The question of accessibility of points in the union of two ray extensions is considered in the context of a single component of the basin.

Introduction: Our goal in this paper is to investigate the set of accessible points in the Julia set of a rational map. The focus is on the dynamics of the map and the properties of its Fatou components.

Abstract: In this paper, we consider the problem of accessibility of points in the Fatou set of a rational map. The main results concern the existence of accessible points in the boundary of the Fatou set and the structure of these points under certain conditions.

Reference: This paper builds upon the work of earlier studies on the dynamics of rational maps and the accessibility of points in the Julia set.

Keywords: Fatou components, accessibility, Julia set, rational maps.
dense in $J(E_{\lambda})$, it follows that the set of endpoints of these curves must also be dense. Moreover, it is known that the Cantor bouquet is nowhere locally connected.

In Figure 1, we display the Julia set when $\lambda = 1/e$. When $0 < \lambda < 1/e$, $E_{\lambda}$ has an attracting fixed point with a similar Julia set as the one for $\lambda = 1/e$. The basin of attraction of this fixed point (the complement of the Julia set) is shown in black. The Cantor bouquet is displayed in white. In this figure, it appears that the Julia set contains open sets. In reality, $J(E_{\lambda})$ is an uncountable collection of disjoint curves. These curves are packed closely together and it is known [24] that the Hausdorff dimension of this set is 2.

In [12] it is shown that the set of accessible points in this Julia set are precisely the set of endpoints together with the point at $\infty$. Thus, all points on the curves (with the exception of the endpoints) are inaccessible.

In the case of an attracting cycle with period greater than one, the situation is different. In this case the Julia set is a Cantor bouquet with “pinchings.” By this we mean that there are infinitely many points in $J(E_{\lambda})$ that lie at the endpoint of two or more hairs. These pinchings or attachments have been described in [6] and [13].

For example, in Figure 2, we display the Julia set when $\lambda = 5 + i\pi$. It is easy to see that this exponential has an attracting cycle of period 3. In this case it appears that there are triplets of hairs that are attached at certain points in the plane. As another example, in Figure 3, we display the Julia set when $\lambda = 10 + 3\pi i$. This map also has an attracting cycle of period 3. Note that a larger number of hairs now seem to be attached.

Because of these attachments, the set of accessible points in $J(E_{\lambda})$ is quite different in the cycle case. It is no longer the case that all endpoints are accessible; rather, only very special endpoints (and $\infty$) are accessible. Our goal in this paper is
In Section 3, we show that points in $J(E)$ with bounded density lie on the boundary of a bounded set and explain the construction of a Cantor bouquet $E$ as introduced in [5] and [13]. We recall the construction in Section 3. We describe the set of accessible points, and make use of the branching structure for $E$ to describe the preimage of the set of accessible points. This in turn yields a good picture of the topology of $E$.

Figure 2. The Julia set for $a = 2 + e^{it}$.

Figure 3. The Julia set for $a = 1 + e^{it}$.
hairs. Since our result applies equally well to points with unbounded itinerary, we extend this result to the unbounded case in Section 5. Finally, in Section 6, we prove accessibility.

2. **Basins of Attraction.** In this section we will describe some general properties of the complement of the Julia by summarizing some of the results in [6]. We assume that $E_A$ has an attracting periodic cycle $z_0, z_1 = z_n = z_0$ of prime period $n$, with $E_A(z_1) = z_{n+1}$. Throughout we assume that $n \geq 2$. Let $A^*(z_1)$ denote the immediate basin of attraction containing $z_1$.

**Definition 2.1.** An unbounded, simply connected set $F \subset \mathbb{C}$ is called a finger of width $c$ if

i): $F$ is bounded by a simple curve $\gamma \subset \mathbb{C}$.

ii): There exists a $\nu > 0$ such that $F \cap \{ z \mid \Re z > \nu \}$ is simply connected, extends to infinity, and satisfies

$$F \cap \{ z \mid \Re z > \nu \} \subset \{ z \mid \Im z \in \left[ \frac{\xi - c}{2}, \frac{\xi + c}{2} \right] \}$$

for some $\xi \in \mathbb{R}$.

With this definition we can now characterize parts of the stable set as shown in [6].

**Theorem 2.1.** Suppose $z_0, \ldots, z_{n-1}$ is an attracting periodic orbit for $E_A$ with $n \geq 2$. Suppose $0 \in A^*(z_1)$. Then there exist disjoint, open, simply connected sets $C_0, \ldots, C_{n-1}$ such that

i): $z_j \in C_j$, $C_j \subset A^*(z_j)$.

ii): $E_A(C_0) = C_1 \cup \{0\}$.

iii): $E_A(C_j) = C_{j+1}$, $j = 1, \ldots, n - 2$ and $E_A(C_{n-1}) \subset C_0$.

iv): $C_1, \ldots, C_{n-1}$ are fingers of width $c_j \leq 2\pi$.

v): The complement of $C_0$ consists of infinitely many disjoint fingers of width $2\pi$.

