On a measure theoretic characterization of polynomial Julia sets

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The Julia set $J(f)$ of a non-constant rational map f of degree $d \geq 2$, acting on the Riemann sphere $\widehat{\mathbb{C}}$, can be considered as the set where the iterates of f behave in a chaotic manner. We may ask if there are any conditions which could distinguish if a given Julia set came up from a polynomial map or from a genuine rational map, that is a meromorphic function on $\widehat{\mathbb{C}}$ with at least one finite pole. In 1965 Brolin proved that, if f is a polynomial, then the equilibrium measure ν associated to $J(f)$ and the measure of maximal entropy $\mu(f)$ of f are equal. Intuitively, the equilibrium measure of $J(f)$ resembles a mass charge distribution produced when a set of charges reaches an equilibrium in $J(f)$. On the other hand the measure of maximal entropy $\mu(f)$ measures the proportion of preimages in $f^{-n}(a)$ that accumulate about a given subset of $J(f)$, when $n \to \infty$ and a has an infinite backward orbit. In 1984 Lopes proved that for a rational map f satisfying $f(\infty) = \infty \in J(f)$, if the equilibrium measure and the maximum entropy measure coincide then f is a polynomial. And in 2011 Okuyama and Stawiska [OS11] relaxed Lopes' hypothesis by requiring that ∞ lies in a forward invariant Fatou component. The present work is aimed to give enough details to comprehend the next result.

Theorem (Okuyama & Stawiska)**.** *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. If ∞ lies in a forward invariant Fatou component and $\mu(f) = \nu$ then f is a polynomial. Conversely, if f is a polynomial then $\mu(f) = \nu$.*

We also explore several consequences of the theory covered: For example, given a rational map f, using Lemma 3.3 from [OS11] we were able to find an upper bound for the logarithmic potential $p_\mu(f)$ restricted to the Julia set. Particularly, when f is a polynomial, studying this upper bound guided us to find another proof of the equivalence $\mu(f) = \nu$. The proof relies strongly in Lemma 3.1 also proved by [OS11]. We also state several queries or questions that we found interesting for further exploration on the subject.