MATH 751: HOMEWORK 4

Due on November 20.

Problem 1 Give an example of a local homeomorphishm which is not a covering map.

Problem 2 Show that the map $p : S^1 \to S^1$ given by $p(z) = z^n$ is a covering. (Here we represent S^1 as the set of complex numbers z of absolute value 1.)

Problem 3 Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply-connected, then p is a homeomorphism.

Problem 4 Let $p : E?B$ be a covering map and let $b_0 \in B$ and $e_0 \in E$ such that $p(e_0) = b_0$. Show that $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

Problem 5
(1) Show that if $n > 1$ then any continuous map $f : S^n \to S^1$ is nullhomotopic.
(2) Show that any continuos map $f : \mathbb{RP}^2 \to S^1$ is nullhomotopic.

Problem 6 Classify all coverings of the Möbius band up to equivalence.

Problem 7
(1) Show that the torus T^2 is a two-fold cover of the Klein bottle.
(2) Is it possible to realize the Klein bottle as a two-fold cover of itself?
(3) Find the universal cover of the Klein bottle.

Problem 8 Let $p : E \to B$ be a covering map with E simply-connected. Show that given any covering map $r : Y \to B$, there is a covering map $q : E \to Y$ such that $r \circ q = p$.

Problem 9 Show that if G is a finite group with a fixed-point free action on a Hausdorff space X, the quotient map $p : X \to X/G$ is a covering.

Problem 10 Let $\mathbb{Z}_6$ act on $S^3 = \{(z, w) \in \mathbb{C} \mid |z|^2 + |w|^2 = 1\}$ via $(z, w) \to (\epsilon z, \epsilon w)$, where ϵ is a primitive sixth root of unity. Denote by L the quotient space $S^3/\mathbb{Z}_6$.
(1) What is the fundamental group of L?
(2) Describe all coverings of L.
(3) Show that any continuous map $L \to S^1$ is nullhomotopic.

Problem 11 Prove that the following property is a characterization of regular covering maps: for any loop in the base space, if its lift starting at some point of the fibre is a loop, then any of its other lifts, starting at any other point of the fibre, is also a loop.

Problem 12 Define a *1-complex* as an identification space $C = (V, E)/\sim$ where $V = \{v_i\}$ is a collection of points, the vertices of C, and $E = \{I_j\}$ is a collection of intervals, $I = [0, 1]$, the edges of C, with $0 \in E_j$ identified with some v_i and $1 \in E_j$ identified with some v_i, with C having the quotient topology.

C is a *finite* 1-complex if it has finitely many vertices and edges.

A *subcomplex* of C is a subset of C that is a complex.

A *tree* is a contractible 1-complex. A maximal tree in C is a subcomplex of C which is a tree, and it is not contained in other subtree of C. Note the maximal tree always exist although it may not be unique.

(1) Given a maximal tree T of a finite 1-complex C, prove that the fundamental group of C is a free group with generators in a 1:1 correspondence with the edges of C but not in a maximal tree T of C.

(2) Prove that for any covering map $p : E \to C$, the covering space E is also a 1-complex.

(3) Prove that any subgroup G of a free group H is a free group. Moreover, if G has k generators and H is a subgroup of order n, then H has $(k-1)n+1$ generators.

(4) Find all subgroups of index 2 of $\mathbb{F}_2$, the free group with two generators.

(5) Is there a subgroup of index 3 of the $\mathbb{F}_2$ isomorphic to $\mathbb{F}_4$? Why? In affirmative case give an example of a normal and a non normal subgroup.