FUNDAMENTA
MATHEMATICAE
222 (2013)

A combinatorial invariant for escape time Sierpiński rational maps

by

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Dedicated to Joyce T. Macabea, in loving memory

Abstract. An escape time Sierpiński map is a rational map drawn from the McMullen family $z \mapsto z^n + \lambda/z^n$ with escaping critical orbits and Julia set homeomorphic to the Sierpiński curve continuum.

We address the problem of characterizing postcritically finite escape time Sierpiński maps in a combinatorial way. To accomplish this, we define a combinatorial model given by a planar tree whose vertices come with a pair of combinatorial data that encodes the dynamics of critical orbits. We show that each escape time Sierpiński map realizes a subgraph of the combinatorial tree and the combinatorial information is a complete conjugacy invariant.

1. Introduction. In this paper we consider the McMullen family of rational maps

$$F_\lambda(z) = z^n + \lambda/z^n$$

with $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \geq 2$. These maps were first considered by McMullen [10] and have been extensively studied by Devaney and coauthors (see for example [5], [3] and the references therein), and more recently by Roesch [13], Steinmetz [14], Qiu et al. [12], among others.

Due to the symmetries exhibited by these maps (and discussed in more detail in the following section), these maps have essentially a single free critical orbit. Indeed, a straightforward computation shows the existence of $2n$ finite, nonzero critical points given by the roots $\lambda^{1/2n}$ and only two critical values given by $2\lambda^{1/2}$. The orbits of the critical values may collide into a single orbit or behave symmetrically since $F_\lambda(-z) = (-1)^n F_\lambda(z)$.

The point at infinity is a superattracting fixed point for any n and any λ. When critical orbits are trapped by its basin, the Escape Trichotomy Theo-

2010 *Mathematics Subject Classification*: Primary 37F10; Secondary 37F20.
Key words and phrases: rational maps, Julia sets, combinatorial invariants.

DOI: 10.4064/fm222-2-1

rem completely determines the topology of the Julia set [7]. In particular, if the critical values take more than one iteration to enter the immediate basin of infinity, then the Julia set is homeomorphic to the *Sierpiński curve continuum*. That is, it is a locally connected plane continuum whose boundary components are Jordan curves that are pairwise disjoint [15].

If λ is a parameter for which the critical values of F_λ take $\tau \geq 2$ iterations to enter for the first time the immediate basin of infinity, we say that λ is an *escape time Sierpiński parameter* (or ETS parameter) and call τ its *escape time*. According to [8] and [13], there are $(n-1)(2n)^{\tau-2}$ hyperbolic components of ETS parameters in the parameter space. Each component (called a *Sierpiński domain*) contains a single parameter λ (known as the *center* of the hyperbolic component) that satisfies the equation $F_\lambda^\tau(c) = 0$, for c any root of $\lambda^{1/2n}$. See Figure 1 for several examples of parameter spaces when $n = 2, 3, 4$ and 5.

An interesting problem is to determine when two Sierpiński parameters belong to the same (topological) conjugacy class when their maps are restricted to their Julia sets. It is known that maps within the same hyperbolic domain are quasiconformally conjugate [10], and that τ itself is a conjugacy invariant [8]. So the question reduces to know when two parameter values drawn from distinct Sierpiński domains belong to the same conjugacy class. The Escape Time Conjugacy Theorem (also found in [8]) provides an algebraic relation among those parameters in the same conjugacy class. We summarize it in the following result.

THEOREM 1.1. *For a fixed $n \geq 2$, let λ and μ be the centers of two distinct Sierpiński domains with the same escape time $\tau \geq 2$. Then F_λ and F_μ are topologically conjugate on their Julia sets if and only if the parameters satisfy*

$$\lambda = \beta^{2j}\mu \quad \text{or} \quad \lambda = \beta^{2j}\bar{\mu}, \tag{1.1}$$

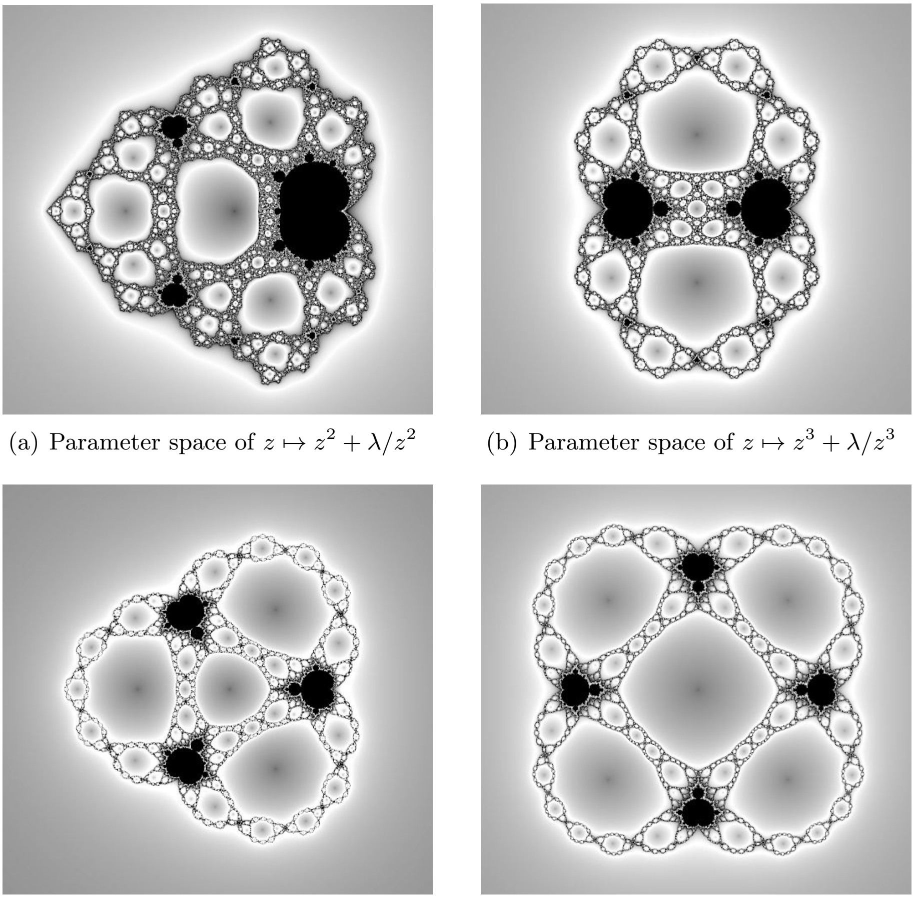
for β an $(n-1)$th primitive root of unity and some integer j. Moreover, if λ and μ are ETS maps drawn from distinct Sierpiński domains of the same escape time, then F_λ and F_μ are topologically conjugate on their Julia sets if and only if the centers of those domains satisfy (1.1).

As a consequence of the Escape Time Conjugacy Theorem, the authors derived a precise count of the number of conjugacy classes, given by

$$(2n)^{\tau-2} \qquad \text{if } n \text{ is odd,} \tag{1.2}$$

$$\frac{(2n)^{\tau-2}}{2} + 2^{\tau-3} \qquad \text{if } n \text{ is even.} \tag{1.3}$$

In this paper we construct a combinatorial model for ETS maps consisting of a planar tree with combinatorial information on its vertices, and show that for each center of a Sierpiński domain, its associated map realizes



(a) Parameter space of $z \mapsto z^2 + \lambda/z^2$ (b) Parameter space of $z \mapsto z^3 + \lambda/z^3$

(c) Parameter space of $z \mapsto z^4 + \lambda/z^4$ (d) Parameter space of $z \mapsto z^5 + \lambda/z^5$

Fig. 1. Parameters in hues of grey are associated to escaping critical orbits. With the exception of the central bounded domain in (b), (c) and (d), the rest of the bounded regions correspond to Sierpiński domains.

a minimal tree in the dynamical plane which is homeomorphic to a subtree of the combinatorial model. Each vertex in the combinatorial model comes with a pair of combinatorial data that encodes the dynamics of critical orbits. We also show that this information is a full combinatorial invariant for ETS maps, thus giving us a combinatorial version of the Escape Time Conjugacy Theorem and an alternative way to count conjugacy classes. Our work is mostly done for postcritically finite ETS maps, although the results extend to ETS parameters in general.

1.1. Statements of the results. In order to state our main results, let us introduce a few definitions and notation. A *k-tree* is a planar tree,

$T_k = (V_k, E_k)$, that exhibits $2n$ rotational symmetry and comes equipped with a *coloring map*, $c : V_k \to \underline{A}KS$, where $\underline{A}KS$ denotes a set of finite words in $4n$ symbols. The description of $\underline{A}KS$ and the realization of a k-tree as a geometric object are presented in §3.4 and §4. Similarly, we denote by $\mathbb{T}_k = (\mathbb{V}_k, \mathbb{E}_k)$ a planar tree in the dynamical plane of a postcritically finite ETS map. Its set of vertices are the points in the backward orbit of the origin up to the $(k-1)$-preimage, while its set of edges is given by the arcs of *extended rays* (defined in §3.3). To each point z in the backward orbit of the origin we can associate a *kneading sequence*, $\kappa(z)$, which is a finite word in the set $\underline{A}KS$ (see §3.4). Denote by z_{0_k} a vertex in $\mathbb{T}_k$ with kneading sequence 0_k, a word of k zeros, and for $0 < \alpha < 1/4$, the number $1 + \alpha + \cdots + \alpha^{k-1}$ denotes a vertex in T_k.

THEOREM 1 (Dynamical k-tree). *For each positive integer $k \leq \tau - 1$ for which $\mathbb{T}_{k-1}$ is a $(k-1)$-tree and critical values do not lie in it, we have:*

(1) *The set*
$$\mathbb{T}_k := \mathbb{T}_0 \cup F_\lambda^{-1}(\mathbb{T}_{k-1})$$
is a connected planar tree whose set of vertices
$$\mathbb{V}_k := \bigcup_{j=0}^{k+1} F_\lambda^{-j}(0)$$
has cardinality $((2n)^{k+2} - 1)/(2n-1)$ and $\mathbb{V}_k^* = \mathbb{V}_k - \{0\}$ is colored by the kneading sequences of its elements.

(2) *There exists a homeomorphism of rooted trees*
$$\varphi_k : (\mathbb{T}_k, z_{0_k}) \to (T_k, 1 + \alpha + \cdots + \alpha^{k-1})$$
that preserves the rotational ordering of edges and sends $\mathbb{V}_k$ to V_k in such a way that $\varphi_k(z_{0_k}) = 1 + \cdots + \alpha^{k-1}$.

Moreover, φ_k can be chosen to be compatible with the colorings, so for each vertex $v \in \mathbb{V}_k^*$, $c \circ \varphi_k(v) = \kappa(v)$.

Denote by $\mathbb{T}_\lambda$ the smallest dynamical tree whose set of vertices contains the critical values of F_λ. Let φ_λ denote the homeomorphism of rooted trees from $\mathbb{T}_\lambda$ to the combinatorial model $T_{\tau-2}$ (or to a subgraph of the tree, see Proposition 5.6). We write v_λ for a preferred critical value which is determined by a *basic configuration* of critical values and fixed points of F_λ as described in §3.1. Moreover let
$$\kappa_\lambda = \kappa(v_\lambda) \in \underline{A}KS$$
denote the kneading sequence of that critical value and
$$\delta_\lambda = \delta_1 \ldots \delta_t \in \underline{A}KS$$

denote the *direction* of the vertex $\varphi_\lambda(v_\lambda)$ along $T_{\tau-2}$ (δ_λ essentially identifies the position of v_λ in a rooted k-tree, see Definition 4.4). The pair $(\kappa_\lambda, \delta_\lambda)$ is the *combinatorial information* of the map F_λ.

Our main result is the following.

THEOREM 2 (Realization Theorem). *Fix any $n \geq 2$ and $k \geq 0$. Let T_k denote the k-tree with $2n$ rotational symmetry and color map c. For any given vertex $z \in V_k - \{0\}$, let $c(z)$ and $\delta(z)$ denote its color and direction. Then $(c(z), \delta(z))$ is realized as the combinatorial information (with respect to the basic configuration) of a postcritically finite ETS map of degree $2n$ if and only if $\delta_1 = \lfloor n/2 \rfloor$.*

As a consequence of the Realization Theorem, we provide a combinatorial version of Theorem 1.1.

THEOREM 3 (Conjugacy invariant). *Let F_λ and F_μ be two postcritically finite ETS maps of same degree $n \geq 2$. Then the maps are topologically conjugate on their Julia sets if and only if the maps have the same combinatorial information, that is, $\delta_\lambda = \delta_\mu$ (and thus $\kappa_\lambda = \kappa_\mu$).*

In Corollary 6.6 we provide an alternative derivation of formulas (1.2) and (1.3) by counting all those vertices in the combinatorial model that can be realized by ETS maps.

