# Math 751 Notes

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**Theorem 1.** *Let $E$ be path connected and let $p : E \to B$ be a covering map with $p(e_0) = b_o$. Then,*

- *$\gamma \in \Omega(B, b_0)$ lifts to a loop $\tilde{\gamma} \in \Omega(E, e_0)$ if and only if $\gamma \in p_*(\pi_1(E, e_0))$.*
- *$\Phi_{e_0} : \frac{\pi_1(B,b_0)}{p_*(\pi_1(E,e_0))} \to p^{-1}(b_0)$ is a bijection, where $[\gamma] \to \tilde{\gamma_{e_0}}(1)$. In particular, $\#p^{-1}(b_0) = [\pi_1(B, b_0) : p_*(\pi_1(E, e_0))]$.*

*Proof.*

- First part is trivial.
- $\Phi_{e_0}$ is well defined; i.e., $\Phi_{e_0}([\delta][\gamma]) = \Phi([\gamma])$ for any $[\delta] \in p_*(\pi_1(E, e_0))$. Indeed, we have that $\Phi_{e_0}([\delta][\gamma]) = \tilde{\gamma}_{e_0}(1)$. Recall E path connected $\implies \Phi_{e_0}$ is surjective. Let us check $\Phi_{e_0}$ is injective. If $\Phi_{e_0}([\gamma_1]) = \Phi_{e_0}([\gamma_2])$. Then $(\tilde{\gamma}_1)_{e_1} * (\tilde{\gamma}_2)^{-1}_{e_2}$ is a loop based at $e_0 \in E$. The loop $(\tilde{\gamma}_1)_{e_1} * (\tilde{\gamma}_2)^{-1}_{e_2}$ is a lift of $\gamma_1 * \tilde{\gamma}_2$. Then $\gamma_1 * \tilde{\gamma}_2 \in p_*(\pi_1(E, e_0))$. Finally $[\gamma_1] = [\gamma_1 * \gamma_2^{-1} * \gamma_2] = [\gamma_1 * \gamma_2^{-1}][\gamma_2]$. i.e. $[\gamma_1]$ and $[\gamma_2]$ are the same class in $\frac{\pi_1(B,b_0)}{p_*(\pi_1(E,e_0))} \to p^{-1}(b_0)$.

$\square$

**Theorem 2.** *(Lifting lemma) Let E,B and Y be path connected and locally path connected spaces. Let $p : E \to B$ be a covering map. Let $e_0 \in p^{-1}(b_0)$. Let $f : Y \to B$ be a continuous map such that $f(y_0) = b_0$, for some $y_0$. Then, there exists $\tilde{f} : Y \to E$ continuous at $p \circ \tilde{f}$ and $\tilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$.*

*Proof.* $\Leftarrow$

$$\begin{tikzcd}
Y, y_0 \arrow[d, dotted, "\exists \tilde{f}"'] \arrow[dr, "p"] & \\
E, e_0 \arrow[r, "f"] & B, b_0
\end{tikzcd}$$