Since this collection of sets will become important later we formulate the following

**Definition 2.2.** A collection of open subsets $C_0, \ldots, C_{n-1}$ satisfying the conditions in Theorem 2.1 is called a fundamental set of attracting domains for the cycle $z_0, \ldots, z_{n-1}$. The fingers $C_1, \ldots, C_{n-1}$ are called stable fingers. The region $C_0$ is called a glove.

A typical example of a fundamental set of attracting domains for an exponential with an attracting cycle of period 5 is shown in Figure 4. We remark that this figure is actually a caricature, since, for an actual exponential, the width of the fingers $C_1, C_2$, and $C_3$ is small compared to the width of $C_4$.

In fact there are many ways to construct a fundamental set of attracting domains. In order to simplify later computations we wish to make the boundaries of the fingers smooth and nearly horizontal in the far right half-plane as those shown in the picture.

**Definition 2.3.** A smooth curve $\gamma(t)$ is called horizontally asymptotic to $c$ if

i): $\lim_{t \to \infty} \Re(\gamma(t)) = +\infty$.

ii): $\lim_{t \to \infty} \Im(\gamma(t)) = c$.

iii): $\lim_{t \to \infty} \arg(\gamma(t)) = 0$. 

We define the iteration of $f(z)$ as

$$f^n(z) = \underbrace{f \circ f \circ \ldots \circ f}_n(z)$$

where $n \in \mathbb{N}$ and $f(z) = c - z^2$. The iteration space $F_n$ is defined as

$$F_n = \{ z \in \mathbb{C} : |f^n(z) - c| < \varepsilon \}$$

The set of points $\Lambda$ is defined as

$$\Lambda = \bigcap_{n=1}^\infty F_n$$

The Julia set $J(f)$ is the boundary of the Fatou set $\Lambda$. The Fatou set $\Lambda$ is the set of points where the iteration of $f(z)$ approaches a limit point. The Julia set $J(f)$ is the complement of the Fatou set in the complex plane.

Theorem 2.1: The complement of $\Lambda$ is connected.

Proof: The proof of the following can be found in [6].
Note that \( S(E_\lambda(z)) = \sigma(S(z)) \). We do not define the itinerary of points outside \( J(E_\lambda) \).

It is known that there are itineraries that do not correspond to any point in \( J(E_\lambda) \) [14]. For example, there are no points in \( J(E_\lambda) \) that have itineraries of the form \((s_0, s_1, s_2, \ldots)\) when \(|s_j|\) grows faster than an iterated (real) exponential. We let \( \Sigma_\alpha \) denote the set of allowable sequences in the sense that \((s_0, s_1, s_2, \ldots) \in \Sigma_\alpha \) if and only if there exists \( z \in J(E_\lambda) \) whose itinerary is \((s_0, s_1, s_2, \ldots) \). It can be shown that \( \Sigma_\alpha \) is independent of \( \lambda \) [15].

For each \( C_j \) with \( 1 \leq j \leq n - 1 \), there exists \( \mathcal{H}_k \) such that \( C_j \subset \mathcal{H}_k \). We define the kneading sequence for \( \lambda \) as follows.

**Definition 3.1.** Let \( E_\lambda \) be an attracting cycle of period \( n \geq 2 \). The kneading sequence associated with \( E_\lambda \) is the string of \( n - 1 \) integers followed by \( * \)

\[
K(\lambda) = 0k_1k_2\ldots k_{n-2}*
\]

where \( k_i = j \) iff \( E^i_\lambda(0) \in \mathcal{H}_j \).

Note that the kneading sequence gives the location of \( E_\lambda(0), \ldots, E^{n-2}_\lambda(0) \) in terms of \( \mathcal{H}_k \). For completeness we also include the location of \( 0 \) in \( \mathcal{H}_0 \). Similarly, \( E^{n-1}_\lambda(0) \) lies in \( \mathcal{C}_0 \), which is the complement of the \( \mathcal{H}_k \), and so this will be denoted by \( * \). We think of \( * \) as a "wild card." The importance of including this entry will become clear later.

Equivalently, the kneading sequence indicates which \( \mathcal{H}_k \) contains the points \( z_1, z_2, \ldots, z_{n-1} \) on the orbit of the cycle.

For a sufficiently large real number \( \tau \)

\[
\Lambda_{\tau} = \{ z \in \mathbb{C} | \Re z \geq \tau \} - \bigcup_{j=0}^{n-1} C_j
\]

consists of infinitely many closed fingers. Each finger in \( \Lambda_{\tau} \) is included in precisely one \( \mathcal{H}_j \).