The presentation of this paper is the following: we review some essential properties of the McMullen family in §2, then we describe in §3 a partition of parameter and dynamical spaces that allow us to define kneading sequences of critical orbits. We also discuss the basic configuration of critical values and fixed points that define a marking within the $2n$ degree families. §4 explains the construction of the combinatorial model, while §5 shows how to construct dynamical k-trees and provides the proof of Theorem 1. The proofs of Theorems 2 and 3 are given in §6, while §7 contains some final remarks and open questions.

2. Preliminaries. The proofs of most of the results presented here can be found in [3], which is itself a good reference and a starting point in the study of the dynamics and topology of the McMullen family.

Each map $F_\lambda(z) = z^n + \lambda/z^n$ has a critical point of order $n-1$ at $z = \infty$. There exist $2n$ distinct, finite, nonzero and simple critical points given by the $2n$th roots of λ. As the fate of their orbits is determined by the parameter, we call them *free critical points*. For $\xi \in \mathbb{R}/\mathbb{Z}$, let $\arg(\lambda) = 2\pi\xi$ and ω be the primitive $2n$th root of unity with the smallest positive argument. Then for each $j = 0, 1, \ldots, 2n-1$,

$$c_j = |\lambda|^{1/2n} \exp(i\pi\xi/n)\omega^j$$

is a free critical point. These points map alternately to two distinct critical values, namely

$$F_\lambda(c_j) = (-1)^j 2|\lambda|^{1/2} \exp(i\pi\xi/n).$$

Denote by v_+ and v_- the critical values $F_\lambda(c_j)$ when j is even and odd, respectively. Observe that $v_- = -v_+$ and since $F_\lambda(v_-) = (-1)^n F_\lambda(v_+)$, we say F_λ is essentially a unicritical rational map as the $2n$ free critical orbits merge into a single orbit or become two symmetrically behaving orbits, depending on the parity of n.

F_λ exhibits several symmetries: *$2n$-fold symmetry* that for all j,

$$F_\lambda(\omega^j z) = \omega^{jn} F_\lambda(z),$$

and the *involution symmetries*. That is, if $I_\lambda(z)$ denotes one of the n branches of $z \mapsto \lambda^{1/n} z^{-1}$, then $F_\lambda(I_\lambda(z)) = F_\lambda(z)$ for all z. Depending on the branch selected, each involution fixes a line containing two free critical points and reflects the plane through the *critical circle*

$$C_\lambda = \{z \mid |z| = |\lambda|^{1/2n}\}.$$

The point at infinity is also a superattracting fixed point; denote by $B_\lambda = B_\lambda(\infty)$ its *immediate basin*. The origin is a pole of order n and there exist $2n$ *prepoles* lying on the critical circle C_λ and given by the roots $(-\lambda)^{1/2n}$. Denote by w_0 the prepole whose principal argument is the smallest in absolute value. Then, label the rest of the prepoles in increasing order while traversing C_λ in a positive (counterclockwise) direction.

If the free critical points do not lie in B_λ, it is known that $F_\lambda^{-1}(B_\lambda)$ consists of two simply connected components, namely B_λ itself and T_λ, which contains the pole at the origin and maps n-to-1 onto B_λ. If the critical orbits eventually escape, they must enter T_λ before mapping into B_λ, thus T_λ is commonly known as the *trap door* of the basin of attraction.

Let us concentrate on escaping free critical orbits.

THEOREM 2.1 (Escape Trichotomy Theorem [7]). *Suppose the orbits of the free critical points of F_λ tend to infinity.*

(1) *If one of the critical values lies in B_λ, then J_λ is a Cantor set and $F_\lambda|J_\lambda$ is a one-sided shift on $2n$ symbols. Otherwise J_λ is connected and the preimage T_λ is disjoint from B_λ.*
(2) *If one of the critical values lies in T_λ, then J_λ is a Cantor set of simple closed curves (quasicircles).*
(3) *If one of the critical values lies in a preimage of T_λ, then J_λ is a Sierpiński curve.*

DEFINITION 2.2. A parameter value satisfying the last condition in the Escape Trichotomy Theorem will be called an *escape time Sierpiński parameter*, or succinctly, an *ETS* parameter. Analogously we say F_λ is an *ETS* map.

As was mentioned in the introduction, the integer τ stands for the *escape time* of the free critical orbits, more precisely, τ represents the number of iterations required for the critical points to enter B_λ for the first time. The escape time is an open condition that defines simply connected (Sierpiński) domains in the parameter space. Each domain has a unique center, that is, a parameter value that is a simple zero of the equation $F_\lambda^\tau(c) = 0$ for c any root $\lambda^{1/2n}$. In terms of the critical values, this equation becomes

$$F_\lambda^{\tau-1}(v) = 0 \quad \text{for } v = v_+, v_-.$$

For each $\tau \geq 0$ define the set

$$H_\tau = \{\lambda \in \mathbb{C} \mid F_\lambda^\tau(v) \in B_\lambda \text{ where } v \text{ is either } v_+ \text{ or } v_-\}.$$

When $\tau \geq 2$, H_τ consists of $(n-1)(2n)^{\tau-2}$ Sierpiński domains [8]. The set H_0 represents the *Cantor locus* of the family, that is, the set of parameter values satisfying the first condition in the Escape Trichotomy Theorem. If $n \geq 3$, the set H_1 consists of a single simply connected component known as the *McMullen domain* [7]. When $n = 2$, it follows from the Grötzsch inequality that H_1 is empty [10].

If λ belongs to the connectedness locus of the family of rational maps of degree $2n$, then standard techniques in holomorphic dynamics show that B_λ is a simply connected domain whose Böttcher uniformization

$$\varphi_\lambda : B_\lambda \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$$

conjugates the action of $F_\lambda | B_\lambda$ with $z \mapsto z^n$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Observe that φ_λ is unique up to multiplication by a $(n-1)$th root of unity. Giving an angle $\theta \in \mathbb{R}/\mathbb{Z}$, the *external ray of angle* θ is defined as the set

$$R_\theta(t) = \varphi_\lambda^{-1}(t \exp(i2\pi\theta)), \qquad t > 1.$$

If the limit of $R_\theta(t)$ exists as t decreases to 1 and equals p, we say *the ray lands at* p. If θ is a rational angle, standard arguments in the theory of external rays can be applied to show $R_\theta(t)$ lands at a point in ∂B_λ. Moreover, if θ is periodic under $\theta \mapsto n\theta \bmod 1$, then p is a periodic point for F_λ (see, for example, [11]).

Finally, when λ is an ETS parameter, the Fatou set coincides with the basin of the point at infinity and the Julia set, denoted by J_λ, is given by

$$J_\lambda = \widehat{\mathbb{C}} - \bigcup_{j \geq 0} F_\lambda^j(B_\lambda).$$

3. Partitions and kneading sequences. Our aim in this section is to describe a partition of the Riemann sphere through pullbacks of a forward F_λ-invariant curve containing a Cantor set of points in the Julia set. This partition will generate a suitable labeling of Fatou components, so (free) critical orbits can be combinatorially described by kneading sequences.

3.1. Partitions and markings. We begin by defining a partition of the parameter space into $n-1$ rotationally symmetric open sectors given by

$$\mathcal{S}_j = \left\{ \lambda \in \mathbb{C} \;\middle|\; \frac{j}{n-1} < \frac{\arg(\lambda)}{2\pi} < \frac{j+1}{n-1} \right\}, \tag{3.1}$$

with $j = 0, 1, \ldots, n-2$ and $0 \leq \arg(\lambda) < 2\pi$. Denote the left- and right-hand boundaries of $\mathcal{S}_j$ by $\partial^- \mathcal{S}_j$ and $\partial^+ \mathcal{S}_j$ respectively.

Let $k_0 \geq 1$ be equal to the integer $\lfloor n/2 \rfloor$. Observe that the negative real line has nonempty intersection with $\mathcal{S}_{k_0-1}$ (whenever n is even) or $\partial^- \mathcal{S}_{k_0-1} = \mathbb{R}^-$ (if n odd). Moreover, $\partial^- \mathcal{S}_0 = \mathbb{R}^+$, regardless of the parity of n.

Lemma 3.1. *For $n \geq 2$ and any $0 \leq j \leq n-2$, no ETS parameter lies in the boundary of $\mathcal{S}_j$.*

Proof. By rotational symmetry it suffices to show that no ETS parameter lies in $\mathbb{R}^+$. If $\lambda > 0$ then $c_0 \in \mathbb{R}^+$ and $F_\lambda|\mathbb{R}^+$ is a unicritical map that leaves invariant the positive real line. In particular, F_λ maps $(0, +\infty)$ onto $[v_+, +\infty)$. Thus, for any $\lambda \in \mathbb{R}^+$ and $\tau \geq 2$, the equation $F_\lambda^{\tau-1}(v_+) = 0$ has no solution, implying that λ cannot be an ETS parameter. ∎

Now we define a *static partition* of the dynamical plane for any $\lambda \in \mathcal{S}_j \setminus H_0$ and any j. This partition will be modified in Section 5 to adjust it to the dynamics of F_λ. For each free critical point c_k, its *critical ray* is given by $\eta_k(t) = tc_k$, with $t \geq 0$. Similarly, denote by $\ell_\pm(t) = tv_\pm$ with $t \geq 1$ the *critical value rays*. As is customary, we let η_k and $\ell_\pm$ denote the curves they parametrize.

The critical rays divide the dynamical plane into $2n$ rotationally symmetric open sectors

$$S_k = \left\{ z \in \mathbb{C} \;\middle|\; \frac{\xi + k - 1}{2n} < \frac{\arg(z)}{2\pi} < \frac{\xi + k}{2n} \right\} \tag{3.2}$$

for $k = 0, \ldots, 2n-1$, where $0 \leq \arg(z) < 2\pi$.

Since J_λ is connected, by the conjugacy of F_λ with $z \mapsto z^n$ in B_λ and the landing of external rays with rational angles, there exist $n-1$ fixed points in ∂B_λ that correspond to the landing points of $R_{\theta_k}(t)$ for the angles $\theta_k = k/(n-1), k = 1, \ldots, n-1$, which in turn are fixed by $\theta \mapsto n\theta$ mod 1. Denote by p_0 the fixed point whose principal argument is the smallest in absolute value. Then, label the rest of the fixed points in increasing order by following the cyclic order of external rays. Observe $p_0 \in \mathbb{R}^+$ if and only if $\lambda \in \mathbb{R}^-$.

The location of critical values and fixed points in ∂B_λ with respect to the partition S_k is relevant for our construction. On one hand, each of them lies in its own sector, thus defining a configuration of these points with respect to the partition (Lemma 3.2). On the other hand, each p_k gives rise to an invariant Cantor set in the Julia set [5]. When $n = 2, 3$ there exists a single invariant Cantor set, but when $n \geq 4$, we need to make a choice of the fixed point (and thus, the Cantor set) to be used in the partition (Lemma 3.4). Each lemma is followed by an example that computes the configuration (Example 3.3) and the choice of a fixed point in $\mathcal{S}_j$ (Example 3.6).

LEMMA 3.2 (Location of critical values and fixed points). *Let* $\lambda \in \mathbb{C}^*$ *and* $n \geq 2$.

(1) *If* $\lambda \notin \mathbb{R}^+$, *the critical values lie each in sectors* S_i *and* S_{i+n} *for some* $0 < i < n$.
(2) $\lambda \in \mathcal{S}_k$ *if and only if* $i = k+1$. *The sectors* $S_+ := S_{k+1}$ *and* $S_- := S_{k+1+n}$ *are called the* critical value sectors *for* $\mathcal{S}_k$.
(3) $p_0 \in S_0$ *for all* $\lambda \notin \mathbb{R}^+$. *And for each* $\lambda \in \mathcal{S}_k$, *each fixed point in* ∂B_λ *lies in a sector* S_j *where* j *satisfies one of the following conditions:*
   - j *is even and either* $0 < j < k+1$ *or* $k+1+n < j < 2n-1$.
   - j *is odd and* $k+1 < j < k+1+n$.
(4) *A sector that contains a fixed point in* ∂B_λ *contains only one of them and it is disjoint from the critical value sectors.*

Proof. Due to the symmetries of the critical values, it is sufficient to work with one of them. Assume first $\lambda \notin \mathbb{R}^+$. Working out the inequalities in (3.2), v_+ lies in S_0 if and only if

$$\frac{\zeta + 2n - 1}{2n} < \frac{\zeta}{2} < \frac{\zeta}{2n}.$$

This inequality is equivalent to $\zeta - 1 < n\zeta < \zeta$, and this holds if and only if $\zeta = 0$, so the first claim follows.

Similarly, from (3.1), $\lambda \in \mathcal{S}_k$ if and only if

$$\frac{k+\zeta}{2n} < \frac{\zeta}{2} < \frac{k+1+\zeta}{2n},$$

that is, $v_+ \in S_{k+1}$. From rotational symmetry, v_- lies in S_{k+1+n} and the second assertion follows.

