MATH 751: HOMEWORK 3

Due on November 6th.

Problem 1 There are six ways to obtain a compact surface by identifying pairs of sides in a square. In each case determine what surface one obtains.

Problem 2 The following labeling schemes describe two dimensional surfaces:

- $abc^{-1}b^{-1}a^{-1}c$
- $abc^{-1}c^{-1}ba$
- $a_1a_2\cdots a_na_1^{-1}a_2^{-1}\cdots a_n^{-1}$

In each case determine what standard surface it is homeomorphic to.

Problem 3 Consider the space X obtained from a seven-sided polygonal region by means of the labeling scheme $abaaab^{-1}a^{-1}$. Show that $\pi_1(X)$ is the free product of two cyclic groups.

Problem 4 Let X be the quotient space obtained from an eight-sided polygonal region P by means of the labeling scheme $abcdad^{-1}cb^{-1}$. Let $\pi : P \to X$ be the quotient map.

- Show that π does not map all the vertices of P to the same point of X.
- Determine the space $A = \pi(\partial P)$ (the boundary of P), and calculate its fundamental group.
- Calculate the fundamental group of X. (Hint: first transform the labeling scheme into a standard one by cutting and pasting operations.)
- What surface is X homeomorphic to?

Problem 5 Let X be a space obtained by pasting the edges of a polygonal region together in pairs.

- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2, $\mathbb{RP}^2$, K, nT^2, $nT^2\#\mathbb{RP}^2$, $nT^2\#K$, where K is the Klein bottle and $n \geq 1$.
- Show that X is homeomorphic to exactly one of the spaces in the following list: S^2, $\mathbb{RP}^2$, mK, mT^2, $\mathbb{RP}^2\#mK$, where mK is the m-fold connected sum of K with itself and $m \geq 1$.

Problem 6 Prove the following formula for the Euler characteristic of the connected sum of two surfaces:

$$\xi(S_1\#S_2) = \xi(S_1) + \xi(S_2) - 2.$$

As an application compute the Euler characteristic of the connected sum of n tori, n projective planes, a projective plane and n tori, a Klein bottle and n tori.

Problem 7 Prove that there are only five regular polyhedra (namely, the tetrahedron, cube, octahedronm dodecahedron and icosahedron) by considering subdivisions of the sphere into n-gons (with n fixed) such that exactly m edges (m again fixed) meet at each vertex.

Problem 8 For any triangulation of a compact surface, show that

$$3f = 2e$$
$$e = 3(v - \xi)$$
$$v \geq \frac{1}{2}(7 + \sqrt{49 - 24\xi}).$$

In the case of sphere, the projective plane and the torus, what are the minimal values of the numbers v, e and f?

Problem 9 Prove it is not possible to subdivide the surface of a sphere into regions, each of which has six sides (i.e., it is a hexagon) and such that distinct regions have no more than one side in common.

Problem 10 What surface is represented by a $2n$-gon with the edges identified in pairs according to the symbols

$$a_1 a_2 \ldots a_{n-1} a_n a_1^{-1} a_2^{-1} \ldots a_{n-1}^{-1} a_n$$

and

$$a_1 a_2 \ldots a_{n-1} a_n a_1^{-1} a_2^{-1} \ldots a_{n-1}^{-1} a_n^{-1}$$

Hint: The cases where n is odd and where n is even are different.

This gives alternative normal forms for the representation of compact surfaces as quotient space of a polygon and labeling scheme.