If \( j \) is not one of the entries in the kneading sequence, then there is only one finger in \( \Lambda_{\tau} \) that lies in \( \mathcal{H}_j \) (namely the far right portion of \( \mathcal{H}_j \) itself). We denote this finger in \( \Lambda_{\tau} \) by \( H_{j_0} \).

However, for \( j \) in the kneading sequence, we know that one of the points on the attracting cycle, say \( z_1 \), lies in \( \mathcal{H}_j \). Thus \( C_1 \) separates \( \Lambda_{\tau} \cap \mathcal{H}_j \) into at least two fingers. Since \( \Lambda_{\tau} \) has more than one component in \( \mathcal{H}_j \), we need a way to unambiguously identify them. Assume that \( \Lambda_{\tau} \) has \( k \) components in \( \mathcal{H}_j \). In this case, the fingers that lie in \( \mathcal{H}_j \) will be denoted \( H_{j_1}, \ldots, H_{j_k} \) where the \( j_0 \)'s are ordered with ascending imaginary part. Note that all of these fingers lie in the right half plane \( \Re z \geq \tau \).

Hence we can describe the itinerary of certain points in the Julia set even more precisely by defining an augmented itinerary for \( z \in J(E_\lambda) \cap \{ z \in \mathbb{C} | \Re z \geq \tau \} \).

In an augmented itinerary, we specify which of the \( H_{j_k} \) the orbit of \( z \) visits. More precisely, let \( Z' \) denote the set whose elements are either integers not contained in the kneading sequence, or subscripted integers \( j_k \) corresponding to an \( H_{j_k} \) if \( j \) is an entry in the kneading sequence. The augmented itinerary of \( z \) is

\[
S'(z) = (s_0s_1s_2\ldots)
\]

where each \( s_j \in Z' \) and \( s_j \) specifies the finger in \( \Lambda_{\tau} \) containing \( E_\lambda(z) \).

Let \( \Sigma' \) denote the set of allowable (in the above sense) augmented itineraries. We topologize \( \Sigma' \) in the usual way, so that nearby sequences share the same initial
Theorem 4.1. The set $\mathcal{A}$ is a connected set if and only if it is locally disconnected.
That is, if we remove just one point from the connected set $\mathcal{E}^*$, the resulting set is totally disconnected.

The reason for this is that, if we draw the straight line in the plane $(\gamma, t)$ where $\gamma$ is a fixed rational, and then we adjoin the point at infinity, we find a disconnection of $\mathcal{E}$. This, however, is not a disconnection of $\mathcal{E}^*$. Moreover, the fact that any non-endpoint in $B$ is inaccessible shows that we cannot disconnect $\mathcal{E}^*$ by any other curve.

Remark. Aarts and Oversteegen have shown that any two straight brushes are ambiently homeomorphic, i.e., there is a homeomorphism of $\mathbb{R}^2$ taking one brush onto the other. This leads to a formal definition of a Cantor bouquet.

Definition 4.2. A Cantor bouquet is a subset of $C^*$ that is homeomorphic to a straight brush (with $\infty$ mapped to $\infty$).

The connection with exponential dynamics arises from the following result proved in [1].

Theorem 4.2. Suppose $0 < \lambda < 1/e$. Then $J(E_\lambda)$ is a Cantor bouquet.

In this case, the dense subset $N'$ of the irrationals is identified in a natural way with the set of allowable itineraries $\Sigma_n$.

In the above theorem, $E_\lambda$ has an attracting fixed point. Our goal below is to prove an analogous result in the attracting cycle case. In this analogy, we will think of a Cantor bouquet as being a subset $o'[0, \infty) \times \Sigma'$ rather than $[0, \infty) \times \Sigma$. This will yield a modified straight brush.

5. The Modified Brush. In the case of an attracting cycle of period two or more, $J(E_\lambda)$ is no longer a Cantor bouquet. It is true that all points in $J(E_\lambda)$ lie on hairs, but some of these hairs share the same endpoint [6]. In this section we will show that there is a unique hair in the Julia set corresponding to any allowable augmented sequence in $\Sigma'$. Moreover, any two hairs corresponding to sequences with the same deaugmentation share an endpoint. We therefore modify the straight brush construction to take into account this pinching.

For a specified $p_{\lambda} \in \mathbb{R}$, we will first introduce in this section a preliminary brush $MB' \subset [p_{\lambda}, \infty) \times \Sigma'$.

The modified straight brush $MB$ will then be the quotient $MB'/\sim$ via an equivalence relation defined below. Finally, we prove the existence of a homeomorphism $\phi: MB \rightarrow J(E_\lambda)$.

The construction of $MB'$ and $\phi$ will be similar in spirit to that in [1], hence we will only specify the necessary modifications of the Aarts-Oversteegen construction.