To see the third, let $\lambda \notin \mathbb{R}^+$ and observe that F_λ maps each sector S_j onto $\mathbb{C} - \ell_\pm$. By (1), there exist two sectors that contain a critical value each and fail to completely cover themselves under F_λ. Hence, we discard these sectors in our analysis below and work with the $2n-2$ remaining sectors.

The critical circle C_λ subdivides each S_j into two domains, one bounded and one unbounded; denote the unbounded domain by S_j^u. We want to show that exactly $n-1$ unbounded domains contain a fixed point located in ∂B_λ. To do so, notice that $F_\lambda(C_\lambda)$ is a straight line segment connecting the critical points and passing through the origin, so the boundaries of each S_k^u are mapped onto the straight ray $\ell_+ \cup F_\lambda(C_\lambda) \cup \ell_-$. Let H^- and H^+ denote the open left and right half-planes defined by the complement of the ray. Then $F_\lambda|S_j^u$ is a conformal homeomorphism mapping S_j^u onto either H^- or H^+, depending on the parity of j (indeed, S_0^u is mapped onto H^+, S_1^u maps onto H^- and so on).

Thus, if S_j is compactly contained in H^+ and j is even, then $\overline{S_j^u} \subset F_\lambda(S_j^u) = H^+$. If

$$G_j^u : H^+ \to S_j^u$$

denotes the inverse branch of F_λ taking values in S_j^u, its covering map $\widetilde{G}_j^u : \mathbb{D} \to \mathbb{D}$ is a strict contraction in the Poincaré metric. Thus G_j^u has an attracting fixed point in S_j^u.

By reflection symmetry with respect to the critical circle, we deduce that $S_j^u \cap \partial B_\lambda \neq \emptyset$. And since $S_j^u \cap \partial B_\lambda$ covers itself, it must contain a repelling fixed point for F_λ. In particular, for any $\lambda \notin \mathbb{R}^+$, $p_0 \in S_0^u$.

Now let $\lambda \in \mathcal{S}_k$. Then $S_+ = S_{k-1}$ and $S_- = S_{k+1+n}$ by (2). Thus the integers j for which S_j^u lies in H^+ and cover themselves are those integers that satisfy $0 < j < k+1$ or $k+1+n < j < 2n-1$. The case when j is odd follows similarly. By the conjugacy of F_λ with $z \mapsto z^n$ in a neighborhood of infinity, we deduce the existence of exactly $n-1$ unbounded sectors that cover themselves and give rise to $n-1$ fixed points in ∂B_λ.

From the above analysis, none of the sectors containing a fixed point is a critical value sector. ■

In the next lemma we show that the configuration of critical values and fixed points with respect to the static partition is the same for each sector $\mathcal{S}_j$ up to a rotation. The configuration defined for any $\lambda \in \mathcal{S}_{k_0-1}$ will be called the *basic configuration*. Let us give an example first.

Example 3.3 (Basic configuration for $z \mapsto z^7 + \lambda/z^7$). Let $n = 7$, so $k_0 = 3$. The basic configuration realized by any $\lambda \in \mathcal{S}_2$ is the following. From Lemma 3.2(2) the critical values lie in S_j for $j \in \{3, 10\}$. And Lemma 3.2(3) implies that the fixed points $p_0, \dots, p_5 \in \partial B_\lambda$ lie each on S_j for some $j \in \{0, 2, 5, 7, 9, 12\}$. See Figure 2.

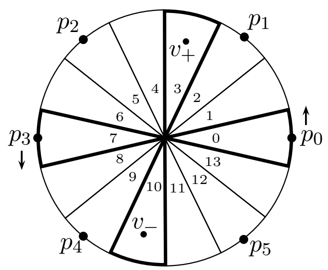


Fig. 2. For the family of maps $z \mapsto z^7 + \lambda/z^7$, the basic configuration of fixed points and critical values with respect to p_0 (or p_3) and with positive orientation

LEMMA 3.4. *Let $\lambda \in \mathcal{S}_k$. Then, whenever i is an integer so that*

$$(3.3)\qquad i = \begin{cases} k+1-k_0 \bmod 2n \text{ and is even, or} \\ k+1+n-k_0 \bmod 2n \text{ and is odd,} \end{cases}$$

or

$$(3.4)\qquad i = \begin{cases} k+1+k_0 \bmod 2n \text{ and is odd, or} \\ k+1+n+k_0 \bmod 2n \text{ and is even,} \end{cases}$$

there exists a fixed point $p_\lambda \in S_i$ that realizes the basic configuration by setting $S_0 = S_i$ and relabeling the sectors in a positive or negative orientation if i is given by (3.3) *or* (3.4), *respectively.*

If n is even, p_λ is unique and realizes the basic configuration with both orientations. Otherwise, both p_λ and $-p_\lambda$ realize the configuration with respect to a single orientation.

Proof. First, note that $k+1+k_0$ and $k+1-k_0$ have the same parity (the same holds for $k+1+n+k_0$ and $k+1+n-k_0$). Thus at least one of these four integers satisfies one of the above conditions. If i is one of these values, Lemma 3.2(3) implies that S_i is a fixed point sector.

By the rotational symmetries of the sectors, $p_\lambda \in S_i$ and $-p_\lambda \in S_{i+n}$ realize the basic configuration, but $-p_\lambda$ is a fixed point only when n is odd. Thus p_λ is unique when n is even. Finally, one or both orientations are realized depending on the parity of n. Indeed, without loss of generality, assume the negative orientation is realized. Then the number of sectors away from S_i to the critical value sectors is either k and $k+n$, or k and $k-n$. So when n is even, k and $k \pm n$ have the same parity, implying that S_i lies the same number of sectors away from the $S_\pm$ sectors, and thus the configuration is realized also in the positive orientation. For n odd, k has different parity from $k \pm n$ so p_λ (and thus $-p_\lambda$) realize the configuration only with respect to the negative orientation. ∎

Thus, it does not matter which parameter sector λ lies in, as long as the partition $\{S_k\}$ is relabeled with respect to p_λ as determined by the above lemma.

DEFINITION 3.5 (Markings). The *marked fixed point* in $\mathcal{S}_j$ is the point p_λ (or $-p_\lambda$) that realizes the basic configuration with respect to the orientations (or orientation) given in Lemma 3.4.

EXAMPLE 3.6 (Marked fixed point for $\lambda \in \mathcal{S}_0$ and $n = 7$). Let $\lambda \in \mathcal{S}_0$. With the initial labeling of sectors, Lemma 3.2(2)&(3) implies that $S_+ = S_1$, $S_- = S_8$ and S_j is a fixed point sector for $j \in \{0, 3, 5, 7, 10, 12\}$. Since n is odd, there are two marked fixed points that realize the basic configuration with respect to the positive orientation, namely $p_2 \in S_5$ and $p_5 \in S_{12}$. If we select $p_\lambda = p_2$, the sectors are relabeled in the positive orientation as $S_j := S_{j+2}$ for $j = 0, \dots, 13$ with addition mod 14.

Standing Assumption. From now on, we work solely with parameters in $\mathcal{S}_{k_0-1}$. In this way,

- the marked fixed point is $p_\lambda = p_0$, which lies in S_0,
- the sectors S_j are labeled as in (3.2) in the positive orientation, and
- the critical value sectors are S_{k_0} and S_{k_0+n}.

3.2. Invariant Cantor sets. With the standing assumption in mind, we describe the main ideas to construct the (marked) *forward-invariant Cantor set* associated to F_λ. Fix $n = 2k_0$ or $n = 2k_0 + 1$ for some integer $0 < k_0 < n$ and consider any parameter $\lambda \in \mathcal{S}_{k_0-1}$. By the previous lemmas, the marked fixed point $p_\lambda = p_0$ lies in the interior of a sector S_0, while S_{k_0} and S_{k_0+n} are the critical value sectors S_+ and S_- respectively.

Select a Böttcher level curve $C \subset B_\lambda$ and denote by C^b and C^u the components in $F_\lambda^{-1}(C)$ that lie in the bounded and unbounded complementary components of C, respectively. We may choose C so that C^b and C^u cut each critical ray in exactly one point. Let A denote the compact annular domain bounded by C^b and C^u and define $W_0 = A \cap \overline{S_0}$ and $W_n = A \cap \overline{S_n}$. Clearly, W_0 and W_n are compact domains that map onto a closed, double-slit topological disk bounded by C with the slits corresponding to segments along $\ell_\pm$. Moreover $W_0 \cup W_n \subset \mathrm{Int}(F_\lambda(W_i))$ for each $i = 0, n$. Denote by $\Sigma = \{0, n\}^{\mathbb{N}}$ the space of infinite sequences endowed with the product topology and denote by $\sigma : \Sigma \to \Sigma$ the right-hand shift. Standard arguments, like those given in Lemma 3.2, establish the following result (cf. [5]).

Lemma 3.7. *For any $n \geq 2$ and any $\lambda \in \mathcal{S}_j$, there exists a forward invariant Cantor set $\Gamma \subset \mathrm{Int}(W_0 \cup W_n)$ and a homeomorphism $h_\lambda : \Gamma \to \Sigma$ that conjugates the action of $F_\lambda|\Gamma$ with $\sigma|\Sigma$. In particular $p_\lambda \in \Gamma$ and the homeomorphism can be chosen so that $h_\lambda(p_\lambda) = \bar{0}$.*

We stress the marking of the Cantor set given above by writing Γ_λ.

3.3. Extended rays. Fix an ETS parameter $\lambda \in \mathcal{S}_{k_0-1}$ and select the uniformization of the immediate basin of infinity $\varphi_\lambda : B_\lambda \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ that is tangent to the identity at infinity. Since every ETS map is hyperbolic, φ_λ has a continuous extension to the boundary of B_λ. Furthermore, its Julia set is locally connected, so for each $\theta \in \mathbb{R}/\mathbb{Z}$ the external ray $R_\theta(t)$ lands at a single point in ∂B_λ. In particular, the external rays of angle $\theta = 0$ and $\theta = 1/2$ land at p_λ and $-p_\lambda$ respectively.

For $j = 0, n$, the inverse map G_j^u defined in Lemma 3.2 has an analytic extension

$$G_j : \mathbb{C} \setminus (\ell_+ \cup \ell_-) \to S_j$$

taking values over noncritical sectors. For any finite string $s_1 \ldots s_r$ of elements $s_i \in \{0, n\}$, denote by $G_{s_1 \ldots s_r}$ the composition of inverse maps

$G_{s_1} \circ \cdots \circ G_{s_n}$ taking values in S_{s_1}. And since S_0 and S_n are compactly contained in $\mathbb{C} \setminus (\ell_+ \cup \ell_-)$, for each $r \geq 1$ the set

$$C_{s_1 \ldots s_r} := \bigcup_{i,j \in \{0,n\}} G_{s_1 \ldots s_{r-1} i}(\overline{S_j}) \tag{3.5}$$

is a chain of 2^{r+1} topological closed disks in $\widehat{\mathbb{C}}$ with either pairwise empty intersection or with trivial intersection at some points in the backward orbit of $z = \infty$. Observe that $C_{s_1 \ldots s_r} \subset C_{s_1 \ldots s_{r-1}}$. Then the set $C_\infty = \bigcap_{r \geq 1} C_{s_1 \ldots s_r}$ is a continuum in $\widehat{\mathbb{C}}$ which is forward invariant under F_λ and contains $\infty, 0$, and its backward orbit restricted to $S_0 \cup S_n$. Moreover, the forward invariance of C_∞ implies that Γ_λ, R_0 and R_n are all contained in C_∞.

We claim that C_∞ is a circle-like continuum. To see this, we use the following characterization given in [2].

Theorem 3.8. *A continuum X is a circle-like continuum if and only if any open cover $\mathcal{U}$ of X with at most four elements has an open refinement $\mathcal{V} = \{V_1, \ldots, V_m\}$ with $V_i \cap V_j \neq \emptyset$ if and only if $|i - j| \leq 1$ or $i, j \in \{1, m\}$.*

Thus, if $\mathcal{U}$ is a given open cover of C_∞, then for $N > 0$ sufficiently large, the chain $C_{s_1 \ldots s_N}$ is also covered by $\mathcal{U}$. To construct an open refinement, denote by D_0 and D_n the open ε-neighborhoods of $\overline{S_0}$ and $\overline{S_n}$, respectively. The value ε is selected to be small enough so $D_0 \cap D_n \neq \emptyset$ and $D_0 \cup D_n$ is disjoint from $\ell_\pm$. If $\mathcal{V}$ denotes the collection of pullbacks under $G_{s_1 \ldots s_N}$ of D_0 and D_n, then their properties guarantee that $\mathcal{V}$ is the sought-for refinement. Thus, we have shown

Proposition 3.9. *The set $C_\infty \subset \widehat{\mathbb{C}}$ is a continuous image of a circle that passes through the origin and the point at infinity.*

The set C_∞ is known as the *extended ray of angle* 0. A rather different construction was given in [4], where extended rays for the maps $z \mapsto z^n + \lambda/z^n$ were first introduced. For our purposes, we modify the definition of an extended ray as follows.

