Homotopy Invariance of Homology Groups

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1 Introduction

Given $f : X \to Y$ is a continuous map, we have an induced homomorphism $f_\# : \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ for all n:

$$\forall \sigma : \Delta^n \to X, \quad f_\#(\sigma) := f \circ \sigma, \tag{1}$$

$$\forall \sum_i n_i \sigma_i \in \mathcal{C}_n(X), \quad f_\#(\sum_i n_i \sigma_i) := \sum_i n_i f_\#(\sigma_i) \in \mathcal{C}_n(Y). \tag{2}$$

Our questions:

1. (*Naturality*) Is this a chain map on the chain complexes? If it is true, then it will induce an homomorphism over homologies.
2. (*Invariance*) Let g be another continuous map that is homotopic to f, does it have the same induced homomorphism over homologies?

2 Naturality

First, we state the concept of chain map in the following lemma:

Lemma 1. *If $f : X \to Y$ is continuous, then $f_\#$ is a chain map, i.e., the following diagram is commutative:*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \mathcal{C}_n(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & \mathcal{C}_{n+1}(Y) \arrow[r, "\partial"] & \mathcal{C}_n(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. By direct computation, let $[v_0, \cdots, v_n]$ be a $(n+1)$-simplex and $\sigma : [v_0, \cdots, v_n] \to X$ be a continuous map, and we have

$$\begin{aligned}
f_\#(\partial(\sigma)) &= f_\#(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \cdots \hat{v}_i \cdots v_n]}) \\
&= \sum_{i=0}^n (-1)^i f(\sigma|_{[v_0, \cdots \hat{v}_i \cdots v_n]}) \\
&= \sum_{i=0}^n (-1)^i (f \circ \sigma)|_{[v_0, \cdots \hat{v}_i \cdots v_n]} \\
&= \partial(f \circ \sigma) = \partial(f_\#(\sigma))
\end{aligned}$$

□

Let $B_n(X)$ be n-boundary of chain complex $\mathcal{C}.(X)$ and $Z_n(X)$ be the n-cycle. We have the following corollaries.

Corollary 1. *If $f: X \to Y$ is continuous, then $f_\#(B_n(X)) \subset B_n(Y)$ and $f_\#(Z_n(X)) \subset Z_n(Y)$.*

Proof. For any $b \in B_n(X)$, there exists some $c \in \mathcal{C}_{n+1}(X)$ such that $\partial(c) = b$. Then by **Lemma.1**, $f_\#(b) = f_\# \circ \partial(c) = \partial(f_\#(c)) \in B_n(Y)$. Therefore, $f_\#(B_n(X)) \subset B_n(Y)$.
For any $z \in Z_n(X)$, we know $\partial z = 0$. Then by **Lemma.1**, $\partial(f_\#(z)) = f_\#(\partial z) = 0$. Then $f_\#(z) \in Z_n(Y)$. Therefore, $f_\#(Z_n(X)) \subset Z_n(Y)$. □

Define an induced map $f_*: H_n(X) \to H_n(Y)$ such that $f_*[\sigma] = [f_\#(\sigma)] \in H_n(Y)$ for any $[\sigma] \in H_n(X)$.

Corollary 2. *If $f: X \to Y$ is continuous, then $f_*: H_n(X) \to H_n(Y)$ is a homomorphism and the following diagram is commutative:*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y) \arrow[r, "\partial"] & H_n(Y) \arrow[r] & \cdots
\end{tikzcd}$$

.

As for this induced map, we have the following properties:

Theorem 1. *Given continuous maps $f: X \to Y$ and $g: Y \to Z$, we have*

1. $(g \circ f)_* = g_* \circ f_*$,
2. $(Id_X)_* = Id_{H_n(X)}$.

3 Invariance

According to **Theorem 1**, we can conclude the invariance of homologies under homeomorphism

Theorem 2. *Homology groups are invariant under homeomorphism*

Proof. If X and Y are homeomorphic, then there exists continuous maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$. Then by **Theorem 1**, we can conclude f_*, g_* are bijective. Hence, $f_*: H_.(X) \to H_.(Y)$ is isomorphism. □

Question: What happens under homotopy instead of homeomorphism?
Remark: In this case, since we can construct homology from chain complexes, we can forget the topological space when comparing homologies.