- Definition of $\tilde{f}(y)$
  $\alpha : I \to Y$ continuous , $\alpha(0) = y_0$, $\alpha(1) = y$, $f \circ \alpha : I \to B$ continuous, $(f \circ \alpha)(0) = b_0$, these two facts tell us that there exists a unique $(\widetilde{f \circ \alpha})_{e_0} : I \to E$ s.t. $p \circ (\widetilde{f \circ \alpha})_{e_0}(0) = (f \circ \alpha)(t)$ and $(\widetilde{f \circ \alpha})_{e_0}(0) = e_0$. We define $\tilde{f}(y)$ to be $(\widetilde{f \circ \alpha})_{e_0}(1)$.
- $\tilde{f}$ lifts f
  $(p \circ \tilde{f})(y) = p(\tilde{f}(y)) = p(\widetilde{f \circ \alpha})_{e_0} = p((\widetilde{f \circ \alpha})_{e_0}(1) = (f \circ \alpha)(1) = f(y)$.
- $\tilde{f}$ is well defined
  Assume $\beta : I \to Y$ continuous $\beta(0) = y_0, \beta(1) = y$. $\widetilde{\alpha * \bar{\beta}}(0) = \alpha * \beta(1)$.
  By hypothesis: $f_*([\alpha * \bar{\beta}]) \in p_*(\pi_1(E, e_0))$; i.e. $(f \circ \widetilde{(\alpha * \bar{\beta})})_{e_0}$ is a loop in E at $e_0$.
- $\tilde{f}$ is continuous, let $U \subset E$ be a neighbourhood of $\tilde{f}(y)$.
  We look for a neighbourhood $V \subset Y$ such that $\tilde{f}(V) \subset U$. Take $U'$ to be an evenly covered neighbourhood of $f(y)$ in B such that $U' \subset p(U)$.
  $p^{-1}(U') = \coprod_{\alpha \in A} W_\alpha$. Call W the component of $p^{-1}(W)$ that contains $\hat{f}(y)$.
  Take U" an evenly covered neighbourhood of $f(y)$ in B such that $U'' \subset p(U \cap W)$.
  $p^{-1}(U'') = \coprod_{\beta \in B} W'_\beta$. Call $W'$ the path connected component of $p^{-1}(U'')$ that continuous. There is $V \subset Y$ neighbourhood of y such that $f(V) \subset U''$. The neighbourhood V can be chosen to be path connected because Y is locally path connected.
  Finally $\tilde{f}(V) \subset U$.

□

**Corollary 3.** *If $Y$ is simply connected, then there always exists a lift $\tilde{f} : Y \to E$.*

**Proposition 4.** *Any two such liftings $\tilde{f}_i : Y \to E$ with $i = 1, 2$ coincide.*

*Proof.* The set $A = \{y \in Y | \tilde{f}_1(y) = \tilde{f}_2(y)\}$ can be shown to be the whole Y.

- $A \neq \emptyset$ because $\tilde{f}_1(y_0) = \tilde{f}_2(y_0) = e_0$
- A is a closed set. Take y such that $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Let $U \subset B$ be an evenly covered neighbourhood of y. Then $U \subset B$ be an evenly covered neighbourhood of y. Then $p^{-1}(U) = \coprod_\alpha \tilde{U}_\alpha$. Let $\tilde{U}_i$ contain $\hat{f}_i(y)$, since $\tilde{f}$ are continuous, $\exists N \subset Y$ neghbourhood of $y$ such that $\tilde{f}_i(N) \subset \tilde{U}_i$, i.e., $N \subset A^c$ or $A$ closed.
- $A$ is open. Analogously, if $\tilde{f}_i(y) = \tilde{f}_2(y)$, we have that $\tilde{f}_1 = \tilde{f}_2$ on $N$ because $f = p \circ \tilde{f}_1 = p \circ \tilde{f}_2$ and $p$ is injective on $\tilde{U}_1 = \tilde{U}_2$. Finally, $A = Y$ because it is both open and closed and not empty.

□

**Definition 5.** Let $p : E \to B$ and $p' : E' \to B$ be covering maps. A homomorphism of coverings $h : (E, p) \to (E', p')$ is a continuous map $h : E \to E'$ such that $p' \circ h = p$.
An isomorphism (or equivalence) of coverings is a homomorphism $h : (E, p) \to (E', p')$ such that $h : E \to E'$ is an homeomorphism.

**Theorem 6.** *Let $p : E \to B$ and $p' : E' \to B$ be covering maps with $p(e_0) = p'(e_0') = b_0 \in B$. Then there is an isomorphism $h : (E, p) \to (E', p')$ with $h(e_0) = e_0'$, if and only if $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ coincide.*

*Proof.* $\Rightarrow$: $h_*(\pi_1(E, e_0)) \cong \pi_1(E', e_0')$ and thus $H = H'$.
$\Leftarrow$: Since $H \subset H'$, then there is a homomorphism $h : (E, e_0) \to (E', e_0')$ because of the lifting lemma. Analogously $H' \subset H$ and there is a homomorphism $k : (E', e_0') \to (E, e_0)$. Now we get the following commutative diagram

$$\begin{tikzcd}
(E, e_0) \arrow[rr, "h"] \arrow[dr, "p"'] & & (E', e_0') \arrow[ll, bend right, "k"'] \arrow[dl, "p'"] \\
& (B, b_0) &
\end{tikzcd}$$

i.e., $p \circ (k \circ h) = (p \circ k) \circ h = p' \circ h = p$. $k \circ h$ and $\mathrm{id}_E$ are two lifts of $p$ that agree at $e_0$. By uniqueness of liftings $k \circ h = \mathrm{id}_E$. Therefore $h$ is an isomorphism. □

