Covering Spaces

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1 Definition of Covering spaces

Definition 1.1 *Let E and B be topological space. A map p $E \mapsto B$ is called a covering map if:*

(1) *p is surjective and continous;*

(2) *For all $b \in B$, there exists an open neighborhood U of b which is evenly covered.(i.e. $p^{-1}(U) = \coprod_\alpha V_\alpha$, where V_α are disjoint and open sets in E and $p|_{V_\alpha}$: $V_\alpha \mapsto U$ is a homeomorphism)*

Remark 1.1 *p: $E \mapsto B$ is a covering map, then p is open and is a local homeomorphism.*

Remark 1.2 *Not all local homeomorphisms gives covering maps.*

Remark 1.3 *$p^{-1}(b)$ is discrete. Let X be a topological space. $M \subset X$ is discrete if for any point $m \in M$, there is a open subset U of X s.t. $m \in U$ and $(M - \{m\}) \cap U = \emptyset$*

Example 1.1 *p: $\mathbb{R} \mapsto S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$.*

Example 1.2 *p: Helix $= \{(\cos 2\pi s, \sin 2\pi s, s) | s \in \mathbb{R}\} \subset \mathbb{R}^3 \mapsto S^1$, $(\cos 2\pi s, \sin 2\pi s, s) \mapsto (\cos 2\pi s, \sin 2\pi s)$.*

Example 1.3 *Id_X: $X \mapsto X$.*

Example 1.4 *p: $S^1 \mapsto S^1$, $z \mapsto z^n$.*

Example 1.5 *p: $S^n \mapsto \mathbb{PR}^n$, $x \mapsto [x] = \{\pm x\}$.*

Example 1.6 *If p_i: $E_i \mapsto B_i$ for $i = 1, 2$ are covering maps, then $p_1 \times p_2$: $E_1 \times E_2 \mapsto B_1 \times B_2$, $(x, y) \mapsto (p_1(x), p_2(x))$ is a covering map.*

Definition 1.2 *Let p_i: $E_i \mapsto B$ for $i = 1, 2$ be covering maps, we say p_1 and p_2 are equivalent if there is a homeomorphism f: $E_1 \mapsto E_2$ s.t. $p_2 \circ f = p_1$.*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Lemma 1.1 *If p: $E \mapsto B$ is a covering map, $B_0 \subset B$ and $E_0 = p^{-1}(B_0)$, then $p|_{E_0}$: $E_0 \mapsto B_0$ is a covering map.*

Example 1.7 *Let p: $\mathbb{R}^2 \mapsto \mathrm{T}^2$ be a covering map. Take $p_0 = (1, 0)$. Let $B_0 = \{p_0\} \times S^1 \cup S^1 \times \{p_0\}$, then $p^{-1}(B_0) = (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \cup \mathbb{Z})$.*

2 Lifting properties

Theorem 2.1 (*Path lifting*) *Let p: $E \mapsto B$ be a covering map, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. If γ: $I \mapsto B$ is a path in B with starting point $b_0 = \gamma(0)$. Then there is a unique path lifting $\widetilde{\gamma}$: $I \mapsto E$, s.t. $\widetilde{\gamma}(0) = e_0$.*

Theorem 2.2 (*Homotopy lifting*) *Let F: $I \times I \mapsto B$ be a homotopy with $b_0 = F(t, 0)$. Then there is a unique lifting $\widetilde{F}$: $I \times I \mapsto E$, s.t. $\widetilde{F}(t, 0) = e_0$.*

Corollary 2.1 *If γ_1, γ_2: $I \mapsto B$ are two loops with $\gamma_1(0) = \gamma_2(0) = b_0$ and homotopic by some F, then $\widetilde{\gamma}_{1e_0} \sim \widetilde{\gamma}_{2e_0}$ by $\widetilde{F}$. In particular, the lifting $\widetilde{\gamma}_{1e_0}$ and $\widetilde{\gamma}_{2e_0}$ have the same end points. i.e. $\widetilde{\gamma}_{1e_0}(1) = \widetilde{\gamma}_{2e_0}(1)$.*

Definition 2.1 *Let $b_0 \in B$, and for $e_0 \in p^{-1}(b_0)$. We define:*

$$\Phi_{e_0} : \pi_1(B, b_0) \mapsto p^{-1}(b_0), \text{ by } [\gamma] \mapsto \widetilde{\gamma}_{e_0}(1)$$

Remark 2.1 *Φ_{e_0} is well-defined by previous corollary.*

Theorem 2.3 *If E is path connected, then Φ_{e_0} is surjective.*

Proof: Let e_0, $e_1 \in p^{-1}(b_0)$ and $e_0 \neq e_1$. Let δ: $I \mapsto E$ be a path with $\delta(0) = e_0$, $\delta(1) = e_1$ (δ exists because E is path connected). Then $p \circ \delta$: $I \mapsto B$ is a path with $(p \circ \delta)(0) = p(0) = b_0$, $(p \circ \delta)(1) = p(e_1) = b_0$.