Definition 3.10. The *extended* 0*-ray*, denoted by $\mathcal{R}_0$, is the set $C_\infty \cap S_0$, where S_0 contains the marked fixed point p_λ. Analogously, the *extended j-ray*, denoted by $\mathcal{R}_j$, is $\omega^j \mathcal{R}_0$ for each $j = 1, \ldots, 2n-1$. Each extended ray can be decomposed into two components, one bounded and one unbounded. The set

$$\mathcal{R}_j^b = \mathcal{R}_j \cap \overline{S_j^b}$$

defines the bounded part of $\mathcal{R}_j$.

3.4. Kneading sequences. For any ETS map, the critical values belong to the backward orbit of the origin, so we are interested in recording their passing from a Fatou component to another before mapping into the trap door. We begin with those preimages of zero in $\mathcal{R}_0 \cup \mathcal{R}_n$. From the previous discussion, observe the inverse maps G_0 and G_n take values over S_0 and S_n, so for each $m \geq 1$ and each choice of $s_j \in \{0, n\}$, $G_{s_1 \ldots s_m}(0)$ is the unique point in $F_\lambda^{-m}(0)$ that lies in C_∞.

Definition 3.11. The *kneading sequence* of a point $z \in F_\lambda^{-m}(0)$ is the finite word $s_1 \ldots s_m$, with $s_j \in \{0, n\}$, if and only if $F_\lambda^{j-1}(z) \in S_{s_j}$ for each $j = 1, \ldots, m$. In other words, $z = G_{s_1 \ldots s_m}(0)$. We write $\kappa(z) = s_1 \ldots s_m$ to denote the kneading sequence of z. For short, we also write $z_{s_1 \ldots s_m}$.

To associate a kneading sequence to a preimage of zero in any given extended k-ray, we observe that from the symmetries of the map, $F_\lambda(\mathcal{R}_k) = F_\lambda(\omega^k \mathcal{R}_0) = \omega^{kn} F_\lambda(\mathcal{R}_0)$. Thus for $u \in \mathcal{R}_k$ such that $F_\lambda^m(u) = 0$, there exists a point $z \in \mathcal{R}_0$ so that $u = \omega^k z$. Then, if $\kappa(z) = 0 s_2 \ldots s_m$, the kneading sequence of u is

$$\kappa(u) = \begin{cases} k(s_2 + kn) s_3 \ldots s_m & \text{if } n \text{ is even,} \\ k(s_2 + kn) \ldots (s_m + kn) & \text{if } n \text{ is odd,} \end{cases} \tag{3.6}$$

with addition mod $2n$ and $k \in A := \{0, 1, \ldots, 2n-1\}$.

Finally, to associate a kneading sequence to preimages of the origin not in $\bigcup_{k=0}^{2n-1} \mathcal{R}_k$, observe that the complement of this union consists of $2n$ rotationally symmetric open sectors, each containing a free critical point. To avoid introducing a new set of $2n$ symbols, we use again the set A and underline its elements whenever the iterates of z belong to a sector in $\widehat{\mathbb{C}} \setminus \bigcup_{k=0}^{2n-1} \mathcal{R}_k$. For $k \in A$, let $E_{\underline{k}}$ be the *extended ray sector* containing the critical point c_k and whose boundaries are $\mathcal{R}_k$ and $\mathcal{R}_{k+1}$. With this convention, for any point $z \in E_{\underline{k}}$ such that $F_\lambda^N(z) = 0$ for some $N \geq 2$, its kneading sequence is written as

$$\kappa(z) = \underline{a_1 \ldots a_r} k s_2 \ldots s_m,$$

where $N = m + r$, $r \geq 1$, $m \geq 1$ (if $m = 1$ then $s_1 = k$), $a_i, k \in A$ and $s_j \in \{0, n\}$, so that $F_\lambda^i(z) \in E_{\underline{a_i}}$ for $i = 0, \ldots, r$ and the point $F_\lambda^{r+1}(z) \in \mathcal{R}_k$ has kneading sequence $k s_2 \ldots s_m$.

3.5. Admissible rules. Observe that not every word of the form

$$\underline{a_1 \ldots a_r} k s_2 \ldots s_m$$

with $a_j, k \in A$ and $s_j \in \{0, n\}$ is realized by a rational map F_λ. For example, the word $\underline{0}0$ is not admissible since the $2n$ preimages of the prepole $w_0 \in \mathcal{R}_0$ lie on $\bigcup_{k=0}^{2n-1} \mathcal{R}_k$, while $E_{\underline{0}}$ is a connected component in $\widehat{\mathbb{C}} \setminus \bigcup_{k=0}^{2n-1} \mathcal{R}_k$.

We use the notation $a \mapsto b$ to denote an *admissible rule*, so the word ab is realized by the dynamics of F_λ. A straightforward analysis of the dynamics of extended rays (and thus of sectors $E_{\underline{a}}$, $a \in A$) gives the following rules:

(1) $0 \mapsto n$ and $n \mapsto 0$.
(2) For any $k \in A$, $k \mapsto 0$ and $k \mapsto n$.
(3) Let $a, k \in A$. Then $\underline{a} \mapsto k$ if and only if
    (a) $\underline{a} = \underline{2j}$ and $k = 1, \ldots, n-1$.
    (b) $\underline{a} = \underline{2j+1}$ and $k = n+1, \ldots, 2n-1$.
(4) Let $a, b \in A$. Then $\underline{a} \mapsto \underline{b}$ if and only if
    (a) $\underline{a} = \underline{2j}$ and $\underline{b} = \underline{0}, \underline{1}, \ldots, \underline{n-1}$.
    (b) $\underline{a} = \underline{2j+1}$ and $\underline{b} = \underline{n}, \ldots, \underline{2n-1}$.

The set of all admissible words $\underline{a_1 \ldots a_r} k s_2 \ldots s_m$ under the rules described above is denoted by the mnemonic $\underline{A}KS$.

Remark 3.12. Combining Lemma 3.2 and the injectivity of the rotation $z \mapsto \omega^k z$, we easily deduce the existence of a unique point in $F_\lambda^{-m}(0) \cap \mathcal{R}_k$ that realizes $k s_2 \ldots s_m$ as a kneading sequence, and vice versa. In contrast, as each sector $E_{\underline{a}_j}$ contains a free critical point, it must map into a simply connected component of $\mathbb{C} \setminus C_\infty$ in a two-to-one fashion. Thus, a sequence $\underline{a_1 \ldots a_r} k s_1 \ldots s_m$ is associated to 2^r points lying in $F_\lambda^N(0) \cap E_{\underline{a}_1}$. In Figure 3 we display the Fatou components associated to $\underline{00}1$ and $\underline{0}1$ along the Julia set of a degree four ETS map.

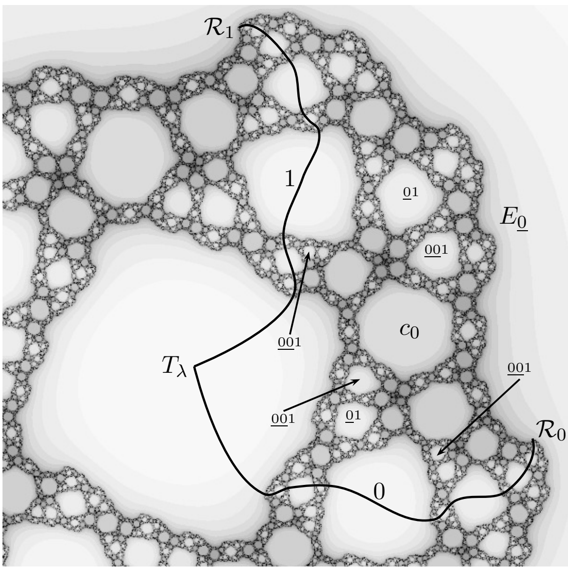


Fig. 3. For $F_\lambda(z) = z^2 + \lambda/z^2$ and a choice of $\lambda \in H_3$ (so that $F_\lambda^3(\pm v) \in B_\lambda$), we display the critical point c_0, the trap door T_λ and several Fatou components labeled by the kneading sequence of preimages of the origin. Lines indicate the extended rays $\mathcal{R}_0$ and $\mathcal{R}_1$ cutting through the Julia and Fatou sets and joined at the origin. $E_{\underline{0}}$ is bounded by the extended rays.

In the following section we introduce the concept of a k-tree, a topological model that will allow us to identify in a combinatorial way each preimage of the origin with the same kneading sequence.

4. The model. In this section we construct for each $n \geq 2$ a family of $2n$-rotationally symmetrical trees whose vertices are labeled by finite words in $4n$ symbols. We also provide each vertex with a *direction* which determines uniquely its position along the branches of the tree.

4.1. k-Trees. Fix $n \geq 2$. For each $k \geq 0$ we define in a recursive way a planar graph $T_k = (V_k, E_k)$ as follows. Let $L := [0,1] \subset \mathbb{C}$ and recall $\omega = \exp(2\pi i/2n)$. Then the 0-*tree* is defined as the set

$$T_0 := \bigcup_{j=0}^{2n-1} \omega^j L.$$

T_0 is a graph in the complex plane with $V_0 = \{0, 1, \omega, \ldots, \omega^{2n-1}\}$ as its set of vertices and $E_0 = \{e_j = \omega^j L \mid 0 \leq j \leq 2n-1\}$ its set of edges. Clearly, $|V_0| = 2n+1$, $|E_0| = 2n$, the origin is its unique vertex of degree $2n$ (we call it a *junction point* of the tree), while the rest of the vertices have degree 1 (we call them *simple* vertices).

Let $k \geq 1$ and denote by αT the contraction of a set T by a constant factor α, with $0 < \alpha < 1/4$. Recursively, the k-*tree* is the plane graph given by

$$T_k := T_0 \cup \bigcup_{j=0}^{2n-1} (\alpha T_{k-1} + \omega^j),$$

where $\alpha T_{k-1} + \omega^j$ denotes the algebraic sum (as sets in $\mathbb{C}$) of αT_{k-1} and the set $\{\omega^j\}$. Each $\alpha T_{k-1} + \omega^j$ is called jth *branch* of the k-tree and it is denoted by t_k^j. The factor α can be taken sufficiently small so the added copies αT_{k-1} do not intersect each other. Consequently, a simple vertex in αT_{k-1} may remain simple or become a vertex of degree 2 in T_k. We call a vertex of degree 2 *double*.

If the contraction factor is sufficiently small, we can guarantee that for each $j = 0, \ldots, 2n-1$, t_k^j lies inside the sector

$$\frac{2j-1}{4n} < \frac{\arg(z)}{2\pi} < \frac{2j+1}{4n}.$$

See Figure 4.

The set V_k can be written as the union of the subsets Ξ_k, Δ_k and Σ_k consisting of the junction, double and simple vertices in T_k. The following relations among the cardinalities of these sets can be derived by counting

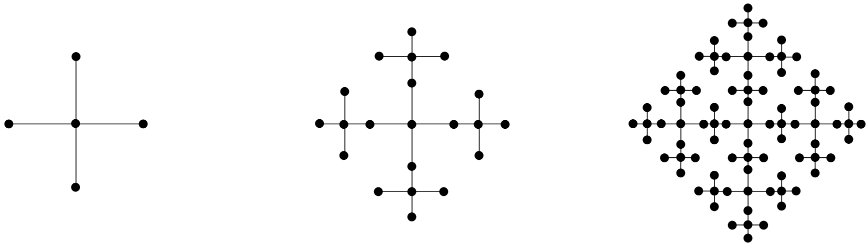


Fig. 4. The 0-, 1- and 2-trees for $n = 2$

the number of simple and double vertices generated from T_{k-1} to T_k:

$$\begin{aligned} |\Sigma_k| &= (2n-1)|\Sigma_{k-1}| + (2n-1)|\Delta_{k-1}|, \\ |\Delta_k| &= |\Sigma_{k-1}| + |\Delta_{k-1}|, \\ |\Xi_k| &= |\Xi_{k-1}| + |\Sigma_{k-1}| + |\Delta_{k-1}|. \end{aligned}$$

Since the cardinality of V_k is the sum of the cardinalities of Ξ_k, Δ_k and Σ_k, it follows from the above equations that

$$|V_k| = |V_{k-1}| + 2n(|\Sigma_{k-1}| + |\Delta_{k-1}|). \tag{4.1}$$

Solving the recursive equation by using $|V_0| = 2n$, $|\Xi_0| = 1$, $|\Delta_0| = 0$ and $|\Sigma_0| = 2n$, we obtain

$$|V_k| = \sum_{m=0}^{k+1} (2n)^m. \tag{4.2}$$

Moreover, $|\Xi_k| = \sum_{m=0}^{k} (2n)^m$ and the total number of simple and double vertices in T_k is

$$|\Sigma_k| + |\Delta_k| = (2n)^{k+1}. \tag{4.3}$$

4.2. Coloring vertices of k-trees. Let $n \geq 2$ be fixed. For each k-tree, we *color* (or label) the set of vertices $V_k^* := V_k - \{(0,0)\}$ by finite words taken from the set of admissible words $\underline{A}KS$ and describe the *coloring map* $c : V_k^* \to \underline{A}KS$ in a recursive fashion. Since the coloring of a k-tree will depend on the parity of n, the description of c will be subdivided into two cases. We remark that the vertex at the origin is not labeled.