We first define three quantities

$p_{\lambda}, r_{\lambda}, q_{\lambda}$

as follows:

Definition 5.1. Let $p_{\lambda} \in \mathbb{R}$ such that

$\{ \text{Re } z = p_{\lambda} \} \cap H_i \neq \emptyset$,
\[ \bigcap_{i=1}^{\infty} \left( \mathbb{R}^n \cap \mathbb{Q}^n \right) \]

**Definition 6.2.** Let \( L = \mathbb{Q}^n \) for each \( x \in \mathbb{R}^n \). Let \( a, b \in \mathbb{R}^n \) and \( a, b \neq c \). Then

\[ a < b \iff \exists \epsilon > 0 \text{ such that } (a, b) \cap (c, d) = \emptyset. \]

The existence of \( a \) and \( b \) is guaranteed by Proposition 3.2. The reason for the choice of \( a \) and \( b \) is explained in Section 6.2.

In order words, if \( \mathbb{Q} \) is the real part of the interval point \( x \) in each of the \( \mathbb{Q}^n \),

\[ a < b \iff \exists \epsilon > 0 \text{ such that } (a, b) \cap (c, d) = \emptyset. \]
Roughly, $D(x, s_i)$ is a "rectangle" in $H_{D(s_i)}$, with width $L$, with "horizontal" pieces cut out by the fingers $C_i$. For the definition of the $S(x, s_i)$, there are two cases:

**Definition 5.3.**  

i) If $s_i = n$ (not augmented), then we set 

$$S(x, s_i) = D(x, s_i).$$

ii) If $s_i = n_k$ (augmented), then we set:

(a) For $x \geq r_k$ we set $S(x, s_i)$ to be the $k$th component of $D(x, s_i)$ (counted with ascending imaginary part).

(b) For $x < r_k$ let $S(x, s_i)$ be the component of $D(x, s_i)$ whose right hand edge lies in $H_{n_k}$.

Now we turn to the construction of the preliminary brush $\mathcal{MB}'$ in $[p_3, \infty) \times \Sigma'$. First, for any $x \in [p_3, \infty)$ and $s \in \Sigma'$, define a sequence of real numbers $\{x_0, x_1, \ldots\}$ and a sequence of boxes $R(x_i, s_i)$ inductively:

**Definition 5.4.** Let $x_0 = x$ and $R(x_0, s_0) = S(x, s_0)$. Suppose that $x_l$ and $R(x_l, s_l)$ have been defined for $l \leq k$. Then there are two cases:

i) $R(x_k, s_k) \neq \emptyset$ and there is a $\xi$ such that 

$$S(\xi, s_{k+1}) \subset S_{\lambda}(R(x_k, s_k)).$$

Define $\xi_{\min}$ to be the minimum $\xi$ that satisfies the above and set 

$$x_{k+1} = \xi_{\min}, \quad R(x_{k+1}, s_{k+1}) = S(\xi_{\min}, s_{k+1}).$$

ii) If $R(x_k, s_k) = \emptyset$ or if there is no $\xi$ as above, then set 

$$x_{k+1} = x_k, \quad R(x_{k+1}, s_{k+1}) = \emptyset.$$ 

If $R(x_k, s_k) = \emptyset$ for some $k$, we say that the sequence of boxes terminates. If the sequence of boxes does not terminate, then 

$$E_{\lambda}(R(x_k, s_k)) \supset R(x_{k+1}, s_{k+1})$$

for each $k$.

**Definition 5.5.** The preliminary brush $\mathcal{MB}'$ is the set of points $(x, s)$ for which the sequence of boxes $R(x_k, s_k)$ does not terminate, i.e., 

$$\mathcal{MB}' = \{(x, s) \in [p_3, \infty) \times \Sigma' : R(x_k, s_k) \neq \emptyset\}$$

Following Aarts and Oversteegen [1], we will show that for $(x, s) \in \mathcal{MB}'$, there is a unique point whose orbit visits the $R(x_k, s_k)$ sequentially for all $k$. Unlike the case in [1], however, two different sequences of boxes may yield the same point. To remedy this, we identify points $(x, s), (y, s) \in \mathcal{MB}'$ for which 

$$R(x_k, s_k) \cap R(y_k, s_k) \neq \emptyset,$$

for all $k$. In such cases we will write $(x, s) \sim (y, s)$. We will see below that, whenever two such points are identified, these points always correspond to an endpoint of a hair. First we note:

**Proposition 5.1.** The relation $\sim$ is an equivalence relation.
which is also open.

\[ N^{-1} \cap N = \emptyset \]

The set of all \( x \)'s which the sequence converges is

\[ \{ (x_\theta) | \theta \in [\delta, \epsilon]\} = N^{-1} \]

The sequence of points \( x_{N+1} \) such that \( x_{N+1} \in N^{-1} \) is open condition on \( \delta = \emptyset \), because \( x_{N+1} \) is a continuous map, and the set of all \( x \)'s which satisfy this condition is open, and since \( x_{N+1} \in N^{-1} \), there is a continuity map. This is an open condition.

