Convergence for a family of discrete Advection-reaction operators

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Abstract. We define a family of discrete Advection-reaction operators, denoted by $A_{a\lambda}$, which associate to a given scalar sequence $s = \{s_n\}$ the sequence given by $A_{a\lambda}(s) \equiv \{b_n\}$, where $b_n = a_{n-2}s_{n-1} + \lambda_n s_n$ for $n = 1, 2, \ldots$. For $A_{a\lambda}$ we explicitly find their iterates and study their convergence properties. Finally, we show the relationship between the family of discrete operators with the continuous one dimensional advection-reaction equation.

1. Introduction

The first order linear partial differential equation

$\frac{\partial u(x,t)}{\partial t} + \alpha(x) \frac{\partial u(x,t)}{\partial x} = \beta(x)u(x,t), \quad (1)$

known as the one dimensional advection-reaction equation [4], models the reactive transport of solutes in fluid dynamics and many other important applications. For example, if $\beta = 0$ then the equation models the behavior of a pigment squirited into a stream moving by at velocity $\alpha(x)$, so the colorant will be advected downstream and without distortion if $\alpha$ is a constant.

A family of linear operators related to (1) can be given as follows: Let us denote by $\mathcal{S}$ the vector space consisting of all real sequences. If $s \in \mathcal{S}$, we write $s = \{s_n\}_{n=1}^{\infty}$. Given $s \in \mathcal{S}$, and two fixed sequences $a = \{a_n\}_{n=0}^{\infty}$ and $\lambda = \{\lambda_n\}_{n=1}^{\infty}$, let $b = \{b_n\}_{n=1}^{\infty}$ be the sequence given by

$$b_n \equiv a_{n-2}s_{n-1} + \lambda_n s_n, \quad n \in \mathbb{N}.$$ Then the linear operator

$A_{a\lambda} : \mathcal{S} \rightarrow \mathcal{S}$

declared by $A_{a\lambda}(s) \equiv b$ is the discrete Advection-reaction operator for $a$ and $\lambda$. The relationship between (1) and (2) will become clearer in later sections.

Our aim in this work is to study convergence properties of the discrete system (2) and to relate them to properties of the continuous system (1), from both, the

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analytical and the computational points of view. Our focus will be in the existence and asymptotic behavior of iterates of (2), we will find explicitly the iterates of the discrete system in order to show their relationship with interpolation theory, particularly with the Lagrange polynomials of a standard polynomial base.

2. Finite dimensional case

We start by discussing the finite dimensional case for the Advection-reaction operator which has a basic feature in this work. So let \( m = 1, 2, \ldots \) and consider \( \mathbb{R}^m \) with the sup-norm \( \|(s_1, \ldots, s_m)\|_\infty \equiv \max\{|s_1|, \ldots, |s_m|\} \). Given \( a \in \mathbb{R}^{m-1} \) and \( \lambda \in \mathbb{R}^m \), the Advection-reaction operator takes the form

\[
A_{a\lambda}(s_1, \ldots, s_m) \equiv (\lambda_1 s_1, a_0 s_1 + \lambda_2 s_2, \ldots, a_m s_m + \lambda_m s_m). 
\]

Notice that \( A_{a\lambda}(\mathbb{R}^m) \subset \mathbb{R}^m \). Take \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_m) \in \mathbb{R}^m \). Then,

\[
|a_{j-2} x_{j-1} + \lambda_j x_j - a_{j-2} y_{j-1} - \lambda_j y_j| \leq \|x - y\|_\infty \max_j \{|a_{j-2}| + |\lambda_j|\}, \quad 1 \leq j \leq m.
\]

It follows that

\[
\|A_{a\lambda}(x) - A_{a\lambda}(y)\|_\infty \leq K(a, \lambda)\|x - y\|_\infty, \quad \forall x, y \in \mathbb{R}^m.
\]

with \( K(a, \lambda) \equiv \max_j \{|a_{j-2}| + |\lambda_j|\} \), which we assume to be less than one. This shows that \( A_{a\lambda} \) is a contraction and so it has a unique fixed point, which is the zero vector. Thus we have proved the following.

**Proposition 1.** If \( s \in \mathbb{R}^m \), then the sequence \( \{A_{a\lambda}^n(s)\}_{n=0}^\infty \) converges uniformly to zero.