Starting with the 0-tree, color the vertices of $V_0^* = \{\omega^k\}_{k \in A}$ by the rule

$$c(\omega^k) := k \tag{4.4}$$

for $k = 0, 1, \ldots, 2n-1$. To color vertices in T_1, consider first the vertices in the 0-branch $t_1^0 = \alpha T_0 + 1$ given by $\mathrm{Vert}(t_1^0) = \{\alpha\omega^k + 1 \mid k \in A\} \cup \{1\}$. If $v = \alpha\omega^k + 1$ then

$$c(v) := \begin{cases} 0k & \text{if } k \in \{0, n\}, \\ \underline{0}k & \text{if } 1 \leq k \leq n-1, \\ \underline{2n-1}k & \text{if } n+1 \leq k \leq 2n-1. \end{cases} \tag{4.5}$$

The coloring of the remaining vertices in $V_1 - V_0$ is given by the following relation. For any $v \in V_1 - V_0$, there exists and integer $m \geq 0$ for which $\omega^{-m}v$ is a colored vertex in $\mathrm{Vert}(t_1^0)$. If $c(\omega^{-m}v) = t_0t_1 \in \underline{A}KS$, then

$$c(v) = (t_0 + m)(t_1 + mn) \tag{4.6}$$

with addition mod $2n$.

The coloring of a k-tree can now be defined in a recursive way. First, begin with a vertex $v \in V_k - V_{k-1}$ in the 0-branch $t_k^0 = \alpha T_{k-1} + 1$. Then there exists $u \in V_{k-1} - V_{k-2}$ so that $v = \alpha u + 1$. If $c(u) = t_0t_1 \ldots t_{k-1}$ (which has been defined in the previous step) the coloring of v is given by

$$c(v) := \begin{cases} 0t_0t_1 \ldots t_{k-1} & \text{if } t_0 = 0, n, \\ \underline{0}t_0t_1 \ldots t_{k-1} & \text{if } 1 \leq t_0 \leq n-1 \\ & \text{or } \underline{1} \leq t_0 \leq \underline{n-1}, \\ \underline{2n-1}t_0t_1 \ldots t_{k-1} & \text{if } n+1 \leq t_0 \leq 2n-1 \\ & \text{or } \underline{n+1} \leq t_0 \leq \underline{2n-1}. \end{cases} \tag{4.7}$$

In this way, all vertices in the branch t_k^0 are done. To color the remaining vertices in $V_k - V_{k-1}$, observe that for each vertex v in the branch $t_k^j = \alpha T_{k-1} + \omega^j$, the vertex $\omega^{-j}v$ is in t_k^0 and has a well defined color. If $c(\omega^{-j}v) = st_0 \ldots t_{k-1}$, then for n even, let

$$c(v) := (s + j)(t_0 + jn)t_1 \ldots t_{k-1}, \tag{4.8}$$

and for n odd, set

$$c(v) := (s + j)(t_0 + jn)(t_1 + jn) \ldots (t_{k-1} + jn), \tag{4.9}$$

with addition mod $2n$. Compare this definition with the one of kneading sequences given in (3.6).

EXAMPLE 4.1. Let us derive the coloring map for the 1-tree when $n = 2$. In this case, $\omega = \exp(\pi i/2)$ and the set of symbols is $A = \{0, 1, 2, 3\}$. The vertices in V_0^* are given by $1, i, -1$ and $-i$ and by (4.4), their colorings are

$$c(1) = 0, \quad c(i) = 1, \quad c(-1) = 2, \quad c(-i) = 3.$$

In order to assign colors to vertices in T_1, we start by coloring the vertices in the 0-branch $t_1^0 = \alpha T_0 + 1$. Its set of vertices is

$$\mathrm{Vert}(t_1^0) = \{1\} \cup \{\alpha\omega^k + 1\}_{k \in A} = \{1, 1 + \alpha, 1 + i\alpha, 1 - \alpha, 1 - i\alpha\},$$

we know $c(1) = 0$, so by (4.5) the remaining colorings are

$$c(1 + \alpha) = 00, \quad c(1 + i\alpha) = \underline{0}1, \quad c(1 - \alpha) = 02, \quad c(1 - i\alpha) = \underline{3}3.$$

The remaining vertices in $V_1 - V_0$ are now derived from the formula in (4.6). For example, the vertices in the 2-branch $t_1^2 = \alpha T_0 + i$ are obtained by multiplying each $\alpha\omega^k + 1$, $k \in A$, by $\omega^m = i$, that is, $m = 1$. Thus, we obtain

$$i(1 + \alpha), \quad -\alpha + i, \quad i(1 - \alpha), \quad \alpha + i,$$

and their colorings are

$$c(i(1+\alpha)) = 12, \quad c(-\alpha+i) = \underline{1}3, \quad c(i(1-\alpha)) = 10, \quad c(\alpha+i) = \underline{0}1.$$

The colorings for the 2-tree when $n = 2$ are presented in Figure 5.

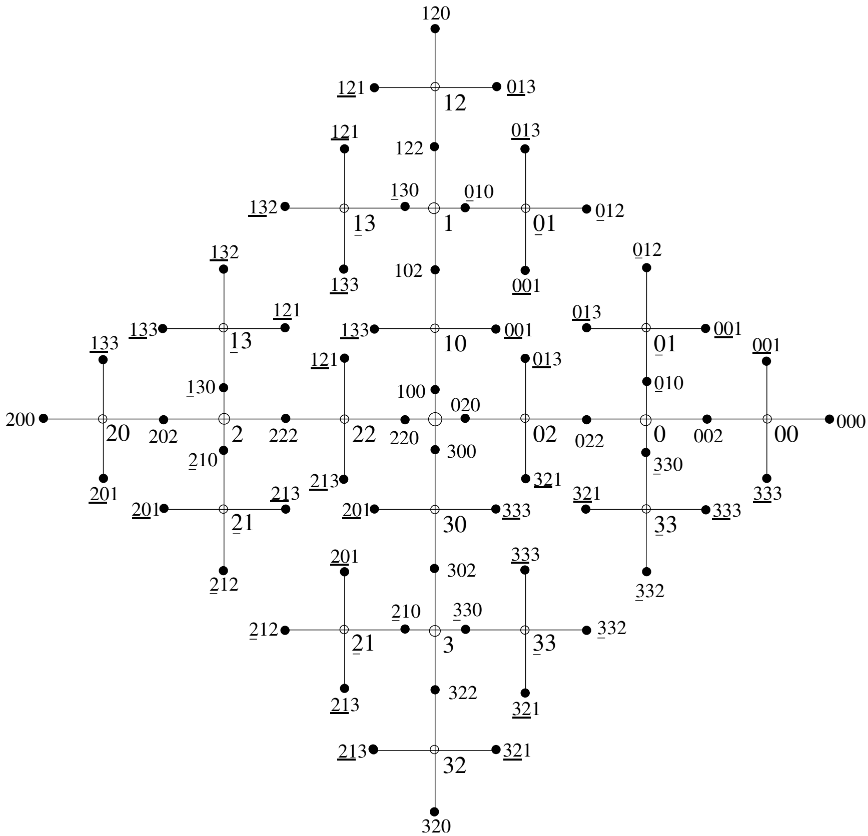


Fig. 5. The 2-tree and coloring of its vertices for $n = 2$

4.3. Paths and directions. Given a color $\underline{a_1 \ldots a_r} ks_2 \ldots s_m \in \underline{A}KS$ of length $r+m \geq 1$, its *class* consists of all those vertices in T_N, $N \geq r+m-1$, that share the same color. We write $v_{\underline{a_1 \ldots a_r} ks_2 \ldots s_m}$ to denote a vertex in this class. It is not difficult to see that for any N-tree, each color $ks_2 \ldots s_m$ has a unique vertex in its class, while $\underline{a_1 \ldots a_r} ks_2 \ldots s_m$ has exactly 2^r vertices. In order to distinguish among vertices in the same class, we introduce the notions of a *path from the origin* and its *direction* along a tree.

Definition 4.2. Consider an N-tree $T_N = (V_N, E_N)$ and a vertex v in the class of $\underline{a_1 \ldots a_r} ks_2 \ldots s_m$ with $r + m - 1 \leq N$. A *path from the origin to the vertex v* is a directed graph $\gamma(0, v) = (V_\gamma, E_\gamma)$ for which the set of vertices, $V_\gamma \subset V_N$, always contains the origin and v. Moreover, the set of edges E_γ, given by the collection of edges $\{e_1, \dots, e_t\} \subset E_N$ satisfy:

(1) The first edge is given by $e_1 = (0, y_1)$ with $y_1 \in V_j^*$ for some $0 \le j \le N$.
(2) Given $e_i = (x_i, y_i)$, then $x_i = y_{i-1}$.
(3) If $e_i = (x_i, y_i)$ and $x_i \in V_j$, then $y_i \in V_N - V_j$.

EXAMPLE 4.3. Consider a path from the origin to the vertex $v_{\underline{11}0} \in V_2$ that branches out of the vertex $v_{\underline{1}0}$ in Figure 5. By the above definition, the path $\gamma(0, v_{\underline{11}0})$ has only two edges instead of three. Indeed, these edges are $e_1 = (0, v_{10})$ and $e_2 = (v_{10}, v_{\underline{11}0})$, so we have discarded the edge $(0, v_{100})$.

DEFINITION 4.4. Given $n \ge 2$, a k-tree T_k and a path from the origin $\gamma(0, v) \subset T_k$ with t edges, the *direction of* v along T_k is a finite word $\delta = \delta_1 \dots \delta_t$ so that $\delta_i = j$ if and only if, by translating the vertex x_i of $e_i = (x_i, y_i) \in E_\gamma$ to the origin, the angle that e_i makes with the positive real line is $\pi j/n$, for $j = 0, \dots, 2n-1$.

EXAMPLE 4.5. Let $n = 2$ and consider the color $\underline{00}1$. The tree T_2 contains 2^2 vertices in the same class of this color. The four paths from the origin to a vertex $v_{\underline{00}1}$ are uniquely distinguished by the set of directions, namely

$$\delta \in \{001, 010, 10, 103\}.$$

The direction of $v_{\underline{11}0}$ given in the previous example is $\delta = 10$.

5. Dynamical k-trees. As stated in Remark 3.12, a kneading sequence $\underline{a_1 \dots a_r} k s_2 \dots s_m$ is associated to 2^r preimages of the origin, and so this combinatorial information is not enough to identify an ETS map F_λ unless $r = 0$. In this section we derive further combinatorial information about critical orbits. The key observation is that, if $r > 0$ and $r + m \le \tau - 1$, then the 2^r preimages of the origin with kneading sequence $\underline{a_1 \dots a_r} k s_2 \dots s_m$ are arranged as vertices of k-trees that arise from pullbacks of the extended rays $\mathcal{R}_j$.

Let F_λ be an ETS map with marked point $p_\lambda \in S_0$. From the construction of extended rays given at the end of Section 3.2, we immediately see that $\mathcal{R}_0 - \{0, \infty\}$ is contained in S_0. From Definition 3.10, the set

$$\mathbb{T}_0 := \bigcup_{j=0}^{2n-1} \mathcal{R}_j^b$$

is a tree-like continuum with $2n+1$ distinguished points or vertices: namely the origin (which is a vertex of degree $2n$) and $2n$ simple vertices at the prepoles. Denote by $\mathbb{V}_0 = \{0, w_0, \dots, w_{2n-1}\}$ the set of vertices and define $\mathbb{V}_0^* = \mathbb{V}_0 - \{0\}$. Also denote by $\mathbb{E}_0 = \{e_0, \dots, e_{2n-1}\}$ the set of edges of $\mathbb{T}_0$ where $e_j = \mathcal{R}_j^b$. It is not hard to see that $\mathbb{T}_0$ is topologically isomorphic (and thus homeomorphic) to a 0-tree. Most important, we can select such a homeomorphism so that the kneading sequences of the prepoles coincide with the coloring in T_0. In that case, we say the homeomorphism is *compatible* with the colorings.