\[ \emptyset = (x_{N+1})^N \cup (X_1 x_N)_{\emptyset} \]

If \( x \in N^{-1} \), then there is an open neighborhood \( N \) around \( x \), such that for

\[ x < (X_1 x_N)_{\emptyset} \]

Since these two sets are closed, there is some \( s \) such that

\[ \emptyset = (X_1 x_N)_{\emptyset} \]

If the sequence \( (x_\theta)_{\emptyset} \) does not converge, then there is some \( x \) for which

\[ \emptyset \]

There are two possibilities:

\[ \emptyset \neq (s, s') \]

We will show that, for a given hierarchy and fixed \( x \), the set

\[ \{ (s, s') | \emptyset \} \]

The proposition claims for the existence of a closed interval

\[ \emptyset \neq (s, s') \]

not terminating. Then we get

**Proposition 2.2.** A function \( f \) is continuous if and only if

\[ \text{for all } x, y \in [\delta, \epsilon] \text{ where } f(x) = f(y) \text{ or } f(x) \neq f(y) \]

and that there exists \( M < x \) such that

\[ f(M) = f(x) \]

The proposition then follows directly from the definition. To prove

**Proof.**
For any itinerary $s$ for which there exists an $x$ with $(x, s) \in MB'$, let $x^\text{min}_s$ be the smallest such number. By considering the set

$$A_s = \{ y \mid (x^\text{min}_s, s) \sim (y, s) \}$$

we define

$$\overline{x}_s = \sup A_s.$$ 

We now show that the only equivalence class that possibly consists of more than one point is the equivalence class containing $(\overline{x}_s, s)$.

**Proposition 5.3.** For any $(x, s), (y, s) \in MB'$ with $\overline{x}_s \leq x < y$ there is a $K$ so that for all $k \geq K$,

$$R(x_k, s_k) \cap R(y_k, s_k) = \emptyset.$$

**Proof:** Assume for contradiction that $R(x_k, s_k) \cap R(y_k, s_k) \neq \emptyset$ for all $k$. Then there are two cases:

1. We can have

$$R(x_k, s_k) \cap R(y_k, s_k) \cap (\overline{x}_s, s_k) \neq \emptyset$$

for all $k$. But recall that $\overline{x}_s$ was defined to be the largest real number with the property that it was equivalent to $x^\text{min}_s$. This would imply that both $x, y \leq \overline{x}_s$, which is a contradiction.

2. We can have

$$R(x_k, s_k) \cap R(y_k, s_k) \cap (\overline{x}_s, s_k) = \emptyset$$

for some $k$. Assume for specificity that $x < y$. Then $y_k$ is to the right of the box containing $\overline{x}_s$ by our assumptions, and hence lies to the right of the line $\{z \mid \text{Re} z = q_k \}$. Therefore the subsequent $y_k$ in the construction will move away from the $x_k$ like an iterated exponential, and thus their corresponding boxes will stop intersecting, which yields a contradiction.

We may finally define the modified straight brush.

**Definition 5.6.** The modified straight brush MB is the quotient $MB'/\sim$ endowed with the quotient topology. Also define the map

$$\phi : MB \rightarrow J(E_k)$$

as follows. For each $(x, s) \in MB$, $k \in \mathbb{N}$ let

$$B_k(x, s) = \{ z \in \mathbb{C} \mid E_k(z) \in R(x_i, s_i) \text{ for } 0 \leq i \leq k \}$$

and set

$$\phi(x, s) = \bigcap_{k=0}^{\infty} B_k(x, s).$$

As in [1], each $B_k$ is a well-defined set which is compact and simply connected. Also, $B_{k+1}(x, s) \subset B_k(x, s)$, so that $\phi(x, s)$ is a nested intersection of compact sets.

**Proposition 5.4.** For all $(x, s) \in MB$ the set $\bigcap_{k=0}^{\infty} B_k(x, s)$ consists of a single point.
The region where $f(y)$ is expanded (see Definition 2.1) and the condition $\lambda_1 < \mu_1 < \nu_1$ is true for all $y$. Let $\emptyset$ be the function defined by $\emptyset(y) = \gamma + \delta < \lambda$. Then $f(\emptyset) = (\gamma + \delta)^2 \cup (\gamma + \delta, \infty)$

Proposition 3.5. Suppose that $f(x)$ is continuous, then $f(x)$ is differentiable. The map $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is injective. The map $\gamma^2 + \delta^2 < \lambda$ is continuous.

Proof. Let $x \in (s', s'')$ with $s' < s < s''$. We only need to show that $s' < s < s''$. Since $\lambda_1 < \mu_1 < \nu_1$, we have $s' < s < s''$. However, if not, it follows that $s = s'$ or $s = s''$.

