Singular Homology Notes for Topology 751

Jacob Gloe

Definition 1. *A standard n-simplex is a set of the form*

$$\Delta^n := \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, 0 \leq t_i \leq 1 \right\}.$$

For example:

- $n = 0$: $\Delta^0 = 1 \in \mathbb{R}$
- $n = 1$: Δ^1 =the segment joining $(1, 0)$ and $(0, 1)$
- $n = 2$: Δ^2 =the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Definition 2. *In general, an n-simplex is the convex span of $\mathbb{R}^{n+1}$ of $n+1$ points in the* **general position** *(i.e. the $n+1$ points do not lie in a hyperplane). Equivalently, if the $n+1$ points are $v_0, \cdots, v_n$, we want $v_0 - v_1, v_0 - v_2, \cdots, v_0 - v_n$ to be linearly independent.*

Notation: If $\Delta^n \to [v_0, v_1, \cdots, v_n]$ is a map given by $(t_0, t_1, \cdots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i$, then $[v_0, v_1, \cdots, \hat{v_i}, \cdots, v_n]$ is called the i-th face of $[v_0, \cdots, v_n]$ and is given by letting $t_i = 0$.

Definition 3. *A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.*

We call $C_n(X) :=$ the free abelian group generated by the singular n-simplex on X to be the "group of n-chains on X". That is,

$$C_n(X) = \left\{ \sum_{i=1}^{m} n_i \sigma_i : \sigma_i \text{ is a singular } n\text{-simplex, } n_i \in \mathbb{Z} \right\}$$

Boundary maps are maps of the form $\partial_n : C_n(X) \to C_{n-1}(X)$ given by

$$(\sigma : \Delta^n \to X) \mapsto \partial_n \sigma := \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}$$

and extended using linearity.

Lemma 1. *If ∂_n and ∂_{n+1} are boundary maps, then $\partial_n \circ \partial_{n+1} = 0$ for all n. Equivalently, $Im\partial_{n+1} \subset \ker \partial_n$ for all n.*

If we take this lemma to be true, we then have a chain complex of free abelian groups

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

with the property that $\partial_n \circ \partial_{n+1} = 0$ for all n.

Definition 4. *Consider the chain complex given above. Let $Z_n = \ker \partial_n$ be the n-cycles for all n and $B_n = Im\partial_{n+1}$ be the n-boundaries. We then call the n-th homology group*

$$H_n(X) := \ker \partial_n / Im\partial_{n+1} = Z_n / B_n.$$

Proposition 1. *If $X \neq \emptyset$ is a path-connected topological space, then $H_0(X) \cong \mathbb{Z}$.*

Proof. Consider the chain complex given by

$$\cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

We have that $H_0(X) := \ker \partial_0 / \mathrm{Im}\partial_1 = C_0(X)/\mathrm{Im}\partial_1$. Now consider $\epsilon : C_0(X) \to \mathbb{Z}$ to be the augmentation map given by $\sum n_i \sigma_i \mapsto \sum n_i$. Then ϵ is clearly surjective and hence $C_0(X)/\ker \epsilon \cong \mathbb{Z}$. Let us check that $\ker \epsilon = \mathrm{Im}\partial_1$.
If $\sigma : \Delta^1 \to X$ is continuous, then $\epsilon(\partial_1(\sigma)) = \epsilon(\sigma|_{[v_0,v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$. So $\ker \epsilon \subset \mathrm{Im}\partial_1$. Now if $\epsilon(\sum n_i \sigma_i) = 0$, this implies $\sum n_i = 0$. Let $\tau_i : \Delta^1 \to X$ be continuous such that $\tau_i((1,0)) = \sigma_0$ and $\tau_1((0,1)) = \sigma_i$. Then

$$\partial_1 \left(\sum n_i \tau_i\right) = \sum n_i(\sigma_i - \sigma_0) = \sum n_i \sigma_i - \sum n_i \sigma_0 = \sum n_i \sigma_i.$$

So $\ker \epsilon \subset \mathrm{Im}\partial_1$ as well. Hence $H_0(X) = C_0(X)/\mathrm{Im}\partial_1 = C_0(X)/\ker \epsilon \cong \mathbb{Z}$. □

Proposition 2. *Let $(X_\alpha)_{\alpha \in A}$ be path-connected complexes of X. Then $H_n(X) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha) \cong \bigoplus_{\alpha \in A} \mathbb{Z}$.*

As an example, if $X = \{x_0\}$ is a point, then

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We note that $C_n(X) \cong \mathbb{Z}$ is generated by the constant singular n-simplex $\sigma_n : \Delta^n \to X$ and the boundary maps are given by $\partial_n : C_n(X) \to C_{n-1}(X)$ where

$$\sigma_n \mapsto \partial_n(\sigma_n) = \sum (-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The chain complex is then given by

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \to \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

So $H_{odd}(X) = \mathbb{Z}/\mathbb{Z} \cong 0$ and $H_{even}(X) = \begin{cases} \mathbb{Z}/0 \cong \mathbb{Z}, & n = 0 \\ 0, & n \neq 0. \end{cases}$ □

We will now consider what is called the Reduced or Modified Chain Map. Consider the chain map given by

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

For $n \geq 1$, $\partial_n \circ \partial_{n+1} = 0$ just as before. We also have $\epsilon \circ \partial_1 = 0$. We may then define the homology groups based on this new chain map.

Definition 5. *For the chain map given above, we have the reduced homology groups given by*

$$\tilde{H}_n(X) := \begin{cases} H_n(X) & \text{if } n > 0 \\ \ker \epsilon / Im\partial_1 & \text{if } n = 0. \end{cases}$$

Note that ϵ induces $\tilde{\epsilon} : H_0(X) \twoheadrightarrow \mathbb{Z}$ where $\ker \tilde{\epsilon} = \tilde{H}_0(X)$ implies that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.