Math 751 Week 3 Notes- Part II

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Classification of surfaces

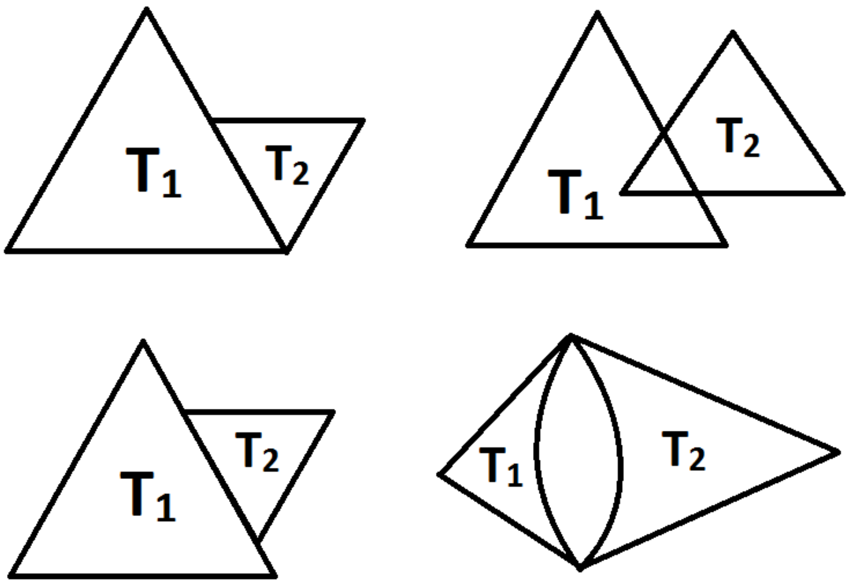
Definition a triangulation of a surface S consists of a finite family of closed subsets $\{T_1, \dots, T_n\}$ such that

1, $\bigcup_{i=1}^{n} T_i = S$

2, $\forall i, \exists$ a triangle T_i' in $\mathbf{R}^2$ and a homeomorphism $\phi_i: T_i' \to T_i$

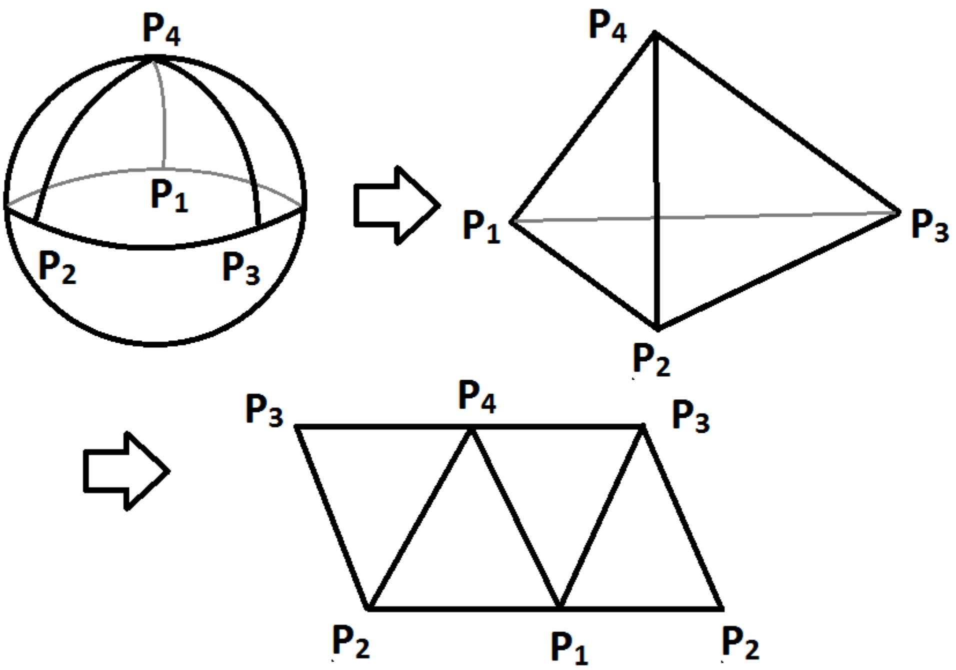
3, $\forall i \neq j, T_i \cap T_j$ is either empty or the image of a vertex or an edge

Remark (not allowed)