Indeed, consider a homeomorphism $\varphi : \mathcal{R}_0^b \to [0,1]$ sending the origin to the origin and w_0 (the prepole that is the endpoint of $\mathcal{R}_0^b$) to 1. Defining $\varphi_0(\mathcal{R}_j^b) := \omega^j \varphi(\mathcal{R}_0^b)$ for each $j = 0, \dots, 2n-1$, we obtain a homeomorphism $\varphi_0 : \mathbb{T}_0 \to T_0$ that preserves the rotational order of edges. Moreover, since $c \circ \varphi_0(w_0) = c(1) = 0$ and $\kappa(w_0) = 0$, for each $j = 1, \dots, 2n-1$ we have

$$c \circ \varphi_0(w_j) = c \circ (\omega^j \varphi_0(w_0)) = c(\omega^j) = j = \kappa(w_j).$$

We have shown

Lemma 5.1 (Dynamical 0-tree). *There exists a homeomorphism of rooted trees*

$$\varphi_0 : (\mathbb{T}_0, w_0) \to (T_0, 1)$$

that preserves the rotational ordering of edges and sends $\mathbb{V}_0$ to V_0 in such a way that $\varphi_0(w_0) = 1$. Moreover, φ_0 can be chosen to be compatible with the colorings, that is, for all $j = 0, \dots, 2n-1$,

$$c \circ \varphi_0(w_j) = \kappa(w_j).$$

Next, we show how to extend the above homeomorphism to subdivisions of 0-trees, that is, graphs with more vertices and only $2n$ branches that run along the extended rays $\mathcal{R}_k$. Let $z_{ks_2 \dots s_m}$ be the unique point in $\mathcal{R}_k$ that is a preimage of the origin with kneading sequence $ks_2 \dots s_m$, where $k \in A = \{0, \dots, 2n-1\}$, $s_j \in \{0, n\}$. Denote by 0_m the word of m zeros.

For any given $m \geq 1$, let $[0, z_{0_m}]$ denote the arc along $\mathcal{R}_0$ with endpoints at the origin and z_{0_m}. Let

$$\mathbb{V}_{0,m} := \Big(\bigcup_{k=0}^{2n-1} \omega^k [0, z_{0_m}] \Big) \cap \Big(\bigcup_{j=0}^{m} F_\lambda^{-j}(0) \Big),$$

and denote by $\mathbb{E}_{0,m}$ the finite collection of all arcs $e = (u, v)$ so that

(1) $e \subset \omega^k [0, z_{0_m}]$ for some $k \in A$, $u, v \in \mathbb{V}_{0,m}$, and
(2) if $p \in e - \{u, v\}$ then $p \notin \mathbb{V}_{0,m}$.

Lemma 5.2 (Subdivision of 0-trees). *The set $\mathbb{T}_{0,m} = (\mathbb{V}_{0,m}, \mathbb{E}_{0,m})$ is a planar tree homeomorphic to a subgraph of T_m given by*

$$T_{0,m} := T_m \cap \bigcup_{k=0}^{2n-1} \omega^k [0, \infty).$$

The homeomorphism $\varphi_{0,m}$ can be chosen to be compatible with the colorings, that is, for all $u \in \mathbb{V}_{0,m}$, $c \circ \varphi_{0,m}(u) = \kappa(u)$.

Proof. From the definition of $\mathbb{V}_{0,m}$ and $\mathbb{E}_{0,m}$ it is enough to verify that $|\mathbb{V}_{0,m}| = |\mathbb{E}_{0,m}|+1$ to conclude this is a planar tree. Observe that the number of preimages of the origin lying on the arc $[0, z_{0_m}]$ is given by

$$1 + \sum_{j=0}^{m} 2^j = 2^{m+1},$$

since each j-preimage is surrounded by two $(j+1)$-preimages for each $0 \leq j \leq m$. Since they are 2^{m+1} vertices along the arc $[0, z_{0_m}]$, by (1) and (2) above, there has to be exactly $2^{m+1} - 1$ subarcs between these points. From rotational symmetry and after subtracting the extra $2n - 1$ copies of the origin, we obtain

$$|\mathbb{V}_{0,m}| = 2n \cdot 2^{m+1} - (2n-1) = 2n(2^{m+1} - 1) + 1 = |\mathbb{E}_{0,m}| + 1.$$

To construct a homeomorphism of rooted trees of the form

$$\varphi_{0,m} : (\mathbb{T}_{0,m}, z_{0_m}) \to (T_{0,m}, 1 + \cdots + \alpha^{m-1}),$$

it is enough to provide its definition along the arc $[0, z_{0_m}]$ and extend it through the rotational symmetries. We proceed by induction on $m \geq 1$. If $m = 1$, $\varphi_{0,1} = \varphi_0$. Now assume $\varphi_{0,m-1}$ has been constructed. Let $\varphi_{0,m}$ coincide with $\varphi_{0,m-1}$ on $\mathbb{V}_{0,m-1}$ and define

$$\varphi_{0,m}([z_{0_{m-1}}, z_{0_m}]) = [1 + \cdots + \alpha^{m-2}, 1 + \cdots + \alpha^{m-1}]$$

in the natural way. As a graph, $T_{0,m}$ contains a subdivision of $T_{0,m-1}$, so for each edge $e = (u_1, u_2) \in E_{0,m-1}$ there exists a unique point $p \in V_{0,m} - V_{0,m-1}$ such that p lies in e.

It follows from (3.5) that the arc $[0, z_{0_m}]$ is covered by 2^m topological disks in $C_{0s_2\ldots s_m}$, each disk containing a single point in $F_\lambda^{-m}(0) \cap \mathcal{R}_0$. In particular, for the pair of vertices $w_1, w_2 \in \mathbb{V}_{0,m-1}$ given by $w_i = \varphi_{0,m-1}^{-1}(u_i)$, there exists a unique $q \in \mathbb{V}_{0,m} - \mathbb{V}_{0,m-1}$, so we set $\varphi_{0,m}(q) = p$. Clearly, $\varphi_{0,m}|[0, z_{0_m}]$ is compatible with the colorings and its extension to $\mathbb{T}_{0,m}$ can be achieved so as to respect the rotational ordering of the j-branches. ∎

Regardless of the value of $m \geq 1$, the sets $\mathbb{T}_{0,m}$ and $T_{0,m}$ will be called 0-trees with the understanding that an extra index indicates a *subdivision at level m* (that is, it contains vertices whose coloring has length m). Under some technical considerations, we generalize Lemma 5.1 in order to construct k-trees in the dynamical plane as follows.

THEOREM 5.3 (Dynamical k-tree). *For each positive integer $k \leq \tau - 1$ for which $\mathbb{T}_{k-1}$ is a $(k-1)$-tree and $v_+ \notin \mathbb{T}_{k-1}$, we have:*

(1) *The set*

$$\mathbb{T}_k := \mathbb{T}_0 \cup F_\lambda^{-1}(\mathbb{T}_{k-1}) \tag{5.1}$$

is a connected plane tree whose set of vertices

$$\mathbb{V}_k := \bigcup_{j=0}^{k+1} F_\lambda^{-j}(0) \tag{5.2}$$

has cardinality $((2n)^{k+2}-1)/(2n-1)$ *and* $\mathbb{V}_k^* = \mathbb{V}_k - \{0\}$ *is colored by the kneading sequences of its elements.*

(2) *There exists a homeomorphism of rooted trees*

$$\varphi_k : (\mathbb{T}_k, z_{0_k}) \to (T_k, 1+\alpha+\cdots+\alpha^{k-1}) \tag{5.3}$$

that preserves the rotational ordering of edges and sends $\mathbb{V}_k$ *to* V_k *in such a way that* $\varphi_k(z_{0_k}) = 1+\cdots+\alpha^{k-1}$.

Moreover, φ_k *can be chosen to be compatible with the colorings, so for each vertex* $v \in \mathbb{V}_k^*$, $c \circ \varphi_k(v) = \kappa(v)$.

To prove the first part, assume $\mathbb{T}_{k-1}$ is a $(k-1)$-tree and critical values do not lie in it. That is, v_+, v_- do not belong to either $\mathbb{V}_{k-1}$ or $\mathbb{E}_{k-1}$. We want to show first that

$$\mathbb{T}_0 \cup F_\lambda^{-1}(\mathbb{T}_{k-1})$$

is a connected set that defines a planar tree. If the critical value rays $\ell_\pm$ have nonempty intersection with $\mathbb{T}_{k-1}$, we cannot guarantee that $F_\lambda^{-1}(\mathbb{T}_{k-1})$ consists of $2n$ connected components, as each inverse branch G_j has $\mathbb{C} \setminus (\ell_+ \cup \ell_-)$ as its domain. This technical issue can be solved by redefining the partition S_j as follows.

Let $\mathcal{P}_\lambda$ denote the postcritical set of F_λ, that is,

$$\mathcal{P}_\lambda = \bigcup_{i>0} \bigcup_{j=1}^{2n-1} F_\lambda^i(c_j).$$

By the Escape Trichotomy Theorem, for each $n \geq 2$ and each postcritically finite ETS parameter λ, $4 \leq |\mathcal{P}_\lambda| < \infty$. Clearly, $F_\lambda : \widehat{\mathbb{C}} - F_\lambda^{-1}(\mathcal{P}_\lambda) \to \widehat{\mathbb{C}} - \mathcal{P}_\lambda$ is an unramified covering map acting on hyperbolic domains, so each curve in $\widehat{\mathbb{C}} - \mathcal{P}_\lambda$ has $2n$ lifts.

Proposition 5.4 (Dynamical partition). *Let* $0 \leq k \leq \tau - 1$ *be the minimal integer for which* $\mathbb{T}_k$ *is a* k*-tree that does not contain critical values and has nonempty intersection with* ℓ_+. *Then there exists a curve* $\tilde{\ell}_+$ *joining* v_+ *and* $z=\infty$ *and homotopic to* ℓ_+ *in* $\widehat{\mathbb{C}} - \mathcal{P}_\lambda$. *The lifts of* $\tilde{\ell}_+, -\tilde{\ell}_+$ *define a partition of the plane into rotationally symmetric open sectors* $\tilde{S}_j$ *so that*

$$\mathbb{T}_k - \{0\} \subset \bigcup_{j=0}^{2n-1} \tilde{S}_j.$$

In particular, the inverse image of $\mathbb{T}_k$ *with respect to the new partition consists of* $2n$ *connected components, each one contained in a sector* $\tilde{S}_j$.

Proof. Let S_+ denote the critical value sector that contains v_+ as defined in Lemma 3.2. Since $\mathbb{T}_k$ has a tree-like structure and $\mathcal{P}_\lambda$ is finite, $S_+ - (\mathbb{T}_k \cup \mathcal{P}_\lambda)$ is an open and connected set, hence pathwise connected. Thus, we can define a continuous curve $\tilde{\ell}_+$ joining v_+ and infinity with the following properties:

(a) off its endpoints, $\tilde{\ell}_+$ is homotopic to ℓ_+ in $S_+ - (\mathbb{T}_k \cup \mathcal{P}_\lambda)$,
(b) $\tilde{\ell}_+ = \ell_+$ in $\overline{B_\lambda}$.

Let $\tilde{\ell}_- := -\tilde{\ell}_+$. The lifts of $\tilde{\ell}_+ \cup \tilde{\ell}_-$ are $2n$ curves $\tilde{\eta}_j$ joining the origin to the point at infinity and passing through a free critical point c_j. Due to the symmetries of F_λ, these new critical rays divide the plane into $2n$ open sectors $\tilde{S}_j$ that remain rotationally symmetric.

Now, $\mathbb{T}_k - \{0\}$ lies in the union of the new sectors. For otherwise, if there exists a point $q \in \mathbb{T}_k \cap \partial\tilde{S}_j$ for some j, then $F_\lambda(q) \in \mathbb{T}_{k-1} \cup \mathcal{R}_0 \cup \mathcal{R}_n$ and at the same time, $F_\lambda(q) \in \tilde{\ell}_+ \cup \tilde{\ell}_-$. By hypothesis, $F_\lambda(q)$ cannot lie in $\mathbb{T}_{k-1}$. Moreover, the 0- and n-extended rays are contained in the closure of $S_0 \cup S_n$, while by properties (a) and (b), $\tilde{\ell}_+ \subset \overline{S}_+$. Lemma 3.2 implies that $F_\lambda(q) = \infty$ and thus the origin is the only point of intersection between $\mathbb{T}_k$ and the closure of the new sectors $\tilde{S}_j$.