**Definition 7.** Any equivalence between $(E, p)$ and itself is called a deck transformation or an automorphism. Automorphisms of $(E, p)$ form a group under deck transformations of $(E, p)$ which is denoted by $D(E, p)$.

**Proposition 8.** *If $h \in D(E, p)$ and $h(x) = x$ for some $x \in E$, then $h = id_E$.*

**Proposition 9.** *Let $(E, p)$ be a covering map and let $e_0$, $e_1 \in E$ s.t. $p(e_0) = p(e_1)$. $\exists h \in D(E, p)$ s.t. $h(e_0) = e_1$ if and only if $p_*(\pi_1(E, e_0)) = p_*(\pi_1(E, e_1))$.*

**Theorem 10.** *Two covering maps $(E, p)$ and $(E', p')$ of $B$ are equivalent if and only if for any $e_0 \in E$ and $e_0' \in E'$ with $p(e_0) = p(e_0') = b_0$ the subgroups $H = p_*(\pi_1(E, e_0))$ and $H' = p_*(\pi_1(E', e_0'))$ are conjugated.*

**Remark.** Do not require equivalence maps $e_0$ to $e_0'$.

*Proof.* $\Rightarrow$: Let $h : E \to E'$ be an equivalence with $h(e_0) = e_0^{''}$ (not necessarily $e_0^{''} = e_0'$). By previous Theorem we have that $H := p_*(\pi_1(E, e_0)) = H^{''} := p'_*(\pi_1(E_1', e_0^{''}))$. Check that $p'(e_0^{''}) = p'(h(e_0)) = p(e_0) = b_0$. i.e., $e_0^{''}$, $e_0^{'} \in (]')^{-1}(b_0)$ and therefore $H^{''}$ and $H' := p'_*(\pi_1(E', e_0'))$ are conjugated. Hence $H$ and $H'$ are conjugated.
$\Leftarrow$: We will use the following lemma

**Lemma 11.** *If $p : E \to B$ is a covering, $p(e_0) = b_0$ and $H = p_*(\pi_1(E, e_0))$, then given any $K \subset \pi_1(B, b_0)$ conjugated to $H$, there is a point $e_1 \in p^{-1}(b_0)$ such that $K = H_1 = p_*(\pi_1(E, e_1))$.*

*proof of lemma.* $\exists \alpha : I \to B$ continuous, $\alpha(0) = \alpha(1) = b_0$ such that

$$H = [\alpha] \cdot K \cdot [\bar{\alpha}].$$

Let $e_1 = \tilde{\alpha}_{e_0}(1)$. Then $H = [p \circ \tilde{\alpha}_{e_0}] H_1 [\overline{p \circ \tilde{\alpha}_{e_0}}]$, hence $K = H_1$. □

By the lemma, there is $e_1 = p^{-1}(b_0)$ such that

$$p'_*(\pi_1(E', e'_0)) = H' = p_*(\pi_1(E, e_1)).$$

By the lifting theorem, there is an equivalence

$$h : E \to E'.$$

□

**Example.** *Let $B$ be the Mobius band. Recall $\pi_1(B) \cong \mathbb{Z}$. Subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for $n \in \mathbb{N}$. Hence we have a covering*

$$S^1 \times I \to B(z, t) \qquad \mapsto (z^k, t)$$

*for each $k \in \mathbb{N}$, with $k$ even. And*

$$S^1 \times I \to B(z, t) \qquad \mapsto (z^k, t)$$

*for each $k \in \mathbb{N}$ odd.*

**Definition 12.** $p : E \to B$ covering map is called a universal covering if $E$ is simply connected.

**Corollary 13.** *If a universal cover $p : E \to B$ exists, then it is unique up to equivalence of coverings.*

*Proof.* Follow from the previous theorem. Since $p_*(\pi_1(E, e_0))$ is the trivial subgroup and its conjugacy class has a unique element. □

