MATH 751: HOMEWORK 1

Due on October 2nd.

Problem 1 Give an example of an arbitrary intersection of open sets which is not open.

Problem 2 Show that the condition f^{-1} be continuous is essential for the definition of homeomorphism.

Problem 3 Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Problem 4 Prove associativity for concatenation of paths.

Problem 5 Let x_0 and x_1 be points of the path-connected space X. Show that $\pi(X, x_0)$ is abelian if and only if for every pair of paths α and β from x_0 to x_1, we have that $\alpha_\# = \beta_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$. Recall that $\alpha_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is the group isomorphism defined by $\alpha_\#([\gamma]) := [\bar{\alpha} * \gamma * \alpha]$.)

Problem 6 Let A be a subspace of $\mathbb{R}^n$. let $h : (A, a_0) \to (Y, y_0)$ be a continuos map of pointed spaces. Show that if f is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h induces the trivial homomorphism on fundamental groups (i.e., h_* maps everything into the identity element).

Problem 7 Show that any two maps from an arbitrary space to a contractible space are homotopic. As a consequence, prove that if X is a contractible space, then any point in X is a (weak) deformation retract. Remark: The statement is not true if one consider strong deformation retracts. See Hatcher, Chapter 0, Exercise 6 or Munkres, Chapter 9, Exercise 8.

Problem 8 Show that if X and Y are path-connected spaces, and $x \in X$, $y \in Y$, then $\pi_1(x \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi(Y, y)$.

Problem 9 Let $A \subset X$ be a subspace and $r : X \to A$ be a retraction of X to A. Show that $r_* : \pi(X, a) \to \pi_1(A, a)$ is a surjective for any point $a \in A$.

Problem 10 Using that the fundamental group of S^1 is $\mathbb{Z}$, show that there are no retractions $r : X \to A$ in the following cases:

- $X = \mathbb{R}^3$, with A any subspace homeomorphic to S^1.
- $X = S^1 \times D^2$, with A its boundary torus $S^1 \times S^1$.
- X is the Möbious band and A its boundary map.

Problem 11 Let V be a finite dimensional real vector space and W be a subspace. Compute $\pi_1(V \backslash W)$.

Problem 12 Let A be a real 3×3 matrix, with all entries positives. Show that A has a positive eigenvalue. Hint: Use Brouwer's fixed point theorem.