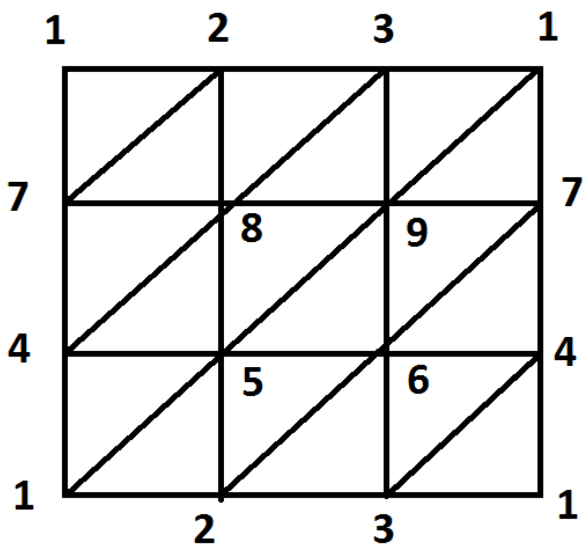


Example

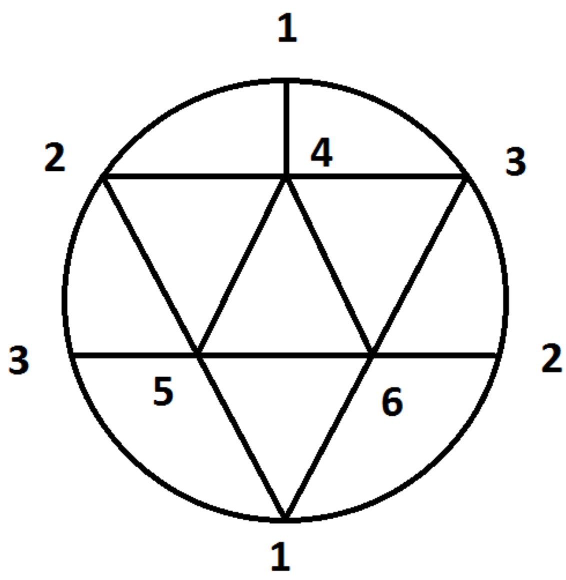
sphere



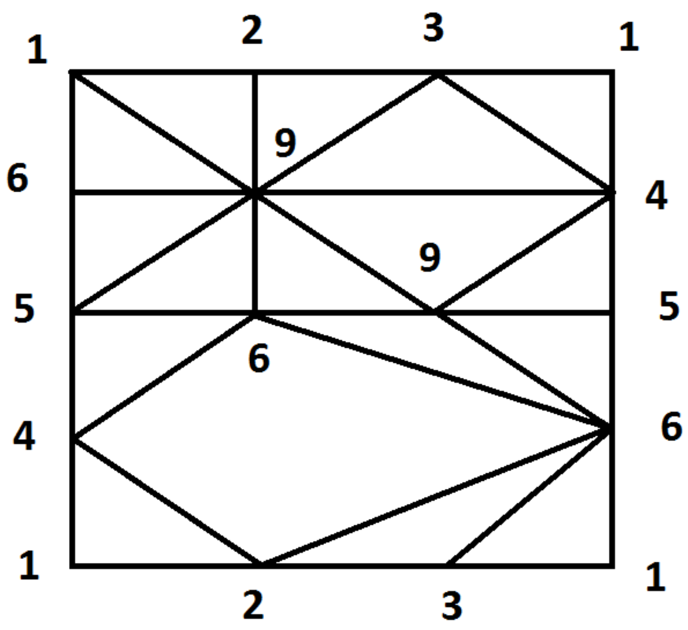
Torus



Projective plane



Klein bottle



Assume S is triangulable by n triangles $T_1, \ldots, T_n$. From the triangulation we get a polygon $P = \bigcup_{i=1}^{n} T_i' \subset \mathbf{R}^2$, with the corresponding homeomorphisms $\phi_i : T_i' \to T_i$ and a regular labeling scheme. Next we are going to present a standard process to reduce the labeling scheme to one of the following canonical forms:

