Rational maps with generalized Sierpinski gasket Julia sets

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Abstract

We study a family of rational maps acting on the Riemann sphere with a single preperiodic critical orbit. Using a generalization of the well-known Sierpinski gasket, we provide a complete topological description of their Julia sets. In addition, we present a combinatorial algorithm that allows us to show when two such Julia sets are not topologically equivalent.

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1. Introduction

In this paper we study a class of postcritically finite rational maps whose Julia sets are given by generalized Sierpinski gaskets. Briefly, a generalized gasket is the limit set obtained by a similar recursive process defined for the Sierpinski triangle, but applied instead to the closed unit disk as starting set and by removing polygons of \( N \) sides.

The class of rational maps considered here have the form:

\[ R_{\lambda,n,m}(z) = z^n + \frac{\lambda}{z^m}, \]

with \( n \geq 2, m \geq 1 \) and \( \lambda \in \mathbb{C} \). This collection of rational maps is special in the sense that each possesses a single critical orbit up to symmetry. Indeed, each map in the family \( F_{\lambda} = R_{\lambda,2,2} \) possesses four critical points (excluding the superattracting fixed point at \( \infty \) and the pole at 0), but each of them lands on the same orbit after the second iteration. For the family \( G_{\lambda} = R_{\lambda,2,1} \), there are three free critical points but the behavior of one of the critical orbits determines the behavior of the other two by symmetry.
The Julia set of a rational map $f$ is the set of points at which the family of iterates fails to be a normal family in the sense of Montel. We denote the Julia set by $J(f)$. In Fig. 1 we display the Julia set of $G_{\lambda}$ when $\lambda \approx -0.59257$ and $-0.03804 + 0.42622i$. In the first case, the Julia set is homeomorphic to the usual Sierpinski triangle whereas, in the second case, note that the removed regions have different configurations in terms of how their vertices lie along the outer boundary of the Julia set. In Fig. 2 we display a similar phenomenon for Julia sets drawn from the family $F_{\lambda}$. Again note how the removed “squares” assume different configurations in terms of how their vertices meet the outer boundary.

In this paper we shall consider the special case where the critical orbits for the maps $R_{\lambda,n,m}$ are all strictly preperiodic to a repelling cycle. Such maps are often called Misiurewicz rational maps. Our main goal is to show that, when these critical orbits also lie on the boundary of the basin of $\infty$, the Julia set of any such map is a generalized Sierpinski gasket. In addition, we provide an algorithm to determine when two such Julia sets are topologically distinct (excluding the obvious symmetric cases).

For simplicity, we restrict most of our analysis to the degree four family $F_{\lambda}(z) = z^2 + \frac{\lambda}{z^2}$. Section 2 consists of basic definitions, a review of the fundamental results for this family and the main assumptions. In Section 3 we show that if $\lambda$ satisfies certain conditions, then the associated Julia set of $F_{\lambda}$ is a generalized gasket. Some other technical results are included in this section. Section 4 contains the description of the algorithm and proofs of the main results. The general case for families of higher order and further remarks appear in Section 5.
2. Preliminaries

2.1. Gasket-like sets

Recall that the familiar Sierpinski gasket (sometimes called the Sierpinski triangle) is obtained by the following iterative process. Starting with a triangle in the plane, remove the open middle triangular region, leaving three congruent triangular regions behind. Then remove the middle triangular regions from each of these remaining triangles, leaving nine triangular regions behind. When this process is carried to the limit, the resulting set is the Sierpinski gasket.

A generalized Sierpinski gasket is obtained by a similar process performed on the closed unit disk \( \Lambda \) and removing homeomorphic copies of a polygon of \( N \) sides with straight edges. Let \( P \) denote the interior of such homeomorphic copy, so the boundary of \( P \) is a simple closed curve with \( N \) distinguished points (or corners) that correspond to the vertices of the original polygon.

At the first stage of the construction, we remove from \( \Lambda \) the region \( P \) having only its corners lying in the boundary of \( \Lambda \). Thus we are left with a connected set composed by the union of \( N \) homeomorphic copies of \( \Lambda \) which we denote by \( \Lambda_1, \ldots, \Lambda_N \). At the second stage, we remove from each \( \Lambda_k \) a smaller copy of \( P \) with only its corners lying in the boundary of \( \Lambda_k \), so we are left with \( N^2 \) copies of \( \Lambda \). If we continue this process to the limit, we obtain a set \( X \) which is compact, connected and locally connected. To complete the definition of a generalized gasket, we add the following conditions to the above construction. See [4] for related constructions.

Definition 2.1. A set \( X \) is a generalized Sierpinski gasket if it is obtained by the recursive process described above in such a way that

1. \( X \) has \( N \)-fold symmetry, and
2. from the second stage and onward, \( m \) corners of a removed region \( P \) lie in the boundary of one of the removed regions in the previous stage, with \( 1 \leq m < N \).

For later use, we call condition 2 above the \( m \)-corners condition.

2.2. Basic properties of \( F_\lambda \)

For clarity, we shall deal mainly with the family

\[ F_\lambda(z) = z^2 + \lambda/z^2 \]

for the remainder of this paper. The results below are easily modified to apply to the family \( G_\lambda(z) = z^2 + \lambda/z \) and, in general, to \( R_{\lambda,n,m} \).

First note that \( F_\lambda(-z) = F_\lambda(z) \) and \( F_\lambda(i z) = -F_\lambda(z) \). It follows that \( J(F_\lambda) \) is symmetric under \( z \mapsto i z \). Also, we have

\[ F_\lambda^*(\bar{z}) = \overline{F_\lambda(z)}. \]

Hence \( J(F_\lambda) \) is homeomorphic to \( J(F_\lambda^*) \) and so we restrict from now on to the case where the imaginary part of \( \lambda \) is nonnegative.

Note that \( F_\lambda \) has critical points at the fourth roots of \( \lambda \). We denote by \( c_1 \) the critical point of \( F_\lambda \) that lies in the first quadrant and set \( c_2 = i c_1, c_3 = -c_1, \) and \( c_0 = -i c_1 \).

There are two other symmetries for \( F_\lambda \), namely the involutions \( H_\lambda^\pm(z) = \pm \sqrt{\lambda}/z \). We have \( F_\lambda(H_\lambda^\pm(z)) = F_\lambda(z) \) for all \( z \). Note that each of these involutions fixes two of the critical points and reflects the plane through the circle of radius \( |\lambda|^{1/4} \) centered at the origin.

