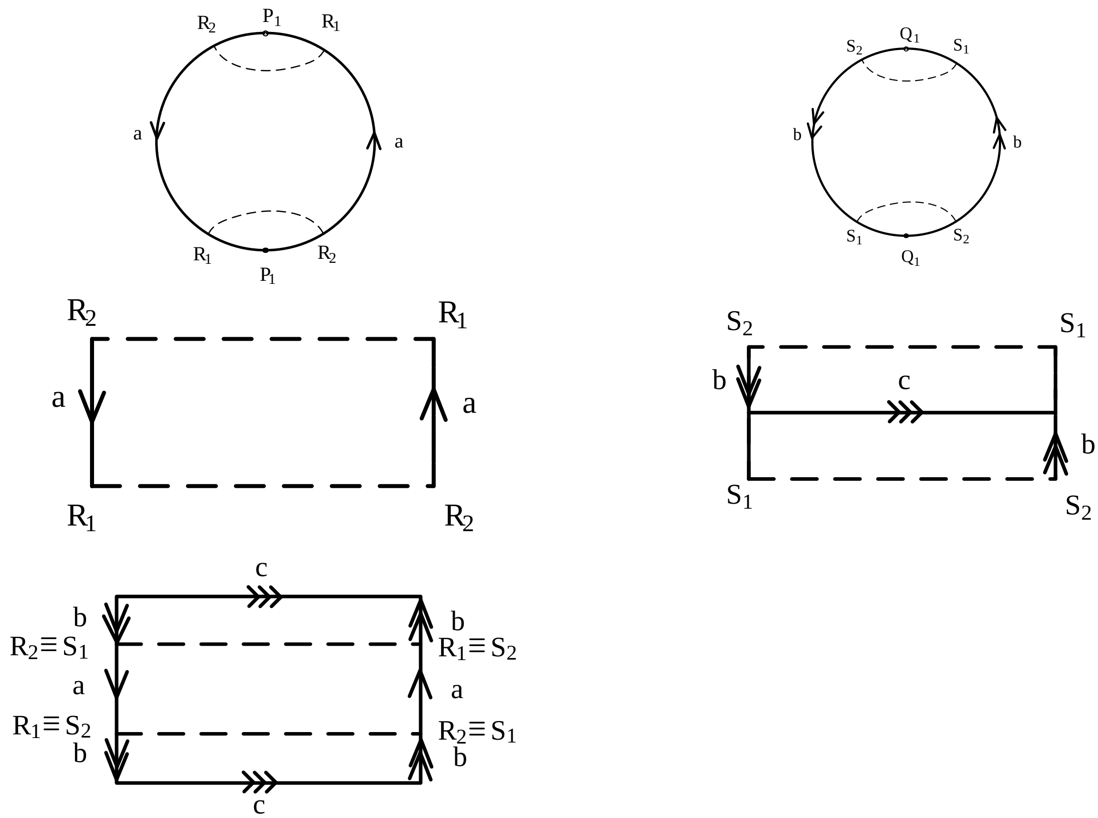








4. $RP^2 \# RP^2 \cong K$



Notation:

$nT^2 := T^2 \# T^2 \# \cdots \# T^2 (n \;\; summands)$

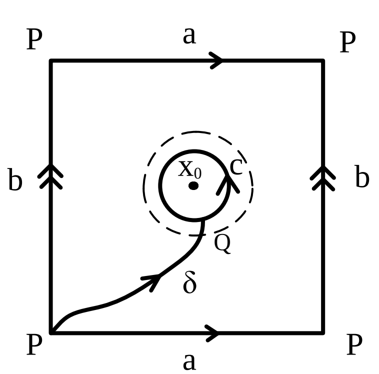
$nRP^2 := RP^2 \# RP^2 \# \cdots \# RP^2 (n \;\; summands)$

Theorem: Any surface is homeomorphic to S^2 , nT^2 or nRP^2 for some $n \in N$.

Fundamental groups of surfaces

1. $\pi_1(S^2) \simeq \{1\} \quad (S^2 \textit{ is simply connected})$

2. $\pi_1(T^2) \quad$ *Let us use Van Kampen Theorem*



$U = T^2 - \{x_0\} \sim S^1 \vee S^1$

$V = B_\varepsilon(x_0) \;\; contractible$

$U \cap V = B_\varepsilon(x_0) - \{x_0\} \sim S^1$

a and b are generators of $\pi_1(U,P) \cong Z * Z$

c generator of $\pi_1(C,Q) \cong Z$ for $Q \in B_\varepsilon(x_0) - \{x_0\}$

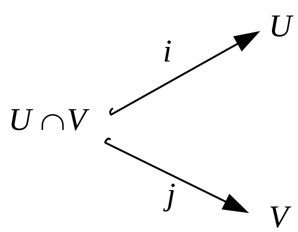
$\delta : I \longrightarrow T^2$ continuous with $\delta(0) = P$ *and* $\delta(1) = Q$