Theorem 3. *Homology groups are invariant under homotopy.*

Assume continuous maps $f, g: X \to Y$ and $F: X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for any $x \in X$. Then we define a map $P: \mathcal{C}_{n-1}(X) \to \mathcal{C}_n(Y)$ (Prism Operator) such that the following diagram is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}_{n+1}(X) \arrow[r, "\partial"] & \mathcal{C}_n(X) \arrow[r, "\partial"] \arrow[dl, "P"] \arrow[d, "f_\#, g_\#"] & \mathcal{C}_{n-1}(X) \arrow[r] \arrow[dl, "P"] & \cdots \\
\cdots \arrow[r] & \mathcal{C}_{n+1}(Y) \arrow[r, "\partial"] & \mathcal{C}_n(Y) \arrow[r, "\partial"] & \mathcal{C}_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

To prove this theorem, we need the following lemma:

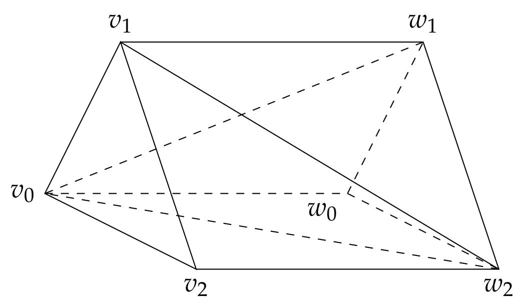
Lemma 2. $g_\# - f_\# = P \circ \partial + \partial \circ P$

Proof of theorem 3. Take any $\alpha = [\sigma] \in H_n(X)$, then $\partial\sigma = 0$. Then $[g_\# - f_\#](\sigma) = P(\partial\sigma) + \partial(P(\sigma)) = \partial(P(\sigma))$ is boundary in $\mathcal{C}_n(Y)$. Therefore, $g_*\alpha = g_*[\sigma] = [g_\#(\sigma)] = [f_\#(\sigma)] = f_*[\sigma] = f_*\alpha$. Hence $g_* = f_*$. $\square$

Now we define the operator P explicitly and prove **Lemma 2**. Consider any continuous map $\sigma : \Delta^n \to X$, let $\Delta^n \times I$ be a cylinder, $[v_0, v_1, \cdots, v_n]$ be its lower surface $\Delta^n \times \{0\}$ and $[w_0, w_1, \cdots w_n]$ be its upper surface $\Delta^n \times \{1\}$. Then $\Delta^n \times I$ can be divided into $n+1$ parts and each part is a $(n+1)$-simplex, i.e.,

$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, v_1, \cdots, v_i, w_i, \cdots, w_n].$$

For example, when $n = 2$, $\Delta^2 \times I$ is divided into 3 parts (see the following figure).



Then we <u>define</u> $P : \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ such that

$$\forall \sigma : \Delta^n \to X, P(\sigma) = \sum_{i=0}^{n} (-1)^i \big(F \circ (\sigma, Id)\big)\big|_{[v_0, v_1, \cdots, v_i, w_i, \cdots, w_n]} \in \mathcal{C}_n(Y).$$

$$\forall \sum_j n_j \sigma_j \in \mathcal{C}_n(X), P(\sum_j n_j \sigma_j) = \sum_j n_j P(\sigma_j)$$

Here, $F \circ (\sigma, Id)$ is the composite function $\Delta^n \times I \xrightarrow{(\sigma, Id)} X \times I \xrightarrow{F} Y$ and $F : X \times I \to Y$ is the homotopy between f and g.

Proof of Lemma 2. We only need to check $[g_\# - f_\#](\sigma) = P(\partial\sigma) + \partial P(\sigma)$ for any $\sigma : \Delta^n \to X$. Then we find

$$\begin{aligned}\partial(P(\sigma)) = &\sum_{0 \le j \le i \le n} (-1)^{i+j} \big(F \circ (\sigma, Id)\big)\big|_{[v_0, v_1, \cdots, \hat{v}_j, \cdots v_i, w_i, \cdots, w_n]} \\ &+ \sum_{0 \le i \le j \le n} (-1)^{i+j+1} \big(F \circ (\sigma, Id)\big)\big|_{[v_0, v_1, \cdots v_i, w_i, \cdots, \hat{w}_j, \cdots w_n]}\end{aligned}$$

In the summation of the first line, when $i = j = 0$, the term becomes $F \circ (\sigma, Id)|_{[w_0, w_1, \cdots w_n]} = g_\#(\sigma)$; When $i = j > 0$, it becomes $F \circ (\sigma, Id)|_{[v_0 \cdots, v_{i-1}, w_i, \cdots w_n]}$. In the summation of the second line, when $i = j = n$, the term becomes $-F \circ (\sigma, Id)|_{[v_0, v_1, \cdots v_n]} = -f_\#(\sigma)$; When $i = j < n$, it becomes $-F \circ (\sigma, Id)|_{[v_0, \cdots v_i, w_{i+1}, \cdots w_n]}$. Note that

$$\sum_{i=1}^{n} F \circ (\sigma, Id)|_{[v_0 \cdots, v_{i-1}, w_i, \cdots w_n]} + \sum_{i=0}^{n-1} -F \circ (\sigma, Id)|_{[v_0, \cdots v_i, w_{i+1}, \cdots w_n]} = 0.$$