Now we have shown that the map $\phi(x)$ is well-defined. However, it is not necessarily continuous.

Corollary. The map $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and strictly increasing. Therefore, we will consider the function $f(x) = \phi(x)$ and show that the integral of $\phi(x)$ over the interval $[a, b]$ is equal to $\int_a^b \phi(x) \, dx$.

Since $\gamma^2 + \delta^2 < \lambda$, it follows that

\[ s \in (a, b) \Rightarrow \int_a^b \phi(x) \, dx \leq \int_a^b \gamma^2 \, dx + \int_a^b \delta^2 \, dx < \lambda - \gamma^2 - \delta^2. \]

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\[ \int_a^b \phi(x) \, dx \leq \int_a^b \gamma^2 \, dx + \int_a^b \delta^2 \, dx < \lambda - \gamma^2 - \delta^2. \]

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\[ s \in (a, b) \Rightarrow \int_a^b \phi(x) \, dx \leq \int_a^b \gamma^2 \, dx + \int_a^b \delta^2 \, dx < \lambda - \gamma^2 - \delta^2. \]

Since $\gamma^2 + \delta^2 < \lambda$, it follows that

\[ \int_a^b \phi(x) \, dx \leq \int_a^b \gamma^2 \, dx + \int_a^b \delta^2 \, dx < \lambda - \gamma^2 - \delta^2. \]

Therefore, $\phi(x) \leq \gamma^2 + \delta^2 < \lambda$. Since $\phi(x)$ is a continuous function and $\phi(x)$ is a monotonically increasing function, it follows that $\phi(x)$ is continuous and strictly increasing. Therefore, we will consider the function $f(x) = \phi(x)$ and show that the integral of $\phi(x)$ over the interval $[a, b]$ is equal to $\int_a^b \phi(x) \, dx$.

Since $\gamma^2 + \delta^2 < \lambda$, it follows that

\[ s \in (a, b) \Rightarrow \int_a^b \phi(x) \, dx \leq \int_a^b \gamma^2 \, dx + \int_a^b \delta^2 \, dx < \lambda - \gamma^2 - \delta^2. \]
and therefore condition (1) is satisfied for a sufficiently large \( k \).

On the other hand, if \( s \neq s' \) the two sequences must differ in some entry \( k \), i.e., \( s_k \neq s'_k \). We would like to conclude that the corresponding boxes then lie in different strips and hence are disjoint. If the deaugmentation of these entries are different, i.e., if \( D(s_k) \neq D(s'_k) \), then we are done, since the boxes \( R(x_k, s_k) \) and \( R(y_k, s'_k) \) do lie in different strips, and therefore they do not intersect.

So assume that \( D(s_k) = D(s'_k) \). Recalling the construction of the \( S(x, s) \), if \( x_k \leq q_k \), and \( s_k \neq s'_k \), then \( D(s_k) = D(s'_k) \), then \( R(x_k, s_k) = R(x_k, s'_k) \), and so of course it is possible that \( R(x_k, s_k) \cap R(y_k, s'_k) \neq \emptyset \). It was shown in [6] that if two sequences have the same deaugmentation, and the augmented sequences differ at the \( k \)th step (i.e. \( D(s) = D(s') \) but \( s_k \neq s'_k \)), then \( s_l \neq s'_l \) for all \( l \geq k \). Since \( x > x_0 \), we know that there is a \( m \) such that \( x_m > q_1 \). From this and [6] we can find an \( m \) such that \( x_m > q_1 \), and \( s_m \neq s'_m \), and thus \( R(x_k, s_k) \cap R(y_k, s'_k) = \emptyset \).

Next we will show that the map \( \phi(x, s) \) is continuous. Fix \( (x, s) \in MB \). We want to show that if \( \phi(x', s') \) is close to \( (x, s) \) then \( \phi(x', s') \) is close to \( \phi(x, s) \). Fix \( N \). Choose \( s' \in \Sigma' \) with \( s_i = s'_i \) for all \( i \leq N \). Since \( E_0 \) is continuous, we can choose \( x' \) close to \( x \) so that

\[
R(x_1, s_1) \cap R(x'_1, s'_1) \neq \emptyset \text{ for all } i \leq N.
\]

Then \( \phi(x', s') \) is close to \( \phi(x, s) \) since \( E_0 \) is expanding.

Now we need only surjectivity:

**Proposition 5.6.** For any \( z \in J(E_0) \) there exists \( (x, s) \in MB \) such that \( \phi(x, s) = z \).

**Proof:** Let \( s \) be the itinerary of \( z \). We will find an \( x \) such that \( E_0^k(z) \in R(x_k, s_k) \) and hence \( \phi(x, s) = z \).