Let us observe that for any two norms in \( \mathbb{R}^m \), their correspondent induced norms or operator norm on the space of \( m \)-by-\( m \) matrices are equivalent [5], therefore there is a condition of the type

\[
K(a_0, a_1, \ldots, a_{m-2}, \lambda_1, \ldots, \lambda_m) < 1
\]

for any induced norm that guarantees the convergence to zero of the sequence \( \{A^n s\}_{n=0}^\infty \) which is called the orbit of \( s \). For example, for the \( l_1 \) norm the condition is \( \max_j \{|a_j| + |\lambda_j|\} < 1 \) and for the Euclidean norm the condition is that \( \max_j \{|\lambda_j|\} < 1 \). With this observation we can state our previous result as:

**Proposition 2.** Let \( \|\| \) be a norm in \( \mathbb{R}^m \), then there is a condition over the numbers \( \{a_n\}_{n=0}^{m-2} \) \( y \{\lambda_n\}_{n=1}^{m} \) such that the orbit under \( A_{a\lambda} \) of any element of \( \mathbb{R}^m \) converges in the induced norm to zero.

3. Infinite dimensional case

We now consider the infinite-dimensional situation. Let us study the orbit of an arbitrary sequence, that is, given a sequence \( s \in S \) let us find the sequence \( \{s, A_{a\lambda}(s), A_{a\lambda}^2(s), \ldots\} \).

In order to state our next result in a condensed form, we use the following notation: Given three natural numbers \( k, m \) and \( N \) let \( P_k[\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+N-1}] \) denote the homogeneous polynomial of degree \( k \) consisting of the sum of all possible monomials with unitary coefficients in the \( N \) variables \( \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+N-1} \). By convention \( P_0[\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+N-1}] = 1 \) and \( P_{-k}[\lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+N-1}] = 0 \) \( \forall k \in \mathbb{N} \).
Theorem 1. Let $s^{(0)} \in S$, then $A^n_{\alpha\lambda}(s^{(0)}) = A^n_{\alpha\lambda}(s_1^{(0)}, s_2^{(0)}, \ldots) = (s_1^{(n)}, s_2^{(n)}, \ldots)$ where

$$s_j^{(n)} = \begin{cases} \lambda^n s_j^{(0)}, & \text{if } j = 1; \\ 
\sum_{m=1}^{j-1} \left( \prod_{k=m}^{j-2} a_k \right) P_{n-j+m}[\lambda_m, \ldots, \lambda_{m+n-1}] s_m^{(0)} + \lambda_j^n s_j^{(0)}, & \text{if } j > 1. 
\end{cases}$$

Proof (by induction over $n$). When $n = 1$ we have

$$s_i^{(1)} = \lambda_i s_i^{(0)} + a_{j-2}s_{j-1}^{(0)} = \lambda_i s_i^{(0)} + \left( \prod_{k=j-2}^{j-2} a_k \right) s_{i-1}^{(0)}$$

$$= \sum_{m=1}^{i-1} \left( \prod_{k=m}^{i-2} a_k \right) P_{n-i+m}[\lambda_m, \ldots, \lambda_{i-1}] s_m^{(0)} + \lambda_i s_i^{(0)} \quad \forall i \in \mathbb{N}.$$ 

Suppose that formula is valid for a fixed $n \in \mathbb{N}$ for all $i \in \mathbb{N}$, then

$$s_i^{(n+1)} = \lambda_i s_i^{(n)} + a_{j-2}s_{j-1}^{(n)}$$

$$= \lambda_i \left[ \sum_{j=1}^{i-1} \left( \prod_{k=j-1}^{i-2} a_k \right) P_{n-i+j}[\lambda_j, \ldots, \lambda_i] s_j^{(0)} + \lambda_i^n s_i^{(0)} \right] +$$

$$a_{i-2} \left[ \sum_{j=1}^{i-3} \left( \prod_{k=j}^{i-2} a_k \right) P_{n-i+j+1}[\lambda_j, \ldots, \lambda_{i-1}] s_j^{(0)} + \lambda_i^{n+1} s_i^{(0)} \right]$$

$$= \sum_{j=1}^{i-2} \left( \prod_{k=j-1}^{i-2} a_k \right) s_j^{(0)} + \lambda_i P_{n-i+j}[\lambda_j, \ldots, \lambda_i] + P_{n-i+j+1}[\lambda_j, \ldots, \lambda_{i-1}] + \lambda_i^{n+1} s_i^{(0)}$$

$$= \sum_{j=1}^{i-1} \left( \prod_{k=j-1}^{i-2} a_k \right) P_{n-i+j+1}[\lambda_j, \ldots, \lambda_i] s_j^{(0)} + \lambda_i^{n+1} s_i^{(0)}. \quad \square$$