Finally, the inverse branches of F_λ, denoted by $\tilde{G}_j$, are now defined over $\mathbb{C} - (\tilde{\ell}_+ \cup \tilde{\ell}_-)$ and take values in $\tilde{S}_j$. Clearly, $\mathbb{T}_k \subset \mathbb{C} - (\tilde{\ell}_+ \cup \tilde{\ell}_-)$ and for each $j = 0, \dots, 2n-1$, $\tilde{G}_j(\mathbb{T}_k)$ is a connected set properly contained in $\tilde{S}_j$. ∎

The next result shows that for the dynamical sectors, those properties described in Lemma 3.2 remain the same.

Corollary 5.5. *If p_λ lies in S_0 then it lies in $\tilde{S}_0$. In particular, for each j, $\mathcal{R}_j - \{0,\infty\}$ is contained in $\tilde{S}_j$. If S_j is a fixed point sector, so is $\tilde{S}_j$. Moreover, $\lambda \in \mathcal{S}_{k_0-1}$ if and only if $v_+ \in \tilde{S}_+ := \tilde{S}_{k_0}$ and $v_- \in \tilde{S}_- := \tilde{S}_{k_0+n}$.*

Proof. The first statement can be derived from (b) above. Indeed, this property implies that $\tilde{S}_j$ coincides with S_j in $\overline{B}_\lambda$, so in particular p_λ and $\mathcal{R}_0^u - \{\infty\}$ lies in $\tilde{S}_0$. Moreover, any fixed point in $S_j \cap B_\lambda$ also lies in $\tilde{S}_j \cap B_\lambda$. Now assume there exists a point $q \in \mathcal{R}_0^b \cap \partial\tilde{S}_0$; then $F_\lambda(q)$ must lie in $\mathcal{R}_0 \cup \mathcal{R}_n$ and in $\tilde{\ell}_+ \cup \tilde{\ell}_-$. The same argument given in Proposition 5.4 shows that $p = 0$ and thus $\mathcal{R}_0 - \{0,\infty\} \subset S_0$. Rotational symmetries of the new sectors and extended rays imply the general case.

To see the final statement, let $\tilde{S}_+ = \tilde{S}_{k_0}$ and recall from Lemma 3.2(2) that $\lambda \in \mathcal{S}_{k_0-1}$ if and only if $S_+ = S_{k_0}$ and $S_- = S_{k_0+n}$. It is enough to show that $v_+ \in S_+ \cap \tilde{S}_+$.

Observe that ∂S_+ is a simple closed curve in $\widehat{\mathbb{C}}$ that surrounds v_+. By property (a), the curves $\tilde{\ell}_+ \cup \ell_+$ and $\tilde{\ell}_- \cup \ell_-$ separate the plane into finitely many domains. Since $\tilde{\ell}_+ \cup \tilde{\ell}_-$ is homotopic to $\ell_+ \cup \ell_-$ rel $\mathcal{P}_\lambda$, there exists a curve, β, that joins $F_\lambda(v_+)$ and the origin and is disjoint from all critical value curves. Since $\mathcal{R}_{k_0}$ lies in $\tilde{S}_+ \cap S_+$, this intersection contains the prepole w_{k_0}, and hence the lift of β that joins w_{k_0} to v_+, as needed. ∎

To avoid introducing more notation, we denote by $S_j, \ell_\pm$ and G_j the (static or dynamical) partition, critical value rays and inverse branch of F_λ that guarantees that, for k as in Proposition 5.4, $G_j(\mathbb{T}_k)$ is a connected set completely contained in S_j for each $j = 0, \dots, 2n-1$.

Returning to the proof of part (1) of Theorem 5.3, we can now assume $\mathbb{T}_{k-1} \cap \ell_\pm = \emptyset$. Thus each $G_j(\mathbb{T}_{k-1})$ lies in a sector S_j. And since G_j is a strict contraction, $G_j(\mathbb{T}_{k-1})$ is a connected set homeomorphic to a $(k-1)$-tree, where $0 \in \mathbb{T}_{k-1}$ is sent to w_j for each $j = 0, \dots, 2n-1$. Thus, the set

$$\mathbb{T}_k := \mathbb{T}_0 \cup F_\lambda^{-1}(\mathbb{T}_{k-1})$$

is a connected plane graph. Moreover, its set of vertices, $\mathbb{V}_k$, is given by the origin and all its preimages up to order $k+1$. Thus, the cardinality of $\mathbb{V}_k$ is

$$\sum_{j=0}^{k+1} (2n)^j = \frac{(2n)^{k+2} - 1}{2n - 1}.$$

Since the origin and every one of its preimages up to order k is a junction point (and thus contributes $2n$ edges), the cardinality of $\mathbb{E}_k$ is

$$2n \sum_{j=0}^{k} (2n)^j = |\mathbb{V}_k| - 1.$$

Hence, $\mathbb{T}_k$ is a planar tree. Finally, each point in $\mathbb{V}_k^*$ has a well-defined kneading sequence as described in Section 3.4.

To show the existence of a homeomorphism between the rooted trees $\mathbb{T}_k$ and T_k that is compatible with colorings, consider the homeomorphism

$$\varphi_{k-1} : (\mathbb{T}_{k-1}, z_{0_{k-1}}) \to (T_{k-1}, 1 + \cdots + \alpha^{k-2})$$

and proceed as in Lemma 5.2. By hypothesis, φ_{k-1} has been chosen to be compatible with the colorings in $\mathbb{V}_{k-1}$. In particular, $\varphi_{k-1}(0) = 0$. Set $\varphi_k = \varphi_{k-1}$ on $\mathbb{V}_{k-1}$. The set $\mathbb{V}_k - \mathbb{V}_{k-1}$ consists of simple and double vertices in $\mathbb{T}_k$ that share an edge with a simple or double vertex in $\mathbb{T}_{k-1}$. Thus, we define φ_k in a recursive way: there are $2n$ vertices in $\mathbb{V}_k - \mathbb{V}_{k-1}$ that share an edge with $z_{0_{k-1}} \in \mathbb{V}_{k-1}$. One of them is z_{0_k}, the root of $\mathbb{T}_k$ so we set $\varphi_k(z_{0_k}) = 1 + \alpha + \cdots + \alpha^{k-1}$. Then, define φ_k at the remaining $2n-1$ vertices by assigning a vertex in T_k adjacent to $1 + \cdots + \alpha^{k-2}$ in positive order.

To see that φ_k is compatible, observe that the $2n$ vertices we have considered before are simple vertices in $G_{0_k}(\mathbb{T}_0)$. Label them in positive rotational order as v_j, $j \in A$, so that $v_0 = z_{0_k}$. Hence, the kneading sequences of those points are

$$\kappa(v_j) := \begin{cases} 0_{m-1}j & \text{if } j \in \{0, n\}, \\ \underline{0}_{m-1}j & \text{if } 1 \le j \le n-1, \\ \underline{2n-1}_{m-1}j & \text{if } n+1 \le j \le 2n-1, \end{cases}$$

which coincides with the coloring for T_m given in (4.5). Similarly, by using the symmetries on both $\mathbb{T}_k$ and T_k, we can define φ_k at the vertices adjacent to each $\omega^j z_{0_k}$ for $j = 1, \dots, 2n-1$, and so on. This concludes the proof of the theorem.

PROPOSITION 5.6 (Subdivision of dynamical k-trees). *Assume* $0 \le k < \tau - 1$ *is the smallest integer for which* $\mathbb{T}_k$ *contains for the first time both critical values not as vertices, but as points along two of its edges. Then there exists a subdivision of* $\mathbb{T}_k$*, namely* $\mathbb{T}_{k,\tau-1}$*, that is homeomorphic* (*as rooted trees*) *to a subtree of* $T_{\tau-1}$ *in such a way critical values become vertices in* $\mathbb{T}_{k,\tau-1}$ *with well defined directions.*

The homeomorphism can be chosen so as to preserve the rotational ordering of edges at every junction point and to be compatible with colorings.

Proof. This is a consequence of Theorem 5.3 applied to the subdivision tree $\mathbb{T}_{0,i}$, with $i = \tau - 1 - k$. In more detail, first assume critical values lie in $\mathbb{T}_0$ but not in $\mathbb{V}_0$. If $\kappa(v_+)$ is of the form $ks_2 \dots s_m$, then it lies along the set $\bigcup_{j=0}^{2n} \mathcal{R}_j$, so there exist subgraphs of the subdivision trees $\mathbb{T}_{0,m}$ and $T_{0,m}$ that are homeomorphic as rooted trees via $\varphi_{0,m}$.

Now assume critical values do not lie in the extended rays $\mathcal{R}_j$, so $0 < k < \tau - 1$, and assume $\mathbb{T}_{0,i}$ has been computed. Clearly, critical values do not lie in this 0-tree, so we can compute

$$\mathbb{T}_{1,i+1} = \mathbb{T}_{0,i} \cup F_\lambda^{-1}(\mathbb{T}_{0,i}),$$

which is homeomorphic to a 1-tree with subdivision at level $i+1$. In particular, $\mathbb{T}_{1,i+1}$ is homeomorphic to a subgraph of T_{i+1}. In a recursive manner, we can compute $\mathbb{T}_{j,i+j}$ for each $0 \le j < k$, as critical values do not lie in any of these sets. Finally, the set

$$\mathbb{T}_{k,\tau-1} = \mathbb{T}_{0,i} \cup F_\lambda^{-1}(\mathbb{T}_{k-1,i+k-1})$$

becomes a dynamical k-tree with subdivisions at level $\tau - 1$, hence the critical values must belong to its set of vertices. The homeomorphism between $\mathbb{T}_{k,\tau-1}$ and the subgraph $T_{k,\tau-1} \subset T_{\tau-1}$ is derived from Lemma 5.2 and Theorem 5.3. ∎

6. Combinatorial invariant. The results in the previous sections lead to the definition of the combinatorial information of a postcritically finite ETS map.

Consider a center ETS parameter λ of escape time $\tau \geq 2$ on any of the parameter sectors $\mathcal{S}_j$. If p_λ denotes the marked fixed point in $\mathcal{S}_j$, then by Lemma 3.4, the sectors S_k have been relabeled with respect to p_λ in the orientation described there. Denote $v_\lambda := v_+ \in S_{k_0}$ and $-v_\lambda \in S_{k_0+n}$.

From the results leading to Proposition 5.6 above, there exist integers $0 \leq k \leq \tau - 1$ and $i \geq 0$, with $k + i = \tau - 1$, so that $\mathbb{T}_{k,k+i}$ is the smallest k-tree with subdivisions at level $\tau - 1$ that contains the critical values as vertices for the first time. We denote this dynamical tree by $\mathbb{T}_\lambda$ from now on. Analogously, denote by φ_λ the homeomorphism of rooted trees from $\mathbb{T}_\lambda$ to either the $(\tau-2)$-tree $T_{\tau-2}$ or to its subgraph $T_{k,\tau-1}$ with subdivisions at level $\tau - 1$.

Definition 6.1 (Combinatorial information). For $\mathbb{T}_\lambda$ given as above, we write

$$\kappa_\lambda = \kappa(v_\lambda)$$

to denote the kneading sequence of the critical value v_λ. If $\gamma(0, v_\lambda)$ is a path in $\mathbb{T}_\lambda$ joining 0 and v_λ, then

$$\delta_\lambda = \delta_1 \ldots \delta_t$$

denotes the direction of the vertex $\varphi_\lambda(v_\lambda)$ along $T_{k,i}$. The pair $(\kappa_\lambda, \delta_\lambda)$ is the *combinatorial information* of F_λ.

Remark 6.2. Observe that whenever $\delta(v) = \delta(u)$, then $u = v$ and thus they have the same coloring. On the other hand, if the colorings are the same, the directions may be different. Yet, if $\kappa(v) \neq \kappa(u)$ then $\delta(v) \neq \delta(u)$.

Our next result shows the existence of a bijective correspondence between postcritically finite ETS parameters and a subset of $V_{k,\tau-1}$ for some $k \geq 0$. Recall from §4 that t_k^j denotes the jth branch of a k-tree; it is a $(k-1)$-tree itself whenever $k > 0$, while t_0^j is just the vertex ω^j. Denote by $\mathrm{Vert}(t_k^j)$ the set of vertices of t_k^j.

Theorem 6.3 (Realization Theorem). *Fix any $n \geq 2$ and $k \geq 0$. Let T_k denote the k-tree with $2n$ rotational symmetry and color map c. For any given vertex $z \in V_k^*$, let $c(z)$ denote its color and $\delta(z) = \delta_1 \ldots \delta_t$ the direction of z. Then $(c(z), \delta(z))$ is realized as the combinatorial information (with respect to the basic configuration) of an ETS map of degree $2n$ if and only if $\delta_1 = k_0$.*

Proof. The necessity can be seen as follows. If F_λ is a $2n$ degree map that realizes $(c(z), \delta(z))$ as its combinatorial pair with respect to p_λ, then by Lemma 3.4, $v_+ \in S_{k_0}$. By Theorem 5.3 (and in particular Proposition 5.4), the branch of the dynamical tree where v_+ lies is completely contained in S_{k_0}. Thus $\delta_1 = k_0$.