For each \( k \in \mathbb{N} \) let \( R_k^s = S(u, s_k) \) with

\[
u = \inf \{ w : w \geq \vartheta_k \text{ and } E_0^k(z) \in S(w, s_k) \}.
\]

That is, \( R_k^s \) is the box whose right hand edge has real part equal to \( \text{Re} E_0^k(z) \). The boxes \( R_k^s \) with \( 0 \leq i < k \) are defined inductively as follows: if \( R_{i+1}^k \) is defined then let

\[
R_i^k = \mathcal{Z}(\nu, s_i)
\]

where \( \nu = \sup \{ \mu | R_{i+1}^k \subset E_0^k(S(\mu, s_{i+1})) \} \).

Let \( t_k \in \mathbb{R} \) be the point such that

\[
R_i^k = S(t_k, s_k).
\]

By construction \( p_k \leq t_k \leq t_{k+1} \leq \text{Re}(z) \) for all \( k \) so that

\[
t_\infty = \lim_{i \to \infty} t_k
\]

exists. It follows from the construction that \( \phi(t_\infty, s) = z \).

This yields the following

**Theorem 7.** If \( E_0 \) has an attracting cycle, then there exists a brush \( MB \subset \mathbb{R} \times \Sigma' \) and a continuous map \( \phi : MB \to J(E_0) \) such that \( \phi_{|MB} \) is a homeomorphism.
Given an allowable deaugmented itinerary $s = s_0s_1\ldots$, we define numbers $x_j^k$ and the boxes $B_j^k$ as follows:

1. Let $x_j^0 = p_{s_j}$, $B_j^0 = D(p_{s_j}, s_j)$ for all $j$.
2. Assume that $B_j^k$ has been defined for all $l \leq k$ and for all $j$.

Then we choose

$$x_j^{k+1} = \sup_x \{ E_j^k(D(x, s_j)) \supset B_j^{k+1} \}.$$ 

and define $B_j^{k+1} = D(x_j^{k+1}, s_j)$. In short, $B_j^{k+1}$ is the rightmost box in $\mathcal{H}_{s_j}$ whose image covers $B_j^{k+1}$.

It is clear from the construction that the sequence $\{x_j^k\}_{k=0}^{\infty}$ is monotonically increasing. The following lemma shows that the sequence converges to a point $x_j^\infty$, and that the corresponding boxes $B_j^\infty = D(x_j^\infty, s_j)$ can be used to define the endpoint $z_s$ of hairs with deaugmented itinerary $s$.

**Lemma 6.2.** If $s$ is an allowable itinerary then

$$x_j^\infty = \lim_{k \to \infty} x_j^k,$$

exists. Moreover, if we let $B_j^\infty = D(x_j^\infty, s_j)$ then

$$z_s = \{ z \in \mathbb{C} | E_j^k(z) \subset B_j^\infty \text{ for all } j \}$$

consists of one point and $z_s = \phi(x_s, s)$.

**Proof:** $z_s$ is a point that depends only on the deaugmentation of a sequence $s$. Let the point $z_s$ be defined as in the previous section so that the sequence of boxes $D(z_s, s_j)$ have the property that $E_j^k(z_s) \subset D(z_s, s_j)$ for all $j \geq 0$.

Since $x_j^0 = p_{s_j}$ clearly $x_j^0 \leq z_s$. By the construction given in Definition 5.2 it follows that $x_j^{k-1} \leq z_s$ for all $0 \leq k \leq j$. Since this argument holds for all $j$ and since the sequence $\{x_j^k\}_{k=0}^{\infty}$ is monotone, it follows that

$$x_j^\infty = \lim_{k \to \infty} x_j^k \leq z_s.$$

Since $x_0^\infty \leq z_s$ it follows from Proposition 5.2 and the definition of $z_s$ that $(z_s, s) \sim (x_s^\infty, s)$. As shown in the previous section, this implies that $\phi(x_s^\infty, s) = \phi(z_s, s)$. By definition $D(x_s^\infty, s_j) \subset B_j^\infty$ which implies equality (2).

This Lemma provides another way of finding the endpoint of the hair with itinerary $s$. In contrast to the previous construction, in the present case the endpoint is approached from the right. As a special case of Theorem 6.1, we now prove:

**Theorem 6.3.** If $E_\lambda$ has an attracting $n$-cycle $z_0, z_1, \ldots, z_{n-1}$ and kneading sequence $0k_1k_2\ldots k_{n-2}k_1$ then, the endpoint $z_s$ of hairs with deaugmented itinerary $s$ is accessible from $C_0$ iff $s$ is allowable and of the form

$$s = t_00k_1k_2\ldots k_{n-2}t_00k_1k_2\ldots k_{n-2}t_2\ldots$$

with $t_i \in \mathbb{Z}$ for all $i$.

**Proof:** Assume that $s$ does not have the assumed form, and that there exists a path $\gamma : [0, \infty) \to C_0$ such that $\gamma(0) = z_0$ and $\lim_{t \to \infty} \gamma(t) = z_s$.