Let us assume from now on that all the elements in the sequence $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ are different. We will calculate $A^n_{\alpha\lambda}(s^{(0)})$ using a different approach and the following two linear operators, $D_\lambda$ and $P_{2\lambda}$ defined on $S$ will play an essential part in this process:

$$D_\lambda(s) \equiv (\lambda_1 s_1, \lambda_2 s_2, \ldots, \lambda_k s_k, \ldots) \quad P_{2\lambda}(s) \equiv \left( s_1, b_{2,m}s_2, \ldots, \sum_{m=1}^{k-1} b_{k,m}s_m, \ldots \right)$$

where

$$b_{k,m} = \frac{\prod_{j=m}^{k-2} a_j}{\prod_{j=m+1}^{k-1} (\lambda_m - \lambda_j)}.$$ 

In the following proofs, we will make use of the Lagrange interpolation polynomial (see [2]) at the points $(\lambda_m, f(\lambda_m)), (\lambda_{m+1}, f(\lambda_{m+1})), \ldots (\lambda_k, f(\lambda_k))$, with $m < k$ for the polynomial $p(\lambda) = \lambda^n$, with $n \in \mathbb{N} \cup \{0\}$, thus
\[
\lambda^n = \sum_{j=m}^{k} \lambda_j \prod_{i=m, j \neq i}^{k} (\lambda - \lambda_i).
\]

**Lemma 1.** The linear operator \( P_{a\lambda} \) is invertible with \( P_{a\lambda}^{-1} \) given by

\[
P_{a\lambda}^{-1}(s) \equiv \left( s_1, c_{2, m} s_2, \ldots, \sum_{m=1}^{k-1} c_{k, m} s_m, \ldots \right)
\]

where

\[
c_{k, m} = \frac{\prod_{i=m-1}^{k-2} a_j}{\prod_{j=m}^{k} (\lambda_k - \lambda_j)}.
\]

**Proof**

Let \( R \) be the linear operator defined on \( S \) by

\[
R(s) \equiv \left( s_1, c_{2, m} s_2, \ldots, \sum_{m=1}^{k-1} c_{k, m} s_m, \ldots \right) \text{ with } c_{k, m} = \frac{\prod_{i=m-1}^{k-2} a_j}{\prod_{j=m}^{k} (\lambda_k - \lambda_j)}.
\]

then

\[
P_{a\lambda} \circ R(s) \equiv \left( s_1, s_2 + d_{2, 1} s_1, \ldots, s_k + \sum_{m=1}^{k-1} d_{k, m} s_m, \ldots \right)
\]

where

\[
d_{k, m} = \left( \frac{\prod_{i=m-1}^{k-2} a_j}{\prod_{j=m}^{k} (\lambda_k - \lambda_j)} \right) \sum_{i=m}^{k} \frac{1}{\prod_{i=m, i \neq j}^{k} (\lambda_k - \lambda_i)}.
\]

We only need to show \( d_{k, m} = 0 \), that is,

\[
\sum_{j=m}^{k} \frac{1}{\prod_{i=m, i \neq j}^{k} (\lambda_k - \lambda_i)} = 0 \quad \forall k, m \in \mathbb{N}.
\]

Using the Lagrange polynomial for \( p(\lambda) = 1 \), in the appropriate nodes, recall (3) with \( n = 0 \), we obtain that

\[
1 = \sum_{j=m}^{k} \frac{\prod_{i=m, j \neq i}^{k} (\lambda_k - \lambda_i)}{\prod_{i=m, j \neq i}^{k} (\lambda_k - \lambda_i)}.
\]

Taking the \( k - m \)-th derivative of (4) with respect to \( \lambda \) we obtain \( d_{k, m} = 0 \). □

**Lemma 2.** \( A_{a\lambda} = P_{a\lambda} \circ D_{\lambda} \circ P_{a\lambda}^{-1} \).

**Proof**

From our previous results we have that

\[
P_{a\lambda} \circ D_{\lambda} \circ P_{a\lambda}^{-1}(s) = \left( \lambda_1 s_1, \lambda_2 s_2 + f_{2, 1} s_1, \ldots, \lambda_k s_k + \sum_{m=1}^{k-1} f_{k, m} s_m, \ldots \right) \text{ where }
\]

\[
f_{k, m} = \left( \prod_{i=m-1}^{k-2} a_j \right) \sum_{j=m}^{k} \frac{\lambda_j}{\prod_{i=m, i \neq j}^{k} (\lambda_k - \lambda_i)}.
\]

Using the Lagrange polynomial for \( p(\lambda) = \lambda \), in the appropriate nodes, recall (3) with \( n = 1 \), we obtain that
λ = \sum_{j=m}^{k} \frac{\lambda_j \prod_{i=m,j \neq i}^{k} (\lambda - \lambda_i)}{\prod_{i=m,i \neq j}^{k} (\lambda_j - \lambda_i)}.