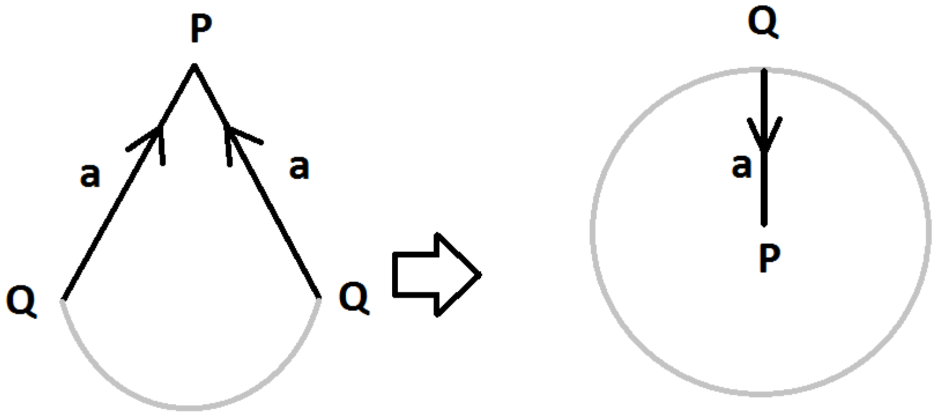
1, aa^{-1}, sphere

2, $a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_na_n^{-1}b_n^{-1}$, n-tori

3, $a_1^2 \ldots a_n^2$, n-projective plane

Notation In a regular labeling scheme we call an edge $\ldots a \ldots a^{-1} \ldots$ the edge of the 1st kind; and call an edge $\ldots a \ldots a \ldots$ the edge of 2nd kind.

Step 1 Eliminate the adjacent edges of the 1st kind

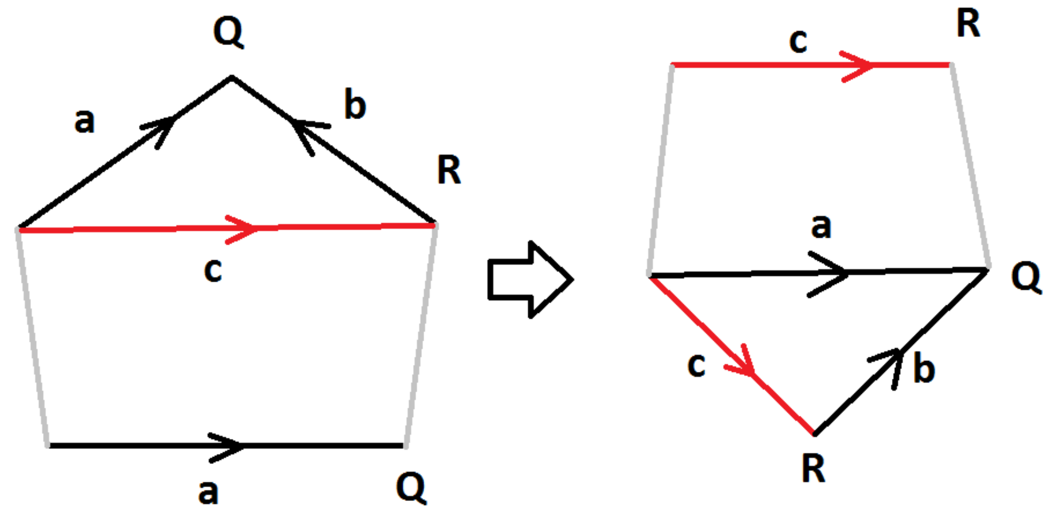


Step 2 Vertex reduction

Transform the polygon to another polygon such that all the vertices are identified.

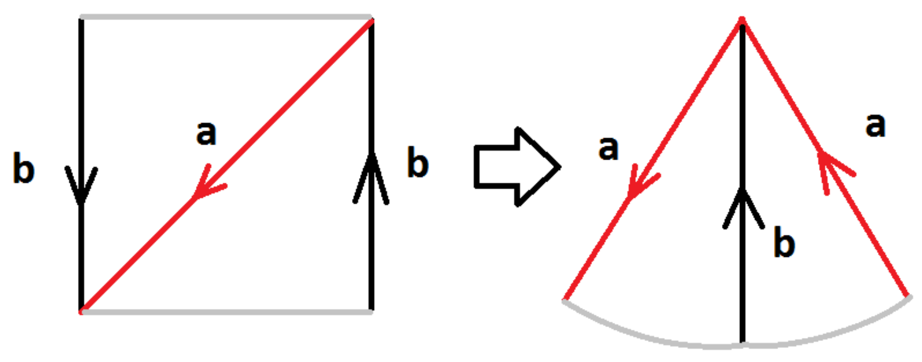
Suppose we have at least one extra vertex in the polygon P. If the extra vertex Q only appears once, it must correspond to a pair of edges of the 1st kind. Then apply step 1.