Now assume the pair $(c(z), \delta(z))$ has been given and $\delta_1 = k_0$. Let $c(z) = \underline{a_1 \dots a_r} k_0 s_2 \dots s_m$, so it has length $r + m \geq 1$, as $r \geq 0$ and $m \geq 1$ (see Section 4.2). Let $\tau = r+m+1$. We show the existence of a bijection between the set

$$\Lambda_\tau = \{\lambda \in H_\tau \cap \mathcal{S}_{k_0 - 1} \mid F_\lambda^{\tau-1}(v_\pm) = 0\}$$

and the subset of vertices in the branch $t_k^{k_0}$ colored by words of length $r+m$, that is,

$$Z_{r+m} = \{v \in \mathrm{Vert}(t_k^{k_0}) \mid |c(v)| = r + m\}.$$

For any $k \geq 0$, each simple or double vertex in the tree T_k is colored by a word of length $k + 1$, and these vertices become junction points in T_{k+1}. Thus, to compute the number of simple and double vertices in T_k, we use the formula (4.3) to obtain

$$|\Sigma_{r+m-1}| + |\Delta_{r+m-1}| = (2n)^{r+m}.$$

Since there are $2n$ branches in T_{r+m-1}, the number of vertices in Z_{r+m} is exactly $(2n)^{r+m-1}$.

On the other hand, it was shown in [8] that the number of center ETS parameters of escape time τ in the $2n$ degree family is $(n-1)(2n)^{\tau-2}$. Thus, on each sector $\mathcal{S}_j$ we have exactly $(2n)^{\tau-2} = (2n)^{r+m-1}$ of these parameters. Hence Λ_τ and Z_{r+m} have the same cardinality.

Let $\psi : \Lambda_\tau \to Z_{r+m}$ be defined by $\psi(\lambda) = \varphi_\lambda(v_\lambda)$, where φ_λ is the homeomorphism between the dynamical tree $\mathbb{T}_\lambda$ and (a subtree of) T_{r+m-1}. In other words, ψ assigns to each parameter in Λ_τ a simple or double vertex in the branch $t_{r+m-1}^{k_0}$ that defines the direction of its critical value $v_\lambda = v_+$, that is, $\delta_\lambda = \delta(\psi(\lambda))$. We show that ψ is one-to-one.

Assume $\lambda, \mu \in \Lambda_\tau$ are given, so that $\psi(\lambda) = \psi(\mu) = z \in Z_{r+m}$. In particular, this implies that critical values v_λ and v_μ have the same kneading sequence, $\kappa(z)$, and the same direction, $\delta(z)$. If this is the case, then F_λ and F_μ have to be *combinatorially equivalent*, that is, there exists a pair of orientation preserving homeomorphisms $\theta_0, \theta_1 : (\widehat{\mathbb{C}}, \mathcal{P}_\lambda) \to (\widehat{\mathbb{C}}, \mathcal{P}_\mu)$ so that $\theta_0 \circ F_\lambda = F_\mu \circ \theta_1$ and θ_0 is isotopic to θ_1 rel $\mathcal{P}_\lambda$.

To see this, assume without loss of generality that both φ_λ and φ_μ are orientation preserving homeomorphisms. Then there exists an orientation preserving homeomorphism $h : \mathbb{T}_\lambda \to \mathbb{T}_\mu$ given by $h = \varphi_\mu^{-1}\varphi_\lambda$ and such that $h(v_\lambda) = v_\mu$. Moreover, h preserves the ordering of edges at each junction point in $\mathbb{T}_\lambda$. Thus, by Theorem 1 in [1], h can be extended to an orientation preserving homeomorphism of the sphere, $H : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, that agrees with h on the dynamical trees. In fact, since $H \circ F_\lambda(v_\lambda) = F_\mu \circ H(v_\lambda) = F_\mu(v_\mu)$, H conjugates F_λ and F_μ on their postcritical sets. That is, F_λ and F_μ are combinatorially equivalent. By Theorem 1.1, this implies that $\lambda = \beta^{2j}\mu$ (or

$\lambda = \beta^{2j}\bar{\mu}$ in the orientation reversing case) for some $j \in \mathbb{Z}$. Then again, as $\lambda, \mu \in \Lambda_\tau$, we conclude $\lambda = \mu$ (or $\lambda = \bar{\mu}$ in the orientation reversing case).

Since ψ is a bijection between Λ_τ and Z_{r+m}, given $(c(z), \delta(z))$ for a vertex $z \in Z_{r+m}$, there exists a unique parameter $\lambda_z = \psi^{-1}(z) \in \Lambda_\tau$ so that F_{λ_z} realizes $(c(z), \delta(z))$ as its combinatorial pair. ∎

The correspondence just defined implies that $(\kappa_\lambda, \delta_\lambda)$ is a full invariant of topological conjugacy. Its proof is based on the algebraic characterization of conjugacy classes given in Theorem 1.1.

THEOREM 6.4 (Conjugacy invariant). *Let F_λ and F_μ be two postcritically finite ETS maps of the same degree $n \geq 2$. Then the maps are topologically conjugate on their Julia sets if and only if the maps have the same combinatorial information, that is, $\delta_\lambda = \delta_\mu$ (and thus $\kappa_\lambda = \kappa_\mu$).*

Proof. Assume first that F_λ and F_μ are conjugate on their Julia sets under an orientation preserving homeomorphism. As shown in [8], this homeomorphism can be extended to the sphere, and in particular F_λ and F_μ are conjugate in $\widehat{\mathbb{C}}$ by a Möbius transformation of the form $M(z) = \beta^j z$, where $\beta^{n-1} = 1$ and $j \in \mathbb{Z}$. From the conjugacy equation

$$M \circ F_\lambda(z) = F_\mu \circ M(z)$$

one can derive that $\mu = \beta^{2j}\lambda$. Moreover, the marked fixed point $p_\lambda \in \partial B_\lambda$ is sent to $M(p_\lambda) = \beta^j p_\lambda$, which is clearly a fixed point in ∂B_μ. In fact, $M(p_\lambda)$ realizes the basic configuration for μ. Indeed, since $v_\mu = 2\sqrt{\mu} = 2\sqrt{\beta^{2j}\lambda} = \alpha^j v_\lambda = M(v_\lambda)$ (and thus $-v_\mu = -\beta v_\lambda$), the sectors containing p_λ and $v_\lambda, -v_\lambda$ are sent to the sectors containing $M(p_\lambda)$ and $v_\mu, -v_\mu$. By Lemma 3.4, $\beta^j p_\lambda$ realizes the basic configuration for μ with (at least) the same orientation selected for λ. Thus $p_\mu = M(p_\lambda)$ if n is even, otherwise we may also have the possibility that $p_\mu = -M(p_\lambda)$.

Because $M(v_\lambda) = v_\mu$, by the Realization Theorem (and regardless of the parameter sector they belong to), both parameters correspond to the same vertex in Z_{r+m}. Thus the ETS maps have the same combinatorial information with respect to their marked fixed points.

Now assume F_λ and F_μ realize the same combinatorial information. That means they have the same escape time $\tau = r + m + 1 \geq 2$. From the proof of the Realization Theorem, F_λ and F_μ are combinatorially equivalent by an orientation preserving (or reversing) homeomorphism that preserves the postcritical sets. Thus, $\mu = \beta^{2j}\lambda$ (or $\mu = \beta^{2j}\bar{\lambda}$) for some $j \in \mathbb{Z}$. ∎

As pointed out in the Introduction, maps associated to parameters in the same Sierpiński domain are quasiconformally conjugate on their Julia sets, so they belong to the same conjugacy class of its domain center. By associating to each parameter in a Sierpiński domain the same kneading sequence and directions defined for its center parameter, we have the following result.

COROLLARY 6.5. *Let F_λ and F_μ be two ETS maps of the same degree.*

(1) *If λ and μ belong to the same Sierpiński domain, then $\delta_\lambda = \delta_\mu$ (and thus $\kappa_\lambda = \kappa_\mu$).*
(2) *If λ and μ belong to Sierpiński domains of distinct escape time, then $\kappa_\lambda \neq \kappa_\mu$ (and thus $\delta_\lambda \neq \delta_\mu$).*

Each $\mathcal{S}_j$ is a sector of angular width $2\pi/(n-1)$ and β is a primitive $(n-1)$th root of unity, so for any $\lambda \in \mathcal{S}_j$, the parameter $\beta^{2j}\lambda$ (or $\beta^{2j}\bar{\lambda}$) belongs to $\mathcal{S}_j$ if and only if $\beta^{2j} = 1$. Counting the number of topological conjugacy classes of maps of escape time τ in any given sector $\mathcal{S}_j$ is thus equivalent to counting the number of parameters in $\Lambda_\tau/\!\sim$ where $\lambda \sim \mu$ if and only if $\mu = \bar{\lambda}$. In terms of the set Z_{r+m}, this is the same as counting the number of vertices in $Z_{r+m}/\!\sim$, where $u \sim v$ if and only if $u = -\bar{v}$.

Recall from the description of the combinatorial model in Section 4 that the branch $t^{k_0}_{r+m-1}$ lies inside the sector

$$\frac{2k_0-1}{4n} < \frac{\arg(z)}{2\pi} < \frac{2k_0+1}{4n}.$$

When $n = 2k_0+1$, then $t^{k_0}_{r+m-1}$ lies in the first quadrant, so there are no identifications in Z_{r+m} under $z \mapsto -\bar{z}$, and we conclude

$$|Z_{r+m}/\!\sim| = (2n)^{\tau-2}.$$

If $n = 2k_0$, the k_0-branch runs along the imaginary axis since $\omega^{k_0}[0, 1 + \cdots + \alpha^{r+m-2}] \subset t^{k_0}_{\tau-1}$ and $\omega^{k_0} = i$. So the only vertices that are identified are half of those lying outside the imaginary axis. Adding the number of vertices of Z_{r+m} in $\omega^{k_0}[0, 1 + \cdots + \alpha^{r+m-2}]$, we obtain

$$|Z_{r+m}/\!\sim| = \frac{(2n)^{\tau-2}}{2} + 2^{\tau-3},$$

as desired. We have derived

COROLLARY 6.6. *The number of distinct topological conjugacy classes of ETS maps of escape time τ is $(2n)^{\tau-2}$ if n is odd, and $(2n)^{\tau-2}/2 + 2^{\tau-3}$ if n is even.*

7. Final remarks. From the bijection constructed in the Realization Theorem, we can identify each Sierpiński component $U \in \mathcal{S}_j$ of escape time τ with the color and direction assigned to a unique vertex in the branch $t^{k_0}_{\tau-2}$.

Now consider a map F_λ such that, after a finite number of iterates, the critical orbit lies on the forward invariant Cantor set Γ_λ. Both Proposition 5.4 and Corollary 5.5 remain valid with the new assumption. Thus, we can associate to F_λ a kneading sequence with periodic s-part (that is, $\underline{a_1 \dots a_r}k_0\overline{s_2 \dots s_l}$) or with infinite s-part ($\underline{a_1 \dots a_r}k_0 s_2 s_3 \dots$). We can also find the smallest dynamical tree $\mathbb{T}_\lambda$ that contains for the first time critical values along its edges. Similarly, there exists a homeomorphism φ_λ between $\mathbb{T}_\lambda$ and a tree T_{r+1}, so if the edge $(u, w) \subset \mathbb{E}_\lambda$ contains v_λ, then $\kappa(u) = \underline{a_1 \dots a_r}k_0$ and $\kappa(w) = \underline{a_1 \dots a_r}k_0 s_2$. Finally, the direction δ_λ is defined as the direction of the vertex $\varphi_\lambda(w) \in T_{r+1}$.

Thus, the necessity of the Realization Theorem holds true for these parameters. We do not attempt to show the sufficiency, although we expect it to be true. We conjecture that parameter values for which the kneading sequence of the map can be associated to words with only infinite $\underline{a}$-part (that is, $\kappa(v_\lambda) = \underline{a_1 a_2 \dots}$) correspond to buried points in the connectedness locus of the family. It is not hard to show that critical orbits must also be buried in the Julia set.

It has been shown in [5] that there exist Cantor necklaces in the parameter plane, so our results suggest that Sierpiński components and parameters whose critical orbits have kneading sequences with infinite s-part, are arranged in the parameter plane following a tree-like structure. A result in this direction can be found in [6] for the case $n = 2$.

Acknowledgements. I want to thank S. Macías for providing me with the reference in [1] and the referee for her/his suggestions that greatly improved the exposition of this article. This work was supported by Consejo Nacional de Ciencia y Tecnología del Estado de Guanajuato (grant CB-2006-01, 07-02-K622-069 A02) and Consejo Nacional de Ciencia y Tecnología (grant CB-2010-01, 153850).

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Received 8 May 2012;
in revised form 12 June 2013