Taking the \( k - m \)-th derivative of (5) with respect to \( \lambda \) we obtain that if \( k = m + 1 \) then \( f_{m+1,m} = a_{m-1} \) and \( f_{k,m} = 0 \) for \( k > m + 1 \). □

**Theorem 2.** \( \Lambda_{a,\lambda}^n : S \to S \) is a linear operator and for \( s \in S \)

\[
\Lambda_{a,\lambda}^n(s) = \Lambda_{a,\lambda}^n \circ D_{a,\lambda}^n \circ P_{a,\lambda}^{-1}(s) = \left( \lambda_1^n s_1, \lambda_2^n s_2 + f_{2,1} s_1, \ldots, \lambda_k^n s_k + \sum_{m=1}^{k-1} f_{k,m} s_m, \ldots \right)
\]

where

\[
f_{k,m} = \left( \prod_{i=m-1}^{k-2} a_i \right) \frac{\lambda_1^n}{\prod_{j=m}^{k} \prod_{i=m,i \neq j}^{k} (\lambda_j - \lambda_i)}
\]

As a byproduct we obtained the following identity.

**Corollary 1.** Let \( \{\lambda_n\}_{n=1}^\infty \) a sequence with different elements, then

\[
P_{n-j+k}[\lambda_k, \lambda_{k+1}, \ldots, \lambda_j] = \sum_{i=k}^{j} \frac{\lambda_i^n}{\prod_{h=k, h \neq i}^{j} (\lambda_i - \lambda_h)} \quad \forall j \in \mathbb{N}.
\]

It follows from theorem 1 that a necessary condition for the iterates of an arbitrary sequence \( s \in S \), \( \Lambda_{a,\lambda}^n(s) \), to converge is that \( |\lambda_i| \leq 1 \) for all \( i \); otherwise, some of the terms of \( \Lambda_{a,\lambda}^n(s) \) will tend exponentially to infinity. In other words, the sequence \( \lambda \) must at least belong to the unit ball of the Banach space of bounded sequences, \( l^\infty \), endowed with the sup norm: \( \|s\|_\infty = \sup \{s_i\} \). This condition is however not sufficient, as we shall now show. Indeed, if we consider the family of sequences \( \{\tilde{e}_j, j \in \mathbb{N}\} \) then, theorem 1 shows that there are only \( n + 1 \) possibly nonzero entries in \( \{\Lambda_{a,\lambda}(\tilde{e}_j)\} \) for every \( j \). Moreover, the last two nonzero entries are simply

\[
(A_{a,\lambda}(\tilde{e}_j))_{n+j-1} = \left( \prod_{k=j+1}^{j+n-1} a_k \right) (\lambda_j + \cdots + \lambda_j + n) \quad \text{and} \quad (A_{a,\lambda}(\tilde{e}_j))_{n+j} = \prod_{k=j+1}^{j+n} a_k.
\]

Thus, the last formula shows that existence of \( \lim_{n \to \infty} \Lambda_{a,\lambda}^n(s) \) also requires that

\[
\lim_{n \to \infty} \prod_{k=1}^{n} a_k
\]

exists. In particular, to avoid oscillation of the products, we shall assume in what follows that \( a_i > 0 \) for all \( i \). Now, product (7) can converge in two different scenarios. First, if \( \sup \{a_i\} < 1 \), then the product always converges to zero. Second, if \( \lim_{i \to \infty} a_i = 1 \), then the limit exists if and only if \( \sum \log a_i \) exists. In particular, this implies that \( \lim_{m \to \infty} a_m = 0 \) and therefore \( \lim a_m = 1 \). Moreover, since this implies that all partial products of the \( a_i \) are bounded, from (6) we conclude that also \( \sum_{i=0}^{\infty} \lambda_i \) must exist.