Suppose we have at least two vertices Q and R that (each) appear more than once. Assume they are adjacent



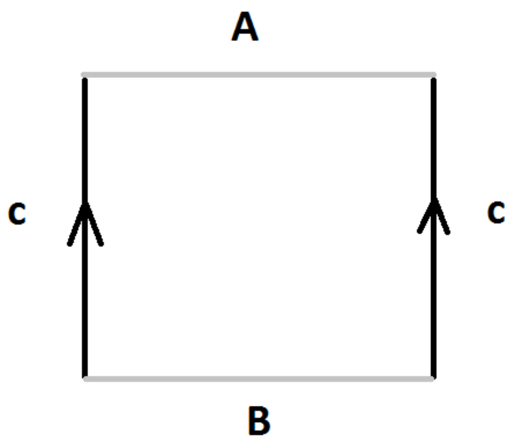
Cut along the segment *c* and glue along the edge *a*. One of Q vertex disappears and one of R appears. After several steps, there will be just one vertex of Q which can be eliminated by the previous step.

Step 3 Make any pair of edges of the 2nd kind adjacent ... b ... b ...



Cut along the segment *a* and glue along the edge *b*.

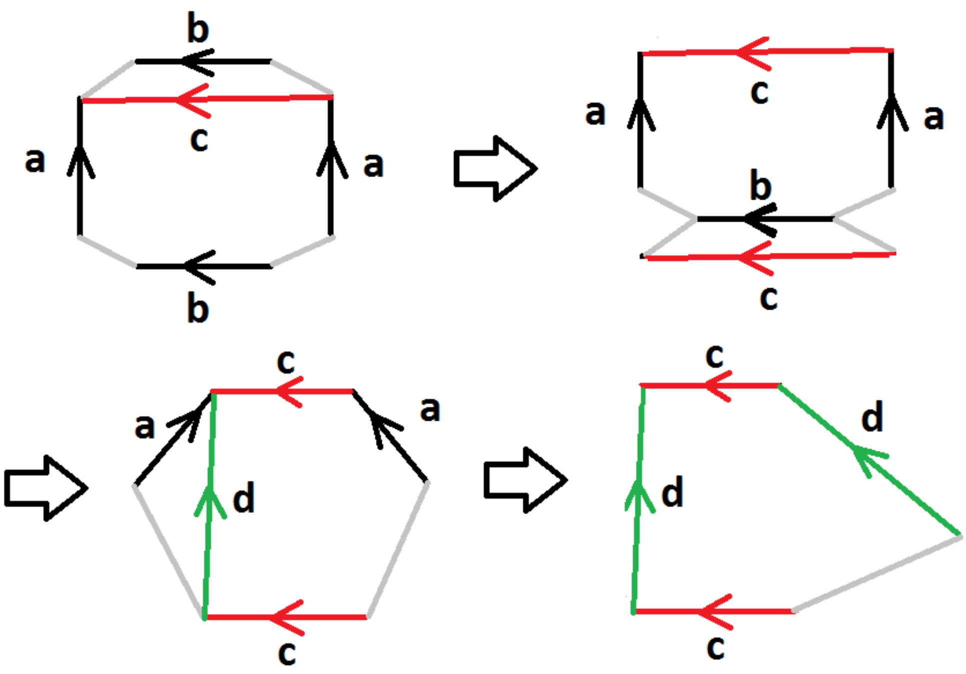
Once all edges of the 2nd kind are adjacent, and there is no edges of the 1st kind, then we reach $a_1^2 \dots a_n^2$, n-projective plane. If there is at least one edge of the first kind, we must have some labeling like $\dots c \dots d \dots c^{-1} \dots d^{-1} \dots$. Otherwise, consider



where no edge in A is identified to an edge in B, and vice versa.

In such assumption, the tail and tip points of "c" are not identified, which contradicts Step 2.

Consequently we have



Repeat this process until all edges of the 1st kind are grouped in pairs as $a_1b_1a_1^{-1}b_1^{-1}$.

If no edges of the 2nd kind, we reach $a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_na_n^{-1}b_n^{-1}$, n-tori.

If there are indeed some edges of the 1st kind, $\ldots a_1b_1a_1^{-1}b_1^{-1}cc \ldots$, we can prove that $\mathbf{RP}^2\#T^2 \simeq 3\mathbf{RP}^2$.