$\delta_\# : \pi_1(U,P) \longrightarrow \pi_1(U,Q)$ is an isomorphism given by

$$[\gamma] \mapsto [\overline{\delta} * \gamma * \delta]$$

Denote $\tilde{a} = \delta_\#(a)$ *and* $\tilde{b} = \delta_\#(b)$

Van Kampen's theorem $\pi_1(T^2) \cong \left\langle \tilde{a}, \tilde{b} \mid i_* c \right\rangle$



$$\begin{aligned} i_* c &\sim \overline{\delta} * a * b * a^{-1} * b^{-1} * \delta \\ &\sim (\overline{\delta} * a * \delta) * (\overline{\delta} * b * \delta) * (\overline{\delta} * a^{-1} * \delta) * (\overline{\delta} * b^{-1} * \delta) \\ &\sim \tilde{a} * \tilde{b} * \tilde{a}^{-1} * \tilde{b}^{-1} \end{aligned}$$

$$\pi_1(T^2) \cong \left\langle \tilde{a}, \tilde{b} \mid \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} \right\rangle \cong Z^2 \quad (abelian)$$

Theorem If X is the identification space of a polygon P and a labelling scheme $a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n}$ with $\varepsilon_i = \pm 1$,such that all the vertices of P are identified by the projection $\pi : P \longrightarrow X$ then

$$\pi(x) = \left\langle a_1 a_2, \cdots a_n \mid a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_n^{\varepsilon_n} = 1 \right\rangle$$

Examples:

$$\pi(nT^2) = \left\langle a_1 a_2, \cdots a_n, b_1 b_2, \cdots b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1 \right\rangle$$

$$\pi(nRP^2) = \left\langle a_1 a_2, \cdots a_n \mid a_1^2 a_2^2 \cdots a_n^2 = 1 \right\rangle$$

Theorem The surfaces S^2, nT^2 *and* nRP^2 have all non isomorphic fundamental group but are neither homotopy equivalent homeomorphic.

Proof Let us consider the abelianized fundamental group

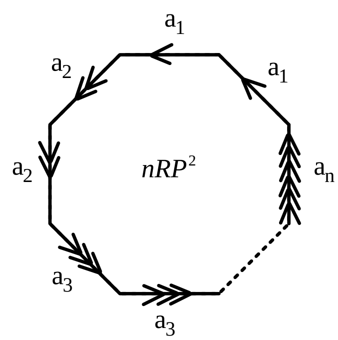
$$\pi_1^{ab}(x) := \pi_1(x) \Big/ [\pi_1(x), \pi_1(x)]$$

where $[G,G]$ denotes the commutator of group G defined as $\left\{ aba^{-1}b^{-1} \mid a, b \in G \right\}$

1[st] $\pi_1(S^2)$ is trivial and so is $\pi_1^{ab}(S^2)$

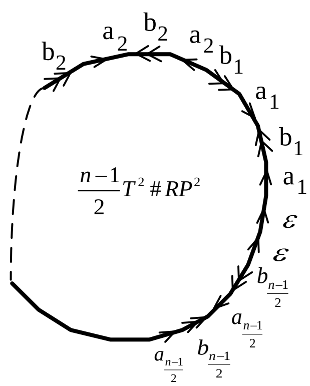
2[nd] $\pi_1^{ab}(nT^2) \cong Z \times \cdots \times Z \cong Z^{2n}$

3[rd] $\pi_1^{ab}(nRP^2) \cong Z^{n-1} \times Z/2Z$



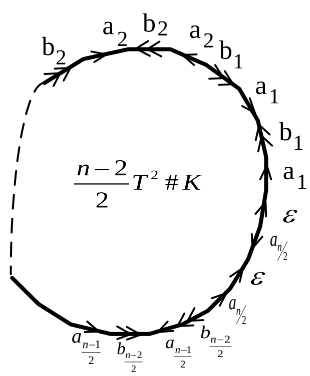
is equivalent to

If n is odd



abelian group
n generators
rank n-1
2-torsion

If n is even



abelian group
n generators
rank n-1
2-torsion

$$\left.\begin{array}{l} a_{n/2}\varepsilon a_{n/2}^{-1}\varepsilon^{-1} \\ a_{n/2}\varepsilon a_{n/2}^{-1}\varepsilon \end{array}\right\} \varepsilon^2 = 1$$

Corollary If a surface X is simply connected then $X \cong S^2$

Remarks: $X \cong S^4$ and $S^2 \times S^2$ are non homeomorphic simply connected 4-manifolds.

For $n = 3$, Theorem (Poincare/Perelman)

If X is a closed, simply connected. 3-manifold then X is homeomorphic to S^3.

For $n = 5$, Theorem

Any simply connected n-manifold homotopy equivalent to S^n is homeomorphic to S^n.