In order to state our next result, let us first introduce some simplifying notation. For any given sequence \( s = (s_0, s_1, \ldots) \), let \( |s| = (|s_0|, |s_1|, \ldots) \) and if \( i < j \) let
Let \( s_{ij} = (s_i, \ldots, s_j) \) be the segment of \( s \) between \( s_i \) and \( s_j \). Also, recall the Banach space \( l^1: \)

\[
l^1 = \{ s = (s_0, s_1, \ldots) \mid \sum_{n=0}^{\infty} |s_n| < \infty \}
\]

endowed with the norm \( \|s\|_1 = \sum |s_n| \), [1].

**Lemma 3.** Let \( \lambda \in l^1 \). For any segment \( \lambda_{j-1, i-1} \), and for any \( n \) such that \( n > i - j \),

\[
|P_n \lambda_{n-(i-j)}| \leq |P_n \lambda_{n-(i-j)}| \leq \|\lambda\|_1^{n-(i-j)}.
\]

In particular, if \( \|\lambda\|_1 \leq 1 \), then \( |P_n \lambda_{n-(i-j)}| \leq 1 \).

The proof is also straightforward, and from the previous arguments we get the following result:

**Theorem 3.** Let \( s \in S \) and two fixed sequences \( a = \{a_n\}_{n=0}^{\infty} \) and \( \lambda = \{\lambda_n\}_{n=1}^{\infty} \) such that

\begin{enumerate}
\item \( \prod_{k=1}^{\infty} a_k \) exists.
\item \( \|\lambda\|_1 \leq 1 \)
\end{enumerate}

then, the limit entry-wise of the iterates \( A^n_{a\lambda}(s) \) as \( n \to \infty \) exists. Moreover, if \( \|\lambda\|_1 < 1 \) this limit is zero.

**Proof** We note that lemma 3 implies that for all \( i, j \) \( P_n \lambda_{n-(i-j)} \) is a sum of positive terms bounded by 1.

The content of this proposition seems mostly algebraic, since we are using only pointwise convergence; however, it provides a natural candidate to understand the asymptotic behavior of the solutions of (1) as we will show in the final section of this work. On the other hand, the same type of arguments provide the following result:

**Theorem 4.** Let \( A_{a\lambda} \) be as in theorem 1. Then every iterate \( A^n_{a\lambda} \) defines a continuous operator from \( l^1 \to l^\infty \).

**Proof** Let \( s \) be any sequence in \( l^1 \), and \( M > \prod_{k=1}^{\infty} a_k \). From theorem 1, we have that for all \( i \) and \( n \)

\begin{align}
|A^n_{a\lambda}(s)|_i & = \left| \sum_j \left( \prod_{l=j}^{i-1} a_l \right) P_{n-(i-j)}(\lambda_{i-1, j-1}, \ldots, \lambda_{i-1})s_j \right| \\
& \leq M\|\lambda\|_1^{n-(i-j)} \sum_{j=1}^{n} |s_j| \leq M\|\lambda\|_1\|s\|_1.
\end{align}

Therefore \( \|A^n_{a\lambda}(s)\|_{\infty} \leq M\|s\|_1 \) as desired.

### 4. A transpose of \( A_{a\lambda} \)

Given \( s = \{s_n\}_{n=0}^{\infty} \in S \) and two fixed sequences \( a = \{a_n\}_{n=0}^{\infty} \) and \( \lambda = \{\lambda_n\}_{n=1}^{\infty} \), let \( c \) be the sequence defined by

\[
c_n = \lambda_n s_n + \lambda_{n+1} a_{n-1}, \quad n \in \mathbb{N},
\]
then the linear operator
\[ A^T: \mathcal{S} \rightarrow \mathcal{S} \]
defined by \( A^T(s) = c \) is a transpose operator of the discrete advection operator \( A_{\alpha \lambda} \) [6]. An explicit result similar to theorem 1 is the following:

**Theorem 5.** Let \( s^{(0)} \in \mathcal{S} \), then
\[ A^{Tn}_{\alpha \lambda}(s^{(0)}) = (s_1^{(n)}, s_2^{(n)}, s_3^{(n)}, ...) \]
where
\[ s_j^{(n)} = \lambda_j^n s_j + \sum_{m=j}^{n+j} \prod_{k=m-1}^{m-2} a_k P_{n+m-j} \lambda_m, \ldots, \lambda_j s_j^{(0)} \]

The proof of this result is similar to the proof of theorem 1. From this result we conclude that convergence properties of \( A^T_{\alpha \lambda} \) can be directly derived from the convergence properties of \( A_{\alpha \lambda} \).