The following facts are known about \( J(F_\lambda) \) (see [1]):

1. The point at infinity is a superattracting fixed point for \( F_\lambda \) and \( F_\lambda \) is conjugate in a neighborhood of \( \infty \) to \( z \mapsto z^2 \). Let \( B_\lambda \) denote the immediate basin of attraction of \( \infty \).
2. If the critical points lie in \( B_\lambda \), then \( J(F_\lambda) \) is a Cantor set; otherwise, \( J(F_\lambda) \) is a connected set. In particular, in the Misiurewicz case, \( J(F_\lambda) \) is locally connected.
In the connected Julia set case, there is an open neighborhood of 0 that is mapped in two-to-one fashion onto $B_\lambda$. This region is called the trap door and is denoted by $T_\lambda$. We have $T_\lambda \cap B_\lambda = \emptyset$.

We denote the boundary of $B_\lambda$ by $\beta_\lambda$ and the boundary of $T_\lambda$ by $\tau_\lambda$. Note that the involution $H_\pm$ interchanges $B_\lambda$ and $T_\lambda$ as well as their boundaries. For the remainder of this paper, we work with the following type of maps.

**Definition 2.2.** If the critical points of $F_\lambda$ satisfy the following conditions,

1. each critical point lies in $\beta_\lambda$, and
2. each of the critical points are strictly preperiodic,

then we say that $F_\lambda$ is a Misiurewicz–Sierpinski map, or, a little more succinctly, an MS map.

Note that the symmetry $H_\pm$ implies that the critical points of an MS map also lie in $\tau_\lambda$. In general, $R_{\lambda,n,m}$ is an MS map if its $n + m$ finite, nonzero critical points satisfy the above conditions.

In Fig. 3 we display the parameter plane for the family $F_\lambda$. The unbounded region consists of those parameter values for which the Julia set is a Cantor set; its complement represents the locus of connectedness of the family $F_\lambda$, which we denote by $D$. If a parameter value is drawn from any of the small copies of the Mandelbrot set in $D$ (regions in black), then the critical orbit is bounded. On the other hand, if $\lambda$ belongs to any of the white bounded regions, then the critical orbit escapes to infinity and the Julia set is homeomorphic to the Sierpinski carpet, [2,5]. We call the white regions Sierpinski holes.

Let $M$ denote the subset of parameter values associated to MS maps. In general, Misiurewicz parameters form a dense subset in the unstable locus of families of rational-like maps, [8]. For the family $F_\lambda$ (and in general, for every family $R_{\lambda,n,m}$), the bifurcation locus contains not only the boundary of $D$, but also every boundary component of the Sierpinski holes and boundary points of the black regions. We claim that the set $M$ is a dense subset in the boundary of the connectedness locus $D$.

3. **Homeomorphisms of MS maps**

3.1. **Topological description of $J(F_\lambda)$**

Our main goal here is to prove the following theorem.
Theorem 3.1. If $F_\lambda$ is an MS map, then $J(F_\lambda)$ is a generalized Sierpinski gasket.

We begin with the following result proved in [2]:

**Theorem.** If the critical points of $F_\lambda$ are preperiodic, then $\beta_\lambda$ is a simple closed curve. If $\beta_\lambda \cap \tau_\lambda$ is nonempty, then the critical points of $F_\lambda$ are the only four points in this intersection.

We call the critical points the **corners** of the trap door. The four corners separate $\tau_\lambda$ into four **edges**. Using the fact that $F_\lambda$ is conjugate to $z \mapsto z^2$ in $B_\lambda$, we may construct four disjoint smooth curves, $\gamma_j$ for $j = 0, 1, 2, 3$, connecting $c_j$ to $\infty$ in $B_\lambda$. Let $v_j$ denote the image of $\gamma_j$ under the involution that fixes $c_j$. Then the curve $\eta_j = \gamma_j \cup v_j$ connects 0 to $\infty$ and meets $J(F_\lambda)$ only at $c_j$. Moreover, the $\eta_j$ are pairwise disjoint (except at 0 and $\infty$). Hence these four curves divide the Julia set into four symmetric pieces $I_0, \ldots, I_3$ where we assume that $c_j \in I_j$ but $c_j$ does not lie in the other three regions. Hence the $I_j$ are neither open nor closed subsets of $J(F_\lambda)$. Later on we will use the $I_j$ to describe the symbolic dynamics generated by $F_\lambda$ on its Julia set.

For $j = 0, 1, 2, 3$, let $I_j$ denote the connected component of the Julia set that lies in one of the four “quadrants” defined by the corners of $\tau_\lambda$ and the four curves $\eta_j$ (see Fig. 4). Let $I_0$ be the component that contains the repelling fixed point $p_\lambda$, which lies in $\beta_\lambda$.

Since there are no critical points in any of the preimages of the trap door, it follows that each of its preimages is mapped in one-to-one fashion onto the trap door by $F_\lambda$. Hence each component of $F^{-k}_\lambda(\tau_\lambda)$ also has four corners and edges, and each of these corners is mapped by $F^{-k}_\lambda$ onto a distinct critical point in $\tau_\lambda$.

We may now show that the Julia set of an MS map is a gasket-like set. Let $K_0 = \mathbb{C} \setminus B_\lambda$ and $K_1 = K_0 \setminus T_\lambda$. Then $K_1 = I_0 \cup I_1 \cup I_2 \cup I_3$. Let $K_{n+1} = K_n \setminus F^{-k}_\lambda(T_\lambda)$. It follows that each $K_n$ is a nested collection of closed and connected subsets of the Riemann sphere with exactly $4^n$ homeomorphic copies of a rectangle removed at each $n$th step [7]. Moreover,

$$J(F_\lambda) = \bigcap_{n=0}^{\infty} K_n,$$

so $J(F_\lambda)$ is a compact and connected set with 4-fold symmetry. Local connectivity follows from subhyperbolicity of the map. To see that the Julia set satisfies the $m$-corners condition, we have the following result.

**Lemma 3.2.** Let $\tau_\lambda^k$ be the union of all of the components of $F^{-k}_\lambda(\tau_\lambda)$ and let $A$ be a particular component in $\tau_\lambda^k$ with $k \geq 1$. Then exactly two of the corner points of $A$ lie in a particular edge of a single component of $\tau_\lambda^{k-1}$.
Proof. The case $k = 1$ is seen as follows. We have that $F_\lambda$ maps each $I_j$ for $j = 0, \ldots, 3$ in one-to-one fashion onto all of $J(F_\lambda)$, with $F_\lambda(I_j \cap \beta_\lambda)$ mapped onto one of the two halves of $\beta_\lambda$ lying between two critical values (which, by assumption, are not equal to any of the critical points). Hence $F_\lambda(I_j \cap \beta_\lambda)$ contains exactly two critical points. Similarly, $F_\lambda(I_j \cap \tau_\lambda)$ maps onto the other half of $\beta_\lambda$ and so also meets two critical points. The preimages of these latter two critical points in $\tau_\lambda$ are precisely the corners of the component of $\tau^1_\lambda$ that lies in $I_j$. Thus we see that each component in $\tau^1_\lambda$ meets the boundary of one of the $I_j$’s in two points lying in $\beta_\lambda$ and two points lying in $\tau_\lambda$. In particular, two of the corners lie in the edge of $\tau_\lambda$ that meets $I_j$.