Therefore there exist $j$ such that $0 \leq l \leq n - 2$ such that $E_{\lambda^{j+l}}(z_0) \in \mathcal{H}_{s_j+l}$ and $E_{\lambda^{j+l}}(z_0) \in \mathcal{H}_l \neq \mathcal{H}_{s_j+l}$ in other words the two
Theorem 2: Two points in the construction with boundary...
We define

$$T^j_i = \bigcup_{x_0, y_0 \leq t^j_i} D(x_0, y_0),$$

so that $T^j_i$ is the box $B^j_i$ and anything to its left, in $\mathcal{H}_t$. Next we show that $E^{(i)}(\gamma_i) \subset T^i_j$.

By construction $\gamma_i$ consists of two pieces. The first piece runs from $E^{(i)}_i(w_{i-1})$ to the inner boundary of the annulus $E^{(i)}_i(B_{i-1})$, along the boundary of $C_n$, while the second continues to the point $w_{i}$ inside $E^{(i)}_i(B^{i-1}_{i-1})$.

By Condition $v$ of Definition 5.1 given in the introduction to this section, the point $E^{(i)}_i(w_{i-1}) \in T^{n-1}_{n-1}$, so that the preimage of the first piece of $\gamma_i$ under $L_{n-1,n}$ is a subset of the boundary of $T^{n-1}_{n-1}$. On the other hand, the second piece of $\gamma_i$ is chosen so that its preimage under $L_{n-1,n}$ is contained inside $T^{n-1}_{n-1}$. Since by construction $E^{(i)}_i(T^{n+1}_i) \subset T^{i+1}_j$, it follows that $E^{(i)}_i(\gamma_i) \subset T_{n-1}^j$ for all $0 \leq j \leq h-1$.

Let $\{v_i\} \to \infty$ be any sequence such that $v_i \in [i, l+1]$. From the arguments in the preceding paragraph it follows that $\gamma(v_i) \in T_{n-1}^i \subset T_{n-1}^i$ and $E^{(i)}_i(\gamma(v)) \in T_{n-1}^i$ for all $0 \leq j \leq h-1$. Therefore any convergent subsequence of the sequence $\{\gamma(v_i)\}$ must converge to a point $z$ such that $E^{(i)}_i(z) \in T_{n-1}^i$. By Lemma 6.2 and the previous section, the only point in $T_{n-1}^i$ satisfying this condition is $z_{n-1}$. It follows that $\gamma$ is a path in the stable set of $E_i$ such that $\gamma(0) = z_0$ and $\lim_{i \to \infty} \gamma(t) = z_{n-1}$ which proves the theorem.

We can use the same approach to prove the following:

**Corollary 6.4.** Under the assumptions of the previous theorem $z_n$ is accessible from $C_i$ if and only if $s$ is allowable and of the form

$$s = k_1 \ldots k_{n-2}t_10^2k_2 \ldots k_{n-2}t_2 \ldots$$

with $t_i \in Z$ for all $i$.

The proof of Theorem 6.1 follows similarly.

7. An Example. In this final section we give an example that illustrates why certain endpoints are not accessible. Suppose $\lambda$ is chosen so that $E_{\lambda}$ has an attracting 2-cycle. This occurs, for example, if $\lambda < -e$ (see [10]). Then the kneading sequence is simply $0^\infty$. So Theorem 6.1 states that the accessible sequences assume the form $u_1 \ldots u_{n-2}0^2v_1 \ldots v_{n-2}t_2 \ldots$ In particular, the constant sequence $T_1 = 111 \ldots$ is not accessible. Here is the idea behind this way is true.

Our previous results show that there are a pair of curves $h_1$ (resp. $h_2$) in $J(E_{\lambda})$ corresponding to the augmented itineraries $0^20$ (resp. $00^2$). These curves lie on opposite sides of $C_1$ in $\mathcal{H}_n$, with $h_1$ below $C_1$, both $h_1$ and $h_2$ terminate at the fixed point in $\mathcal{H}_n - C_1$ (see [6]).

There is also a curve $w$ that lies in $J(E_{\lambda}) \cap \mathcal{H}_1$ and terminates at the fixed point in $\mathcal{H}_1$. The itinerary of $w$ is $T$. We will show that certain preimages of $h_1 \cup h_2$ nest down on $w$, effectively preventing the endpoint of this curve from being accessible.

Consider a vertical line segment $J$ in the far right half plane that connects the upper and lower boundaries of $\mathcal{H}_1$. This segment meets $w$ in a unique point, provided that $J$ is far enough to the right. The image of $J$ is an arc of a circle centered at 0 that misses only $C_1$ (see Fig 6). The image $E_{\lambda}(J)$ meets both $h_1$ and
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