5. The continuous problem

Consider the following initial-boundary value problem for a linear reaction-advection partial differential equation
\[ u_t + \alpha(x)u_x = \beta(x)u, \quad x > 0, \quad t > 0 \]
\[ u(x,0) = f(x), \]
\[ u(0,t) = g(t), \]
where \( \alpha(x), \beta(x), f(x) \) and \( g(t) \) are smooth functions. If \( \alpha(x) > 0 \), then a correct finite difference scheme to approximate solve problem (13)-(15) is given by
\[ v(x,t+k) - v(x,t) \quad + \alpha(x) \frac{v(x,t) - v(x-h,t)}{h} = \beta(x)u(x,t). \]

If (16) satisfies \( \alpha(x) \leq 1 \), then the Courant-Friedrichs-Lewy test holds and \( v(x,t) \rightarrow u(x,t) \) as \( k, h \rightarrow 0 \), [3].

Let \( mh, nk \) with \( m, n = 0, 1, 2, \ldots \) and \( h, k > 0 \) be a rectangular grid. Defining \( v^n_m = v(mh, nk), \alpha_m = \alpha(mh), \beta_m = \beta(mh), \lambda_m = 1 - \alpha_m k + \beta_m k \) and \( a_m = \alpha_m k \), equation (16) is equivalent to
\[ v^{n+1}_m = \lambda_m v^n_m + a_m v^n_{m-1}. \]

A boundary condition (15) consistent with equation (17) is \( u(0,t) = 0 \). Thus, equations (16) and (17) suggest that an iteration of the discrete advection operator corresponds to a one time step of the finite difference scheme for the linear reaction-advection partial differential equation.

Finally, let us give some remarks on the solution of the continuous advection equation. To begin with, recall that the general solution to the homogeneous advection equation may be given in the form:
\[ u(x,t) = \psi \left( t - \int_0^x \frac{1}{\alpha} \right). \]
where $\psi$ is arbitrary. Now, in order to satisfy the boundary condition, we must have $u(0,t) = \psi(t) = g(t)$. In other words, for $y > 0$, we have the condition $\psi(y) = g(y)$, and this fixes the function $\psi$ on the half-line. On the other hand, setting $h(x) = -\int_0^x 1/\alpha$, in order to satisfy the initial condition we must have $u(x,0) = f(x) = \psi(h(x))$. The assumption $\alpha(x) > 0$ implies that $h$ is monotonically decreasing, and in particular, invertible. Therefore, we can rewrite the last equation as:

$$\psi(y) = f(h^{-1}(y)),$$

for $y \in \text{Im} h$. Since $h$ is also negative, $\text{Im} h$ is actually an interval of the form $(\gamma, 0]$, where

$$\gamma = \lim_{x \to \infty} h(x).$$

Thus no ambiguity in the determination of the function $\psi$ (except perhaps at the origin) occurs, and we can write

$$\psi(y) = \begin{cases} g(y) & y > 0 \\ f(h^{-1}(y)) & \gamma < y \leq 0 \end{cases}$$

to construct the unique solution of the problem. The same method applies to the inhomogeneous case: We still use the boundary condition to get $\psi(y) = g(y)$, for $y > 0$, as before; but now, for the initial condition we get the slightly more complicated relation:

$$\psi(x) = e^{-\int_0^x \beta/\alpha} f(x).$$

The right hand side of this last equation involves only known data, and again, since $h$ is invertible, we can change variables to $y = h(x)$ and rewrite this equation as

$$\psi(y) = e^{-\int_{h^{-1}(\gamma)}^{h^{-1}(y)} \beta} f(h^{-1}(y)),$$

which now determines $\psi$ in the interval $(\gamma, 0]$. Now, to relate these facts to the discrete problem, observe that in this description of the solution there might still remain some ambiguity in the determination of $\psi$, since if for instance, $\lim_{x \to \infty} f(x)$ exists and is bounded, we might continue $\psi$ beyond $\gamma$, and hence continue the solution of the problem in a continuous fashion. However, if the integral $\int_0^x 1/\alpha$ converges, then $1/\alpha \to 0$, or $\alpha \to \infty$, as $x \to \infty$. As we have seen, convergence of the iterates $A_n^\alpha$ requires $\prod a_m$ to be convergent, and therefore $a_m$ to be bounded. Since we are setting $a_m = \alpha(m)k$ and taking $h = k$, convergence of the product clearly requires divergence of the integral.

References