Now consider a component in $\tau^k_\lambda$ with $k > 1$. $F^k_\lambda$ maps each component in $\tau^k_\lambda$ onto $\tau_\lambda$ and therefore $F^{k-1}_\lambda$ maps the components in $\tau^k_\lambda$ onto one of the four components of $\tau^1_\lambda$. Since each of these four components meets a particular edge of $\tau_\lambda$ in exactly two corner points, it follows that each component of $\tau^k_\lambda$ meets an edge of one of the components of $\tau^{k-1}_\lambda$ in exactly two corner points as claimed.

As a consequence of the above lemma, we display in Fig. 5 a schematic representation of $\beta_\lambda$, $\tau_\lambda$, and $\tau^1_\lambda$ which holds true for any MS $\lambda$-value.

3.2. Local cut points of $J(F_\lambda)$

The next Theorem provides a topological characterization of the critical points and it will allow us to show that any homeomorphism between two Julia sets of MS maps must send critical points to critical points.

**Theorem 3.3.** The four corners of the trap door are the only set of four points in the Julia set whose removal disconnects $J(F_\lambda)$ into exactly four components. Any other set of four points removed from $J(F_\lambda)$ will yield at most three components.

The proof of this result will follow from Propositions 3.9–3.11. First we introduce several definitions.

**Definition 3.4.** Define the 0-disk to be the whole Julia set $J(F_\lambda)$ and let $\tau^0_\lambda = \tau_\lambda$ denote the boundary of the trap door $T_\lambda$. For any $k \geq 1$, we define a $k$-disk to be a compact and connected subset of a $(k - 1)$-disk such that

- the $k$-disk is mapped in one-to-one fashion onto $J(F_\lambda)$ by $F^k_\lambda$,
- if $A$ is the component of $\tau^{k-1}_\lambda$ that lies in the $(k - 1)$-disk, then two adjacent corner points of $A$ lie in the $k$-disk, and
- a component $B$ of $\tau^k_\lambda$ lies completely in the $k$-disk.
It follows that each $k$-disk is the union of four $(k + 1)$-disks. Moreover, Lemma 3.2 implies that the outside boundary of any $(k + 1)$-disk must contain an edge and two corner points of a single component $A$ in $\tau_{\lambda}^{k}$. Through the rest of this section $D_k$ will denote a $k$-disk.

**Definition 3.5.** The skeleton of the Julia set, denoted by $\mathbb{J}$, is defined by

$$\mathbb{J} := \beta_{\lambda} \cup \bigcup_{k \geq 0} \tau_{\lambda}^{k}.$$ 

Note that $\mathbb{J}$ is an arcwise connected subset of the Julia set whose closure is the whole Julia set. Moreover, a path connecting two points lying in different 1-disks must pass by at least one critical point. So in this sense, the critical points are local cut points of $\mathbb{J}$. Clearly, the skeleton of any $k$-disk is also an arcwise connected set given by $D_{k} := \mathbb{J} \cap D_{k}$ and the corner points of the component $A \subset \tau_{\lambda}^{k}$ that lies in $D_{k}$ are local cut points of the $k$-disk.

Clearly, the Julia set of an MS map does not contain parabolic periodic points or recurrent critical points. Indeed, the $\omega$-limit set of the critical points $c_{\lambda}$ is a repelling periodic cycle that lies in $\beta_{\lambda}$, and by assumption, the critical points are disjoint from the limit set. Then we may apply the following result due to Mañé (see [6]).

**Theorem.** Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a rational map. If a point $z \in J(f)$ is not a parabolic periodic point and is not contained in the $\omega$-limit set of a recurrent critical point, then for all $\varepsilon > 0$ there exists a neighborhood $U$ of $z$ such that for all $n > 0$, every component $V$ of $f^{-n}(U)$ satisfies $\text{diam } V \leq \varepsilon$.

**Proposition 3.6.** $\text{diam } D_{k} \rightarrow 0$ as $k \rightarrow \infty$.

**Proof.** Let $z \in J(F_{\lambda})$, $\varepsilon > 0$, $U$ and $V$ as above. By Montel’s theorem, there exists an $N \gg 1$ so $J(F_{\lambda}) \subset F_{\lambda}^{N}(U)$. Then $U$ contains at least one $N$-disk and consequently, for all $j > 0$,

$$\text{diam } D_{N+j} \leq \text{diam } V \leq \varepsilon.$$  

As a consequence, we have

**Corollary 3.7.** For any two points in the Julia set there exists an integer $k > 0$ such that each point lies in distinct $k$-disks.

Now, let $z$ be any point in $\beta_{\lambda}$ and for any small $\varepsilon > 0$, let $l_{e}$ denote an arc of $\beta_{\lambda}$ that contains $z$ and has length $\varepsilon$. By the invariance of $\beta_{\lambda}$ and conjugacy of $F_{\lambda}$ with $z \mapsto z^{2}$, there exists an integer $N > 0$ such that $\beta_{\lambda} \subset F_{\lambda}^{N}(l_{e})$. This implies that $l_{e}$ contains corner points of components of $\tau_{\lambda}^{N+j}$ for $j \geq 0$. We can easily extend this result to components in $\tau_{\lambda}^{k}$, so we have proved,

**Lemma 3.8.** Any point in $\mathbb{J}$ is an accumulation point of corner points.

The proof of Theorem 3.3 follows by combining the next three propositions.

**Proposition 3.9.** Let $\{p_{1}, \ldots, p_{n}\}$ be a finite collection of non-corner points in $J(F_{\lambda})$. Then $J(F_{\lambda}) \setminus \{p_{0}, \ldots, p_{n}\}$ is connected.

**Proof.** Assume first that each $p_{i}$ is a point in the skeleton $\mathbb{J}$. Then each $p_{i}$ lies in an edge $l_{i}$ of some component of $\tau_{\lambda}^{n_{i}}$. By Lemma 3.8, any small neighborhood of $p_{i}$ contains corner points of higher order accumulating on $p_{i}$, and in particular, along its edge. This implies that $l_{i}$ is pathwise connected to infinitely many $k$-disks of higher order that, in turn, are pathwise connected to lower level disks all the way to $\beta_{\lambda}$.

Assume now each $p_{i}$ is not in $\mathbb{J}$. Let $N$ and $M$ be two open sets in the relative topology of $J(F_{\lambda})$ so $J(F_{\lambda}) \setminus \{p_{0}, \ldots, p_{n}\} = N \cup M$ and $N \cap M = \emptyset$. Without loss of generality assume $\mathbb{J}$ is contained in $N$. If $M$ is not empty, it must contain at least an accumulation point of $\mathbb{J}$. But arbitrarily close to such a point, there must exist points that
Proof. Since $J(F_\lambda)$ belong to $J$, this implies a non-empty intersection of $M$ and $N$, yielding a contradiction. Thus, $M$ is empty and $J(F_\lambda) - \{p_0, \ldots, p_n\}$ is connected.