**Definition 14.** $B$ is semi-locally simply connected if for any $b \in$ there is a neighbourhood $U \subset B$ of $b$ such that the inclusion map $i : U \to B$ induces the trivial map

$$i_A : \pi_1(U, b) \to \pi_1(B, b).$$

**Example.** *1) $B$ simply connected$\Rightarrow$semi-locally simply connected.*

*2) Let $C_n$ be the circle of center $(1/n, 0)$ and radius $1/n$. Let $X = \bigcup_{n \in \mathbb{N}} C_n$. $X$ is not semi-locally simply connected. Take $U \subset \mathbb{R}^2$ neighbourhood of $(0, 0) \in \mathbb{R}^2$ and $n$ large enough s.t. $C_n \subset U$.*

$$\begin{aligned} r : X &\to C_n \text{ retraction} \\ x &\mapsto \begin{cases} x \text{ if } x \in C_n \\ 0 \text{ if } x \notin C_n \end{cases} \end{aligned}$$

*The following diagram commutes:*

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(C_n, 0) \arrow[rr, "j_*"] \arrow[dr, "k_*"'] & & \pi_1(X, 0) \arrow[r, "r_*"] & \pi_1(C_n, 0) \cong \mathbb{Z} \\ & \pi_1(U, 0) \arrow[ur, "i"'] & & \end{tikzcd}$$

*where $j_* = i_* k_*$, $r_* \circ j_* = id_{\mathbb{Z}}$, $i_*$ can't be trivial.*

**Theorem 15.** *Let $B$ path connected, locally path connected and semilocally simply connected. Let $b_0 \in B$ and $H \subset \pi_1$. Then there is a covering map $p : E \to B$ and $e_0 \in p^{-1}(b_0)$ s.t. $p_*(\pi_1(E, e_0)) = H$.*

*Proof.* Let $P = \{\alpha : I \to B \text{ continuous}, \ \alpha(0) \in b_0\}$, $\alpha, \ \beta \in P$. $\alpha \sim \beta$ if and only if $\alpha(1) = \beta(1)$ and $\alpha * \bar{\beta} \in H$.

$$p : E = \{\tilde{\alpha} \mid \alpha \in P\} \to B$$
$$\tilde{\alpha} \mapsto \alpha(1)$$

$p$ is surjective because $B$ is apth connected. Consider the following topology in $E$: $\mathcal{U} = \{U \subset B \mid \text{open and path-connected}, \ \pi_1(U) \to \pi_1(B) \text{ trivial}\}$ is a basis of topology in $B$. Given $U \in \mathcal{U}$ and $\gamma : I \to B$ continuous, $\gamma(0) = b_0$, $\gamma(1) \in \mathcal{U}$. Consider $U_{[\gamma]} = \{\widetilde{\gamma * \eta} \mid \eta : I \to U \text{ continuous } \eta(0) = \gamma(1)\}$. Remark:$\pi_1(U) \to \pi_1(B)$ trivial $\Rightarrow U_{[\gamma]}$ well defined, i.e., only depends on the class of $\gamma$,
$p : U_{[\gamma]} \to U$ surjective and injective.
$U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$,
$\Rightarrow U_{[\gamma]}$ is a basis of a topology in $E$ and $p : E \to B$ continuous. Finally $p_*(\pi_1(E, e_0)) = H$. □

**Theorem 16.** *$B$ has a universal covering if and only if $B$ is path-connected, locally path-connected, semilocally simply connected.*

**Definition 17.** Let $E$ be path-connected. Let $p : E \to B$ be a covering map. $p : E \to B$ is called regular if $p_*\pi_1(E, e_0) \lhd \pi_1(B, b_0)$.

