MATH 751: HOMEWORK 1

Due on October 2nd.

Problem 1 Show that a retract of a contractible space is contractible.

Problem 2 Classify the capital english letters by homeomorphism type and by homotopy type. What are their fundamental groups?

Problem 3 For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternatives holds.

- The solid torus $B^2 \times S^1$.
- The torus T with a point removed.
- The cylinder $S^1 \times I$.
- The infinite cylinder $S^1 \times \mathbb{R}$.
- $\mathbb{R}^3$ with the nonegative x, y, and z axes deleted.
- $\{x \mid \|x\| > 1\}$.
- $\{x \mid \|x\| \geq 1\}$.
- $\{x \mid \|x\| < 1\}$.
- $S^1 \cup (\mathbb{R}_+ \times 0)$.
- $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$.
- $S^1 \cup (\mathbb{R} \times 0)$.
- $\mathbb{R}^2 - (\mathbb{R}_+ \times 0)$.

Problem 4

- Let $A = \{a, b\}$ and let the set of relations consist of two relations
$$abab^{-1} = 1 \qquad \text{and} \qquad b^2a^{-2} = 1.$$
Prove that the word a^4 is equivalent to the empty word.
- Prove that the relation $z^6 = 1$ is deducible in the group
$$\{x, z \mid xz^2xz^{-1}, zx^2zx^{-1}\}.$$
- Identify the group
$$\{a, b, c, d, e \mid d = e^2, bda = 1, ab^{-1}c = 1, ac^{-1}b^{-1} = 1, de = c\}$$
with the cyclic group of order 12.
- Identify the group $G = \{a, b \mid a^3 = b^2 = 1, a^2b = ba\}$ with the symmetric group S_3.

Problem 5 Calculate the fundamental group of the spaces below:

- $\mathbb{R}^3 - \{x - \text{axis and } y - \text{axis}\}$.
- The complement in $\mathbb{R}^3$ of a line and a point not on the line.
- $\mathbb{R}^3$ minus two disjoint lines.
- $T^2 - \{x, y\}$, where x, y are two distinct points on the 2-torus.

- Möbius band. Are the cylinder and the M?obius band homeomorphic?
- The complement in $\mathbb{R}^3$ of a line and a circle. Note: There are two cases to consider, one where the line goes through the interior of the circle and the other where it doesn?t. Are these two spaces homotopy equivalent?
- Two circles S^1 with two common points.
- The two dimensional sphere S^2 with a diameter attached.
- An hemisphere of S^2 with a diameter attached.

Problem 6 Show that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ have the same fundamental group. Are they homeomorphic?

Problem 7 For a given a sequence of continuous maps

$$f_i : X_i \to X_{i+1}$$

define the quotient space

$$M := \left(\bigsqcup_{i \geq 1} X_i \times [0,1] \right) /((x_i, 1) \sim (f_i(x_i), 0))$$

obtained from the disjoint union of cylinders $X_i \times [0,1]$ via the identification of $(x_i, 1) \in X_i \times \{1\}$ with $(f_i(x_i), 0) \in X_{i+1} \times \{0\}$. Compute the fundamental group of M in the case when each X_i is a circle S^1 and $f_i : S^1 \to S^1$ is the map $z \mapsto z^i$ (for each $i \geq 1$).