The general statement now follows easily. 

Next we show that removing the critical points from $J(F_\lambda)$ yields exactly four disjoint components.

**Proposition 3.10.** Let $G$ be the set obtained by removing the two critical points lying in a 1-disk in the Julia set. Let $G = G \cap J$. Then $G$ is a connected set and $G$ is arcwise connected.

**Proof.** Since $F_\lambda$ is a MS map, all the corners of every preimage of the trap door are distinct from the critical points that were removed to obtain $G$. Hence the arcs of $\beta_\lambda$ and $\tau_\lambda$ that lie in $G$ are arcwise connected by the edges of the component of $\tau_\lambda^1$ that lies in $G$. The same is true for any other component: if $A$ is a component of any $\tau_\lambda^i$ then, from Lemma 3.2, $A$ is arcwise connected to a component of $\tau_\lambda^{i-1}$. Similarly, this component is arcwise connected to a component of $\tau_\lambda^{i-2}$ and so on. It follows that $A$ must be arcwise connected to $\tau_\lambda$ and $\beta_\lambda$ and hence $G$ is the largest arcwise connected subset in $G$. Similar arguments as in the previous proposition show that $G$ is connected. 

The proof of Theorem 3.3 reduces now to consider only $J$ and the removal of corner points that are not the critical points.

**Proposition 3.11.** The removal of four corner points (not all critical points) from $J$ results in at most three components.

**Proof.** To establish the claim, we consider the remaining four possible cases, namely when the quartet of points contain one, two, three or four non-critical corner points.

**One non-critical point:** Assume without loss of generality that we remove the critical points $c_0, c_1$ and $c_2$ from $J$. This yields three connected components, $G_1$ and $G_2$ (containing no critical points) and $G_3$ (containing $c_3$). But since $G_3$ is the union of two 1-disks, then $c_3$ is the only local cut point of $G_3$, so we are done.

**Two non-critical points:** Here we may remove two adjacent or non-adjacent critical points. Removing two non-adjacent critical points gives two components with local cut points at the remaining critical points, so again we are done. Assume then that $c_0$ and $c_1$ are two adjacent critical points removed from $J$. This yields two disjoint components $G_1$ (with no critical points) and $G_2$ (containing $c_2$ and $c_3$). From the previous case, we only need to consider removing $p$ and $q$ from either $G_1$. By Corollary 3.7, $p$ and $q$ have to be corner points of some $n$-disk and some $m$-disk respectively. Clearly, if $n > m$ then $G_1$ remains connected. If $n = m$, then $D_{m} - \{p, q\}$ disconnects into two disjoint components. By the structure of $k$-disks, at least one of these components contains a lower level corner in its boundary, hence it is arcwise connected to $G_1$. Thus $G_1 - \{p, q\}$ becomes at most two disjoint components and $G - \{p, q, c_0, c_1\}$ has at most three components.

**Three non-critical points:** Assume without loss of generality that we remove $c_1$ from $J$. This yields a single component with four 1-disks (up to a point). From the previous cases, we only need to consider removing the three remaining non-critical points $p, q$ and $r$ from a single 1-disk. Assume all three points lie in the same $k$-disk, since the other cases are trivial. Then $D_k - \{p, q, r\}$ becomes three disjoint components and one of them must contain a lower level corner. This yields again three components.

**Four non-critical points:** Assume $p_1, p_2, p_3$ and $p_4$ are non-critical points. We consider the general case of removing the four points from the same $k$-disk (here $k \geq 0$). In the worst case scenario, all the points are corner points of a single $n$-disk in $D_k$. Then $D_n - \{p_1, \ldots, p_4\}$ yields four disjoint components with at least two of them containing lower level corners in their outside boundaries and hence, they must be connected to $\beta_\lambda$, yielding at most three components. The other cases are trivial. 

3.3. **Homeomorphisms between Julia sets**

We are now able to state a key result.

**Proposition 3.12.** Suppose $F_\lambda$ and $F_\mu$ are MS maps. If there exists an orientation preserving homeomorphism $h : J(F_\lambda) \to J(F_\mu)$, then
(1) For each \( k \geq 0 \), \( h \) maps the corners of \( F^{-k}_\lambda(\tau_\lambda) \) to the corners of \( F^{-k}_\mu(\tau_\mu) \).
(2) For \( k \geq 1 \), each component of \( F^{-k}_\lambda(\tau_\lambda) \) is mapped to a unique component of \( F^{-k}_\mu(\tau_\mu) \).

**Proof.** Theorem 3.3 establishes that the removal of the corners of \( \tau_\lambda \) disconnects \( J(F_\lambda) \) into exactly four components and no other component of \( \tau^k_\lambda \) for \( k \geq 1 \) has this property.

Hence, the homeomorphism \( h \) cannot take \( \tau_\lambda \) to some component of \( \tau^k_\mu \) when \( k \geq 1 \), since \( \tau_\lambda \) and \( \tau_\mu \) are the only components of all the preimages of the respective trap door whose corner-removal separates their respective Julia set into four disjoint regions. As a consequence, we have that \( h \) maps each of the pieces \( I_0, \ldots, I_3 \) of \( J(F_\lambda) \) to one of the corresponding \( I_j \)'s in \( J(F_\mu) \). As in the proof of Theorem 3.3, the only set of points in the components of \( \tau^k_\lambda \) for \( k \geq 1 \) that may separate one of the \( I_{s_j} \) (that is, a 1-disk) into four pieces is the set of corner points of the four components of \( \tau^1_\lambda \), and each of these components lies in a distinct \( I_{s_j} \). Hence \( h \) maps each of the four preimages of \( \tau_\lambda \) to a distinct preimage of \( \tau_\mu \) under \( F^{-1}_\mu \). Continuing inductively, we see that \( h \) must map each component of \( \tau^k_\lambda \) to a distinct component of \( \tau^k_\mu \). \( \square \)

4. Model

Recall that \( \mathcal{M} \) denotes the set of parameter values corresponding to MS maps. Our goal in this section is to construct a geometric model for \( J(F_\lambda) \) for each \( \lambda \in \mathcal{M} \).

For each \( \lambda \in \mathcal{M} \) define a partition of the Julia set \( J(F_\lambda) \) in four half-open regions \( I_0, I_1, I_2 \) and \( I_3 \) as defined in Section 2. Each point \( z \) in the Julia set has an address \( s_0s_1s_2 \ldots \in \{0, 1, 2, 3\}^\mathbb{N} \) defined in the natural way by its orbit in the regions \( I_k \). For example, the fixed point contained in \( I_0 \) has itinerary \( \hat{0} \); the preimage of the fixed point lying in \( I_2 \) has itinerary \( 20 \) and so on.

