MATH 751: HOMEWORK 5

Due on December 9th (23:59 pm).

Problem 1 (a) Show that if X is a path-connected topological space and $f : X \to X$ is a continuous function, then the induced map $f_* : H_0(X) \to H_0(X)$ is the identity map. (b) Show that $H_0(X, A) = 0$ iff A meets each path-component of X. (c) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each pathcomponent of X contains at most a path-component of A.

Problem 2 A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of finite type if $\mathrm{rank} A_n < \infty$ for all $n \geq 0$. The Euler characteristic of a finite type graded abelian group $A_\bullet$ is the integer $\chi(A_\bullet) := \sum_{n\geq 0}(-1)^n \mathrm{rank} A_n$. A short exact sequence of graded groups $A_\bullet$, $B_\bullet$, $C_\bullet$, is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \qquad n \geq 0.$$

Prove that if

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of graded abelian groups of finite type, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

Problem 3 A pair (X, A) with X a space and A a nonempty closed subspace that is a deformation retract of some neighborhood in X is called a good pair. Show that for a good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ obtained by collapsing A to a point, induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong H_n(X/A)$, for all n.

Problem 4 (a) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions. Are these spaces homeomorphic? (b) Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. (c) On the other hand, show that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

Problem 5 For ΣX the suspension of X, show by a Mayer-Vietoris argument that there are isomorphisms $H_{n+1}(\Sigma X) \cong H_n(X)$ for all n.

Problem 6 For the case of the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs, i.e., there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ so that $g \circ f$ and $f \circ g$ are homotopic to the identity through maps of pairs.

Problem 7 (1) Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$. (2) Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. (3) Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there is a point $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$.

Problem 8 Describe a cell structure on $S^n \vee S^n \vee \cdots \vee S^n$ and calculate its homology groups.

Problem 9 Describe the CW structure of T_n (the connected sum of n tori) and respectively P_n (the connected sum of n projective planes), then use the cellular homology to calculate the homology of these spaces.

Problem 10 For finite CW complexes X and Y, show that

- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.
- $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$ if X is a union of subcomplexes A and B.
- $\chi(Y) = n \cdot \chi(X)$ if $p : Y \to X$ is an n-sheeted covering space.

Problem 11 Show that the closed orientable surface $T_g := T_2 \# \cdots \# T_2$ (g times) is a covering space of T_h if and only if $g = n(h-1) + 1$ for some non-negative integer n.