Obviously, δ is a lift of $p \circ \delta$. By uniqueness of lift of paths $\widetilde{p \circ \delta}_{e_0} = \delta$. So $\Phi_{e_0}([p \circ \delta]) = \widetilde{p \circ \delta}_{e_0}(1) = \delta(1) = e_0$.

This implies that Φ_{e_0} is surjective.

Theorem 2.4 *If $\pi_1(E, e_0)$ is trivial, then Φ_{e_0} is injective.*

Proof: Let γ_1: $I \mapsto B$ and γ_2: $I \mapsto B$ loops with $\gamma_1(0) = \gamma_2(0) = b_0 = \gamma_1(1) = \gamma_2(1)$ s.t.

$$\Phi_{e_0}([\gamma_1]) = \Phi_{e_0}([\gamma_2])$$

Equivalently,

$$(\widetilde{\gamma_1})_{e_0}(1) = (\widetilde{\gamma_2})_{e_0}(1) = e_1$$

and by definition

$$(\widetilde{\gamma_1})_{e_0}(0) = (\widetilde{\gamma_2})_{e_0}(0) = e_0$$

$(\widetilde{\gamma_1})_{e_0} * (\widetilde{\gamma_1})_{e_0}^{-1}$ is a loop in E. $\pi_1(E, e_0)$ is trivial, then

$$(\widetilde{\gamma_1})_{e_0} \sim (\widetilde{\gamma_1})_{e_0}$$

by some F. So we have

$$\gamma_1 \sim \gamma_2$$

by $p \circ F$.

Example 2.1 *p: $S^n \mapsto \mathbb{RP}^n$ be the covering with $p^{-1}([x]) = \{\pm x\}$.*

$$\#p^{-1}([x]) = 2$$

for all $[x] \in \mathbb{RP}^2$. S^n is simply connected when $n \neq 0, 1$. So Φ_{e_0} is injective and surjective. This implies

$$\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$$

for $n \geq 2$.

Example 2.2 *p: $\mathbb{R} \mapsto S^1$, $t \mapsto \mathrm{e}^{2\pi it}$. This induces a bijection Φ_{e_0}: $\pi_1(S^1, b_0) \mapsto \mathbb{Z}$ (fiber of any point).*

Proposition 2.1 *If p $E \mapsto B$ is a covering map with B path connected and b_0, $b_1 \in B$, then there exists a bijection between $p^{-1}(b_0)$ and $p^{-1}(b_1)$.*

Proof: δ*:* $I \mapsto B$ *with* $\delta(0) = b_0$, $\delta(1) = b_1$.

Define

$$f_\delta : p^{-1}(b_0) \mapsto p^{-1}(b_1), e_i \mapsto (\widetilde{\delta})_{e_i}(1)$$

f_δ *is a bijection because* $f_{\overline{\delta}}$ *is its inverse.*

Proposition 2.2 *Let E be path connected and let p: $E \mapsto B$ be a covering map with $p(e_0) = b_0$. Then p_*: $\pi_1(E, e_0) \mapsto \pi_1(B, b_0)$ is injective. Moreover if $e_1 \in p^{-1}(b_0)$ and $e_0 \neq e_1$, we have that the images $p_*(\pi_1(E, e_0))$ and $p_*(\pi_1(E, e_1))$ are conjugated.*

Proof: Take $[\gamma_1]$, $[\gamma_2] \in \pi_1(E, e_0)$.

$$p_*([\gamma_1]) = p_*([\gamma_2])$$

we have

$$p \circ \gamma_1 \sim p \circ \gamma_2$$

by F, then

$$(\widetilde{p \circ \gamma_1})_{e_0} \sim (\widetilde{p \circ \gamma_2})_{e_0}$$

By the uniqueness of lifting

$$\gamma_1 = (\widetilde{p \circ \gamma_1})_{e_0} \sim (\widetilde{p \circ \gamma_2})_{e_0} = \gamma_2$$

so p_* is injective.

Let δ: $I \mapsto E$ be a path with $\delta(0) = e_0$ and $\delta(1) = e_1$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(E, e_0) \arrow[r, "\delta_\#"] \arrow[d, "p_*"'] & \pi_1(E, e_1) \arrow[d, "p_*"] \\
\pi_1(B, b_0) \arrow[r, "(p\circ\delta)_\#"] & \pi_1(B, b_0)
\end{tikzcd}$$

So $p_*(\pi_1(E, e_0))$ and $p_*(\pi_1(E, e_1))$ are conjugated.