Recall that the map \( F_\lambda \) satisfies the following symmetries

\[
F_\lambda(-z) = F_\lambda(z), \quad F_\lambda(iz) = F_\lambda(-iz) = -F_\lambda(z),
\]

so if \( c_1 = \lambda^{1/4} \), then \( c_0 = -ic_1 \), \( c_2 = ic_1 \) and \( c_3 = -c_1 \). If \( S(z) = s_0s_1s_2 \ldots \) gives the address of the point \( z \) in the Julia set, then

\[
S(-z) = (s_0 + 2)s_1s_2 \ldots,
\]
\[
S(iz) = (s_0 + 1)(s_1 + 2)s_2 \ldots,
\]
\[
S(-iz) = (s_0 - 1)(s_1 + 2)s_2 \ldots,
\]

where addition is taken mod 4, and

\[
S(F_\lambda(-z)) = s_1s_2 \ldots,
\]
\[
S(F_\lambda(iz)) = S(F_\lambda(-iz)) = (s_1 + 2)s_2 \ldots.
\]

As an example, assume \( \lambda \approx -0.36428 \). Then the address of \( c_1 \) is 1120 and \( S(c_2) = 2320 \), \( S(c_3) = 3120 \) and \( S(c_4) = 0320 \). On the other hand, for \( \lambda \approx -0.01965 + i0.2754 \), \( S(c_1) = 11120 \) and thus \( S(c_2) = 23120 \), \( S(c_3) = 31120 \) and \( S(c_4) = 03120 \). The Julia sets of these examples are shown in Fig. 2.

For each \( \lambda \in \mathcal{M} \), \( k \geq 2 \), we construct a homeomorphism between the skeleton \( \bar{J}(F_\lambda) \) and a so-called model \( M_k(F_\lambda) = M(F_\lambda, k) \) in the following way. Since \( \beta_\lambda \) is a simple closed curve and \( F_\lambda \) is conjugate to \( z \mapsto z^2 \) on \( \beta_\lambda \), there exists a homeomorphism \( h_0 \) between \( J(F_\lambda) \) and a set \( M_0(F_\lambda) \) such that

\[
h_0(\beta_\lambda) = S^1 \subset M_0(F_\lambda)
\]

and \( h_0 \) is a conjugacy between \( F \) restricted to \( \beta_\lambda \subset \bar{J}(F_\lambda) \) and the angle doubling map \( z \mapsto z^2 \) restricted to \( S^1 \subset M_0(F_\lambda) \). We may assume that \( M_0(F_\lambda) \) satisfies the same symmetry relations as \( F_\lambda \) and that the four half-open regions \( I_0, I_1, I_2 \) and \( I_3 \) are mapped to corresponding regions in \( M_0(F_\lambda) \). We say that \( h_0 \) “straightens” \( \beta_\lambda \) to a circle with the conjugate dynamics on the circle given by angle doubling. In the next step we construct a homeomorphism \( h_1 \) between \( M_0(F_\lambda) \) and a set \( M_1(F_\lambda) \) which is the identity on \( S^1 \subset M_0(F_\lambda) \), straightens \( h_0(\tau_\lambda) \subset M_0(F_\lambda) \) to any given “nice” homeomorphic image of the (boundary of the) trap door \( \tau_\lambda \), and keeps the symmetry. Here
“nice” means that the image of $\tau_{\lambda}$ is a smooth curve except at the four critical points. Successively straightening out $\tau_{\lambda}^1, \ldots, \tau_{\lambda}^k$ with homeomorphisms $h_2, \ldots, h_{k+1}$ which do not alter the preceding changes, we get a homeomorphism

$$h = h_{k+1} \circ \cdots \circ h_0 : J(F_\lambda) \rightarrow M(F_\lambda)$$

between the skeleton $J(F_\lambda)$ and the model $M(F_\lambda) = M(F_\lambda, k)$ (we can even construct $h$ to be a conjugacy between $F_\lambda$ restricted to $\beta_{\lambda} \cup \tau_\lambda \cup \tau_{\lambda}^1 \cup \cdots \cup \tau_{\lambda}^k$ and the induced map on $S^1 \cup h(\tau_\lambda) \cup h(\tau_{\lambda}^1) \cup \cdots \cup h(\tau_{\lambda}^k)$). In pictures of $M(F_\lambda)$ we typically visualize only

$$h(\beta_{\lambda} \cup \tau_\lambda \cup \tau_{\lambda}^1 \cup \cdots \cup \tau_{\lambda}^k) = S^1 \cup h(\tau_\lambda) \cup h(\tau_{\lambda}^1) \cup \cdots \cup h(\tau_{\lambda}^k) \subset M(F_\lambda).$$

To each $z \in \beta_{\lambda} \subset J(F_\lambda)$ we assign the angle $\theta$ of $h(z) \in S^1 \subset M(F_\lambda)$

$$\theta(z) := \frac{\langle h(z) \rangle}{2\pi} \in [0, 1].$$

Note that $\theta(z)$ is well-defined, since $h|_{\beta_{\lambda}} : \beta_{\lambda} \rightarrow S^1$ is the unique orientation preserving conjugacy (up to complex conjugation) with the angle doubling map on $S^1$. Indeed, the itineraries of $z$ under $F_\lambda$ and $e^{2i\pi \theta(z)}$ under $z \mapsto z^2$ (with respect to the partition $\{h(I_0), h(I_1), h(I_2), h(I_3)\}$ of $M(F_\lambda)$) are the same. Note that, since the imaginary part of $\lambda$ is positive, $\theta(c_\lambda) \in [0, \frac{1}{2}].$

Since $F_\lambda$ is an MS map, the forward orbit of $e^{2i\pi \theta(z)}$ under $z \mapsto z^2$ does not intersect the other critical points, i.e., $\theta = \theta(c_1(\lambda))$ satisfies

$$2^j \theta \mod 1 \notin \left\{ \theta, \theta + \frac{1}{4}, \theta + \frac{1}{2}, \theta + \frac{3}{4} \right\} \quad \text{for every } j \geq 1. \quad (4.1)$$

We define $P^1 = (0, \frac{1}{3})$ and use condition $(4.1)$ to define for each $k \geq 2$ the set

$$P^k = \left\{ \theta \in \left(0, \frac{1}{8}\right) : \text{condition (4.1) holds for } 2 \leq j \leq k \right\}.$$

Note that $P^k$ is a union of open intervals $P_{i_k}^k, \ldots, P_{i_2}^k$ for some $i_k \geq 2$, and we may assume that $\sup P^k_i \leq \inf P^k_j$ if $i < j$. For example,

$$p^2 = p^1_2 \cup p_2^2 = \left(0, \frac{1}{12}\right) \cup \left(\frac{1}{12}, \frac{1}{8}\right)$$

since the only 2-periodic orbit under $z \mapsto z^2$ is $\frac{1}{4}, \frac{2}{7}$. Hence, for $(4.1)$ to be violated with $j = 2$, we get the relations

$$\theta \in \left[0, \frac{1}{8}\right],$$

$$\theta + \frac{1}{4} \in \left[\frac{1}{12}, \frac{3}{8}\right] \ni \frac{1}{3};$$

$$\theta + \frac{1}{2} \in \left[\frac{1}{8}, \frac{5}{8}\right],$$

$$\theta + \frac{3}{4} \in \left[\frac{3}{8}, \frac{7}{8}\right]$$

which yields $\theta = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ and $2^2 \theta = \theta + \frac{1}{4}$.