Therefore,

$$\begin{aligned}\partial(P(\sigma)) = &g_\#(\sigma) - f_\#(\sigma) \\ &+ \sum_{0 \le j < i \le n} (-1)^{i+j} \big(F \circ (\sigma, Id)\big)\big|_{[v_0, v_1, \cdots, \hat{v}_j, \cdots v_i, w_i, \cdots, w_n]} \\ &+ \sum_{0 \le i < j \le n} (-1)^{i+j+1} \big(F \circ (\sigma, Id)\big)\big|_{[v_0, v_1, \cdots v_i, w_i, \cdots, \hat{w}_j, \cdots w_n]}\end{aligned}$$

On the other hand, we find

$$\partial\sigma = \sum_{j=0}^{n}(-1)^j \sigma|_{[v_0,\cdots,\hat{v}_j\cdots v_n]}$$

and thus

$$\begin{aligned} P(\partial\sigma) = &\sum_{0\leq j<i\leq n}(-1)^{i+j-1}\big(F\circ(\sigma,Id)\big)\big|_{[v_0,v_1,\cdots,\hat{v}_j,\cdots v_i,w_i,\cdots,w_n]} \\ &+\sum_{0\leq i<j\leq n}(-1)^{i+j}\big(F\circ(\sigma,Id)\big)\big|_{[v_0,v_1,\cdots v_i,w_i,\cdots,\hat{w}_j,\cdots w_n]} \end{aligned}$$

Therefore, $\partial(P(\sigma))+P(\partial\sigma)=g_\#(\sigma)-f_\#(\sigma)$.

$\square$

<u>Remark:</u> Direct calculation when $n=2$

Let $\sigma:[v_0,v_1,v_2]\to X$, then

$$\begin{aligned} P(\sigma) &= F\circ(\sigma,Id)|_{[v_0,w_0,w_1,w_2]} - F\circ(\sigma,Id)|_{[v_0,v_1,w_1,w_2]} + F\circ(\sigma,Id)|_{[v_0,v_1,v_2,w_2]} \\ \partial(P(\sigma)) &= \underbrace{F\circ(\sigma,Id)|_{[w_0,w_1,w_2]}}_{=g_\#} - F\circ(\sigma,Id)|_{[v_1,w_1,w_2]} + F\circ(\sigma,Id)|_{[v_1,v_2,w_2]} \\ &\quad \underbrace{-F\circ(\sigma,Id)|_{[v_0,w_1,w_2]} + F\circ(\sigma,Id)|_{[v_0,w_1,w_2]}}_{=0} - F\circ(\sigma,Id)|_{[v_0,v_2,w_2]} \\ &\quad + F\circ(\sigma,Id)|_{[v_0,w_0,w_2]} \underbrace{-F\circ(\sigma,Id)|_{[v_0,v_1,w_1]} + F\circ(\sigma,Id)|_{[v_0,v_1,w_2]}}_{=0} \\ &\quad - F\circ(\sigma,Id)|_{[v_0,w_0,w_1]} + F\circ(\sigma,Id)|_{[v_0,v_1,w_1]} \underbrace{-F\circ(\sigma,Id)|_{[v_0,v_1,v_2]}}_{=-f_\#} \end{aligned}$$

and

$$\partial\sigma = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]}$$

$$\begin{aligned} P(\partial\sigma) = & \qquad \big(F\circ(\sigma,Id)|_{[v_1,w_1,w_2]} - F\circ(\sigma,Id)|_{[v_1,v_2,w_2]}\big) && (3) \\ &+ \qquad \big(F\circ(\sigma,Id)|_{[v_0,v_2,w_2]} && (4) \\ &- F\circ(\sigma,Id)|_{[v_0,w_0,w_2]}\big) && (5) \\ &+ \big(F\circ(\sigma,Id)| - [v_0,w_0,w_1] - F\circ(\sigma,Id)|_{[v_0,v_1,w_1]}\big) && (6) \end{aligned}$$

Therefore, $\partial(P(\sigma))+P(\partial\sigma)=g_\#(\sigma)-f_\#(\sigma)$ holds for the case $n=2$.