Let $X$ be a topological space and denote by $\mathrm{Hom}(X)$ the set of homeomorphisms of $X$. Let $G \subset \mathrm{Hom}(X)$. The group $G$ acts on $X$ as follows

$$G \times X \to X$$
$$(g, x) \mapsto gx := g(x)$$

**Definition 18.** We say that $G$ acts freely on $X$ if whenever $x = gx$ for some $x \in X$, then $g = e_G$.

**Definition 19.** We say that $G$ acts properly discontinuous on $X$ if for all $x \in X$ there is a neighbourhood $U_x$ such that

$$U_x \cap U_{gx} = \emptyset$$

for all $g \in G - \{e_G\}$. Equivalently, $gU_x \cap hU_x = \emptyset$ for all $g, \ h \in G$, $g \neq h$.

**Definition 20.** We say that $G$ acts transitively on $X$ if for every pair of points $x_1, \ x_2 \in X$ there exists $g \in G$, s.t. $gx_1 = x_2$.

**Proposition 21.** *$p : E \to B$ regular $\Longleftrightarrow D(X, p)$ acts transitively on the fibre.*

**Proposition 22.** *Let $X$ be a Hausdorff topological space and let $G$ be a finite subgroup of Hom($X$). Then $p : X \to X/G$ is a covering if and only if the action of $G$ on $X$ is properly discontinuous.*

**Theorem 23.** *Let X be path connected and locally path connected and let G be a subgroup of Hom(X). Then $p : X \to X/G$ is a covering if and only if G acts properly discontinuous on X. Moreover, in this case, the covering map $p : X \to X/G$ is regular and $D(X, p) \cong G$.*

*Proof.* We first show that p is an open map. If U is open in X, then $p^{-1}p(U)$ is the union of the open sets $g(U)$ of X, for $g \in G$. Hence $p^{-1}p(U)$ is open in X, so that $\pi(U)$ is open in $X/G$ by definition. Thus $p$ is open.

*Step 1:* We suppose that the action of G is properly discontinuous and show that $p$ is a covering map. Given $x \in X$, let U be a neighbourhood of x such that $g_0(U)$ and $g_1(U)$ are disjoint whenever $g_0 \neq g_1$. Then $p(U)$ is evenly covered by $p$. Indeed, $p^{-1}p(U)$ is the disjoint union of open sets $g(U)$, for $g \in G$. Therefore, the map $g(U) \to p(U)$ obtained by restricting $p$ is bijective; begin continuous and open, it is a homeomorphism. Therefore, $p$ is a covering.

*Step 2:* We suppose now that $p$ is a covering map and show that the action of G is properly discontinuous. Given $x \in X$, let V be a neighbourhood of $p(x)$ that is evenly covered by $p$. That is, $p^{-1}(V) = \coprod_{\alpha \in A} U_\alpha$ where $U_\alpha$ is open and the restriction of $\pi$ on each $U_\alpha$ is a homeomorphism. Now assume that $x \in U_\alpha$, we want to show that $g(U_\alpha) \cap U_\alpha \neq \emptyset$ if and only if $g = Id_x$. Suppose not, then there exists some $g \neq Id_x$ and $y, z \in U_\alpha$ such that $g(y) = z$. Clearly, if we restrict $p$ to $U_\alpha$, this will not be injective, which violates our assumption. Therefore, G acts properly discontinuous on X.

*Step 3:* Finally, we show that if $p$ is a covering map, then G is its group of covering transformation and $p$ is regular. We need to show that $G \cong D(E, p)$. The homomorphism is naturally defined.

First, we show that $G \subset D(E, p)$: this is because, $\forall g \in G$, we have $p \circ g = g$ by definition.

Next, we show that $G \supset D(E, p)$: this is because, $\forall h \in D(E, p)$ with $h(x_1) = x_2$, we could find a $g \in G$ such that $g(x_1) = x_2$, then the uniqueness of lifting tells us that $h = g$ since both $h \circ g$ and $g \circ h$ are liftings of themselves.

It therefore follows that $p$ is regular because for any $p(y) = x$, we have $y = g(x)$ for some g. Then consider $p_*(\pi_1(X, y))$, since $[b^{-1}]p_*(\pi_1(X, y))[b] = p_*(\pi_1(X, y))$ for all $[b] \in \pi_1(X/G)$, p is regular by definition.

$\square$