The only 3-periodic orbits under $z \mapsto z^2$ are $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$ and $\frac{3}{7}, \frac{6}{7}, \frac{5}{7}$, hence for $(4.1)$ to be violated with $j = 3$ we get the relations

$$\theta \in \left[0, \frac{1}{8}\right],$$

$$\theta + \frac{1}{4} \in \left[\frac{1}{4}, \frac{3}{8}\right] \ni \frac{2}{7};$$

$$\theta + \frac{1}{2} \in \left[\frac{1}{2}, \frac{5}{8}\right] \ni \frac{4}{7}.$$
\[ \theta + \frac{3}{4} \in \left[ \frac{3}{4}, \frac{7}{8} \right] \Rightarrow \frac{6}{7} \]

which yields the angles \( \frac{1}{28}, \frac{3}{28} \) with \( 2^3 \theta = \theta + \frac{1}{4}, 2^2 \theta = \theta + \frac{1}{2}, 2 \theta = \theta + \frac{3}{4} \). Hence

\[
P^3 = P^3_1 \cup \cdots \cup P^3_5 = \left( 0, \frac{1}{28} \right) \cup \left( \frac{1}{28}, \frac{1}{14} \right) \cup \left( \frac{1}{14}, \frac{1}{12} \right) \cup \left( \frac{1}{12}, \frac{3}{28} \right) \cup \left( \frac{3}{28}, \frac{1}{8} \right).
\]

Note that \( P^j \subseteq P^k \) if \( j > k \) and \( \bigcap_{j=2}^\infty P^j \) consists exactly of those angles in \( (0, \frac{1}{7}) \) which do not lie on a \( j \)-periodic orbit after \( j \)-times angle doubling for any \( j \geq 2 \). Now for each \( k \geq 2 \) the sets \( P^k_1, \ldots, P^k_k \) define a partition of the set \( \mathcal{M} \) of MS parameter values

\[
\mathcal{M} = \bigcup_{i=1}^{i_k} \mathcal{M}^k_i, \quad \mathcal{M}^k_i := \{ \lambda \in \mathcal{M} : \theta(c_1(\lambda)) \in P^k_i \}.
\]

Choose \( \lambda \in \mathcal{M}^k_1 \) and \( \mu \in \mathcal{M}^k_2 \). Now we display \( M(F_\lambda) \) as follows:

1. Draw \( h(c_0), h(c_1), h(c_2), h(c_3) \) and the (homeomorphic image of the) trap door \( h(\tau_0) \). Note that \( 0 < \theta = \theta(c_1) < \frac{1}{12} \).
2. Draw the component of \( h(\tau_1^1) \) in \( h(I_0) \), starting with the corners \( x_0, x_1, x_2 \) and \( x_3 \) (see Fig. 6). Note that, due to Lemma 3.2, two corners are in \( h(\tau_0) \), say \( x_0, x_1 \), and then \( F_\lambda(h^{-1}(x_0)) = c_2, F_\lambda(h^{-1}(x_1)) = c_3 \). The other two corners \( x_2, x_3 \) are on \( S^1 \subset M(F_\lambda) \) and as preimages under angle doubling they have the angles \( \frac{\theta}{2} \) and \( \theta + \frac{3}{4} + \frac{1}{2}(1 - (\theta + \frac{3}{4})) = \frac{\theta}{2} + \frac{7}{8} \).
3. Draw the other components of \( h(\tau_1^1) \) in \( h(I_1), h(I_2), h(I_3) \) using the symmetry.
4. Draw the components of \( h(\tau_2^1) \) in the “triangle” \( h(I_0) \) with corners \( x_0, x_3, h(c_1) \), starting with the corners \( y_0, y_1, y_2, y_3 \). Note that due to Lemma 3.2 two corners are in \( h(\tau_2^1) \cap h(I_0) \), say \( y_0, y_1 \), and \( F_\lambda(h^{-1}(y_0)) = h(ix_0), F_\lambda(h^{-1}(y_1)) = h(ix_1) \). To find the location of \( y_2, y_3 \), note that the (short) arc in \( S^1 \) from \( x_3 \) to \( h(c_1) \) maps under angle doubling to the (short) arc from \( h(c_1) = e^{2\pi i \theta} \) to \( e^{2\pi i 12\theta} \). The angle of the corner \( ix_2 \) is \( \frac{\theta}{2} + \frac{7}{8} + \frac{3}{4} \mod 1 = \frac{\theta}{2} + \frac{7}{8} \) which is larger than \( 2\theta \) (since \( \theta < \frac{1}{12} \)), showing that no preimage of \( ix_2 \) and \( ix_3 \) is contained in the arc from \( x_3 \) to \( h(c_1) \). This fact or a similar direct argument shows that \( y_2, y_3 \) lie on \( h(\tau_2) \) and \( F_\lambda(h^{-1}(y_2)) = h(ix_2), F_\lambda(h^{-1}(y_3)) = h(ix_3) \).
5. Draw the remaining three components of \( h(\tau_2^1) \) in \( h(I_0) \) using similar arguments. Draw the other components of \( h(\tau_2^1) \) in \( h(I_1), h(I_2), h(I_3) \) using the symmetry.

Fig. 6. The model \( M(F_\lambda) \) of Julia set for \( \theta(c_1(\lambda)) = 5/64 \).
Fig. 7. The topological model and the Julia set are displayed on top when $\mu \approx -0.246 + i0.15913$. The address of $c_1(\mu)$ is $112320$ and $\theta(c_1(\mu)) = 3/32 > 1/12$. The bottom images show the model and the Julia set when $\lambda \approx -0.12713 + i0.21384$. The address of $c_1(\lambda)$ is $1111320$ and $\theta(c_1(\lambda)) = 5/64 < 1/12$.

Next we display $M(F_\mu)$. For simplicity let $h$ denote again the homeomorphism. The only change is step (4) when the locations of $y_2, y_3$ are determined. We replace it with

(4') To find the location of $y_2, y_3$, again note that the (short) arc in $S^1$ from $x_3$ to $h(c_1)$ maps under angle doubling to the (short) arc from $h(c_1) = e^{2\pi i \theta}$ to $e^{2\pi i 2\theta}$. The angle of the corner $ix_2$ is $\frac{\theta}{2} + \frac{1}{8}$ which is less than $2\theta$ (since $\theta > \frac{1}{12}$), showing that the preimage of $ix_2$ is contained in the arc from $x_3$ to $h(c_1)$ and hence $y_2$ lies on $\beta_\mu$. This fact or a similar direct argument show that $y_3$ lies on $h(\tau_\mu)$.

Two examples of the models and their Julia sets corresponding to parameters $\lambda \in \mathcal{M}_1^2$ and $\mu \in \mathcal{M}_2^2$ are displayed in Fig. 7.

**Angles of bifurcation**

Before we continue showing that $J(F_\lambda)$ and $J(F_\mu)$ are not homeomorphic, we observe that our model makes sense even for the angle $\theta = \frac{1}{12}$, although the parameter associated to this angle does not correspond to a Misiurewicz parameter value. Indeed, numerical evidence shows that for this parameter there exists a parabolic cycle of period two and the critical orbit lies in the basin of attraction of the cycle. Therefore, the Julia set is not homeomorphic to our
model. Nevertheless, if we identify each Fatou component of the parabolic cycle to a point, we conjecture that the resulting set is homeomorphic to the corresponding model.

For \( \theta = \frac{1}{12} \), the point with angle \( \theta \) (which we still denote by \( h(c_1) \)) is a preimage (under angle-doubling) of \( ix_2 \) with angle \( \frac{\theta}{2} + \frac{1}{8} = 2\theta \), hence \( y_2 = h(c_1) \) (cp. with Fig. 6). We will see that \( J(F_\lambda) \) and \( J(F_\mu) \) are not homeomorphic, hence we have a (topological) bifurcation of the model at \( \theta = \frac{1}{12} \), more precisely we call \( \theta \in (0, \frac{1}{8}) \) a

**level-\( k \) bifurcation**, if \( \theta \in P^{k-1} \) and \( \theta \notin P^k \)

At every level-\( k \) bifurcation, \( k \geq 2 \), the point \( h(c_1) \) in the model (and therefore also \( h(c_0), h(c_2), h(c_3) \) by symmetry) equals a corner point in a component of \( h(\tau^k_\lambda) \) by definition of \( P^k \). Now for the model \( M(F_\lambda) \)

the points \( h(F^{-i}_\lambda((c_0, \ldots, c_3))) \cap h(\tau^i_\lambda) \) are called level-\( i \) corners

e.g. \( h(c_1) \) in Fig. 6 is a level-0 corner, and \( x_0, \ldots, x_3 \in h(\tau^1_\lambda) \cap h(I_0) \) are level-1 corners. Then at a level-2 bifurcation a 0-corner equals a 2-corner. In fact more can be said. Since each level-\( i \) corner, \( i \geq 1 \), is a preimage of a level-(\( i - 1 \)) corner, we immediately get the following lemma:

**Lemma 4.1.** At each level-\( k \) bifurcation, \( k \geq 2 \), infinitely many corners pairwise meet. For each \( i, j \in \mathbb{N}_0 \) with

\[ |i - j| = k \]

a level-\( i \) and a level-\( j \) corner coincide.

Now we can formulate the first case of our main result.

**Theorem 4.2.** \( J(F_\lambda) \) and \( J(F_\mu) \) are not homeomorphic for \( \lambda \in M^1_2 \) and \( \mu \in M^1_2 \).

**Proof.** Assume that \( H \) is a homeomorphism between \( J(F_\lambda) \) and \( J(F_\mu) \), let \( \tilde{H} = h_\mu \circ H \circ h_\lambda^{-1} \) denote the induced homeomorphism between \( M(F_\lambda) \) and \( M(F_\mu) \) with the homeomorphisms \( h_\lambda \) and \( h_\mu \) between the corresponding Julia sets and models.

\[
\begin{array}{ccc}
J(F_\lambda) & \xrightarrow{h_\lambda} & M(F_\lambda) \\
H & \downarrow & \tilde{H} \\
J(F_\mu) & \xrightarrow{h_\mu} & M(F_\mu)
\end{array}
\]

Since \( H \) maps corners of \( \tau_\lambda \) to corners of \( \tau_\mu \) by Lemma 3.2, it follows that two adjacent level-0 corners \( h_\lambda(c_i) \), \( h_\lambda(c_{i+1 \mod 4}) \) in \( M(F_\lambda) \) are mapped by \( \tilde{H} \) onto two level-0 corners in \( M(F_\mu) \) and they also have to be adjacent because otherwise a path in \( M(F_\lambda) \) connecting \( h_\lambda(c_i) \), \( h_\lambda(c_{i+1 \mod 4}) \) with no other level-0 corner on it would be mapped to a path in \( M(F_\mu) \) with at least one other level-0 corner on it, a contradiction. So far we have the following picture for \( \tilde{H} \) restricted to \( h_\lambda(\tau_\lambda \cap \beta_\lambda) \) within one of the four sectors.

\[
\begin{array}{c}
\text{0} \\
\text{0} \\
\text{0} \\
\text{0}
\end{array}
\xrightarrow{\tilde{H}}
\begin{array}{c}
\text{0} \\
\text{0} \\
\text{0} \\
\text{0}
\end{array}
\]
The 0’s indicate level-0 corners. Now Lemma 3.2 implies the following picture for the level-1 corners and their connecting paths in \( h_\lambda(\tau_1^1) \)

where the unknown homeomorphism \( \tilde{H} \) is restricted only by the fact that it maps level-\( i \) corners on level-\( i \) corners. Without loss of generality we assume that \( \tilde{H} \) maps the upper level-0 corner to the upper level-0 corner. From the construction of the model \( M(F_\lambda) \) we get

\[
\begin{array}{c}
0 \\
1 \\
2 \\
2 \\
1 \\
0
\end{array}
\xrightarrow{\tilde{H}}
\begin{array}{c}
0 \\
1 \\
2 \\
2 \\
1 \\
0
\end{array}
\]

a contradiction, since in \( M(F_\lambda) \) there exists a path from a level-0 corner to a level-1 corner with exactly two level-2 corners on it whereas the \( \tilde{H} \) image of this path contains only one level-2 corner. This proves that \( J(F_\lambda) \) and \( J(F_\mu) \) are not homeomorphic.

We now state our main result.

**Theorem 4.3.** For any two MS maps \( F_\lambda \) and \( F_\mu \) with \( \lambda \neq \mu \), their respective Julia sets are not topologically equivalent.

The existence of a conjugacy between two MS maps implies the existence of a homeomorphism of the Julia sets. Thus it follows from the above Theorem,

**Corollary 4.4.** If \( F_\lambda \) and \( F_\mu \) are two conjugated MS maps, then either \( \lambda = \mu \) or \( \lambda = \mu \).

Proving that any two MS Julia sets are not topologically equivalent reduces to combinatorics for computation of the sets \( P_i^k \) and uses the fact that for any two MS parameter values \( \lambda, \mu \in \mathcal{M} \) there exists \( k, i, j \) such that

\[ \lambda \in P_i^k \quad \text{and} \quad \mu \in P_j^k. \]

It is clear that there always exists a \( k \geq 2 \) such that \( \lambda \) and \( \mu \) are not in the same \( \mathcal{M}_i^k \), and therefore the model undergoes at least one bifurcation which changes the topology when passing from \( \lambda \) to \( \mu \). The same argument then shows that the level-\( k \) models \( M(F_\lambda, k) \) and \( M(F_\mu, k) \) are not homeomorphic.

To explain the ideas for the general case, we do it explicitly for all bifurcations up to level 3, the angles at which the bifurcations occur are displayed in the following figure from \( \frac{1}{8} \) down to 0 because the angle \( \theta(c_1(\lambda)) \) is decreasing if the real part of \( \lambda \in \mathcal{M} \) is increasing.

<table>
<thead>
<tr>
<th>Level</th>
<th>( P_5^3 )</th>
<th>( P_4^3 )</th>
<th>( P_3^3 )</th>
<th>( P_2^3 )</th>
<th>( P_1^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
For $\lambda \in \mathcal{M}_i^3$ the angle $\theta(c_1(\lambda))$ is in $P_{i}^3$. To see that $J(F_{\lambda})$ and $J(F_{\mu})$ are not homeomorphic for $\lambda \in \mathcal{M}_i^3$, $\mu \in \mathcal{M}_j^3$, $i \neq j$, we can apply Theorem 4.2 if $(i, j) \in \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$ because then $\lambda \in \mathcal{M}_i^2$, $\mu \in \mathcal{M}_j^2$.

If $(i, j) \in \{(1, 2), (1, 3), (2, 3), (4, 5)\}$ then it is enough to look at the following subsets of the models which are displayed in the next figure.

In each case we get a contradiction, since either in $M(F_{\lambda})$ or $M(F_{\mu})$ there is a path with a sequence of corners of specific levels which cannot be found in the other model. More precisely

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>unique path</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2)</td>
<td>0 ——— ——— ——— 1</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>0 ——— ——— ——— 1</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>0 ——— ——— ——— 1</td>
</tr>
<tr>
<td>(4, 5)</td>
<td>0 ——— ——— ——— 2</td>
</tr>
</tbody>
</table>

We just proved the following extension of Theorem 4.2.

**Theorem 4.5.** $J(F_{\lambda})$ and $J(F_{\mu})$ are not homeomorphic for $\lambda \in \mathcal{M}_i^3$ and $\mu \in \mathcal{M}_j^3$ with $i \neq j$.

5. **The general case**

As it was mentioned in the introduction, the family $G_{\lambda}(z) = z^2 + \lambda/z$ is a special family of rational maps that besides sharing many of the dynamical properties of the family $F_{\lambda}$, it also possesses a unique critical orbit up to symmetry. To see this, note that $G_{\lambda}$ has three finite, nonzero critical points given by the cubic roots of $\lambda/2$. Let $\omega = \cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3})$, so $\omega^3 = 1$. As before, denote by $c_1$ the critical point that lies in the first quadrant and set $c_2 = \omega c_1$ and $c_0 = \omega^2 c_1$. It is not difficult to see that $G_{\lambda}(\omega z) = \omega^2 G_{\lambda}(z)$ and thus $G_{\lambda}(\omega^2 z) = \omega G_{\lambda}(z)$. 
Due to these symmetries, the behavior of one critical point determines the behavior of the other two. As an example, consider the parameter \( \lambda \approx -0.59257 \) corresponding to the Julia set homeomorphic to the Sierpinski triangle (see Fig. 1). \( G_{\lambda} \) maps \( c_1 \) into the fixed point that lies in the boundary of the immediate basin of infinity exactly after two iterations. On the other hand, \( c_0 \) and \( c_2 \) land in a two periodic cycle after the same number of iterations.

A similar analysis can be performed to a more general class of rational maps of the form \( R_{\lambda,n,m}(z) = z^n + \lambda/z^m \), with \( m, n > 1 \). Straightforward computations show the point at infinity is again a superattracting fixed point, the origin is a pole of order \( n + m \) and there exist \( n + m \) finite, simple, nonzero critical points with a unique critical orbit up to symmetry. These symmetries are given by the equation

\[
R_{\lambda,n,m}(\omega^k z) = \omega^{nk} R_{\lambda,n,m}(z),
\]

for \( k = 1, 2, \ldots, n + m - 1 \) and \( \omega \) the \((n+m)\)th root of unity.

As before, let \( \beta_{\lambda} \) denote the boundary of the immediate basin at infinity, \( \tau_{\lambda} \) denote the boundary of the trap door and \( \tau_{k,\lambda} \) denote the union of all of the components of \( R_{\lambda,n,m}^{-k}(\tau_{\lambda}) \). It is known that all these boundaries are simple closed curves whenever \( \lambda \) is a Misiurewicz value (see [3,2]).

Note that \( R_{\lambda,n,m} \) is conjugate to the map \( z \mapsto z^n \) in a neighborhood of infinity, and again, this conjugacy can be extended to \( \beta_{\lambda} \). Hence, we may extend the results of the previous sections in order to conclude the existence of MS maps for the \( R_{\lambda,n,m} \) families.

In Fig. 8 we display two Julia sets corresponding to MS maps for the families \( R_{\lambda,3,2} \) and \( R_{\lambda,2,3} \). Although both rational maps have degree five, the reader can clearly distinguish different configurations of the five corners of \( \tau_{\lambda} \) along the boundary of \( \beta_{\lambda} \) and \( \tau_{\lambda} \). Indeed, while the Julia set of \( R_{\lambda,3,2} \) has two corners lying in the boundary of its trap door, \( R_{\lambda,2,3} \) has only three corners. In general, we may characterize the associated generalized Sierpinski gaskets of the families \( R_{\lambda,n,m} \) by the following lemma, which is a generalization of Lemma 3.1 and we state it without proof.

**Lemma 5.1.** If \( A \) is a component in \( \tau_{k,\lambda} \) with \( k \geq 1 \), then exactly \( m \) corners of the \( n + m \) corners of \( A \) lie in an edge of a single component of \( \tau_{k,\lambda}^{k-1} \).

In the same fashion, a generalization of the algorithm described for the family \( F_{\lambda}(z) = z^2 + \lambda/z^2 \) in Section 4 can be performed to distinguish when two Julia sets of MS maps in the family \( R_{\lambda,n,m} \) are not homeomorphic.

**References**

