SULLIVAN'S PROOF OF FATOU'S NO WANDERING DOMAIN CONJECTURE

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ABSTRACT. A self-contained and simplified version of Sullivan's proof, following N. Baker and C. McMullen.

§1. Set up. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Let $J(f)$ and $F(f)$ denote the Julia set and the Fatou set of f, respectively. Recall that the open set $F(f)$ consists of all points near which the family of iterates $\{f^{\circ n}\}$ is normal, and $J(f) = \widehat{\mathbb{C}} \smallsetminus F(f)$. The Julia set also coincides with the closure of the set of repelling periodic points of f. Every connected component of $F(f)$ is called a *Fatou component*. The image $f(U)$ of a Fatou component U is itself a Fatou component and the mapping $f : U \to f(U)$ is proper of some degree $\leq d$.

Theorem (Sullivan). *Every Fatou component U of f is eventually periodic, that is, there exist $n > m > 0$ such that $f^{\circ n}(U) = f^{\circ m}(U)$.*

The idea of the proof is as follows: Assuming there exists a *wandering* Fatou component U (or simply a *wandering domain*), we change the conformal structure of the sphere along the grand orbit of U to find an infinite-dimensional family of rational maps of degree d, all quasiconformally conjugate to f. This is a contradiction since the space Rat_d of rational maps of degree d, as a Zariski open subset of $\mathbb{CP}^{2d+1}$, is finite-dimensional.

Remark. The corresponding statement for entire maps is false. For example, the map $z \mapsto z + \sin(2\pi z)$ has wandering domains.

§2. A reduction. The following observation drastically simplifies part of Sullivan's original argument.

Lemma (Baker). *If U is a wandering domain, then $f^{\circ n}(U)$ is simply-connected for all large n.*

Proof. Let $U_n = f^{\circ n}(U)$. Replacing U by U_k for some large k if necessary, we may assume that no U_n contains a critical point of f, so that $f^{\circ n} : U \to U_n$ is a covering map for all n. We can also arrange that $\infty \in U$. Since the U_n are disjoint subsets of $\mathbb{C} \smallsetminus U$ for $n \geq 1$, we have $\mathrm{area}(U_n) \to 0$. But $\{f^{\circ n}|_U\}$ is a normal family, so every

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convergent subsequence of this sequence must be a constant function. In particular, $\text{diam}(f^{\circ n}(K)) \to 0$ for every compact set $K \subset U$.

Now take any loop $\gamma \subset U$ and set $\gamma_n = f^{\circ n}(\gamma) \subset U_n$. By the above argument $\text{diam}(\gamma_n) \to 0$. If B_n is the union of the bounded components of $\mathbb{C} \smallsetminus \gamma_n$, it follows that $\text{diam}(B_n) \to 0$ also. Since $f(B_n)$ is open, $\partial f(B_n) \subset \gamma_{n+1}$, and $\text{diam}\, f(B_n) \to 0$, we must have $f(B_n) \subset \overline{B_{n+1}}$ for large n. In particular, the iterated images of B_n are subsets of $\mathbb{C} \smallsetminus U$ for large n. Montel's theorem then implies $B_n \subset F(f)$, which gives $B_n \subset U_n$. Thus γ_n is null-homotopic in U_n for large n. Since $f^{\circ n} : U \to U_n$ is a covering map, we can lift this homotopy to U. This proves that U is simply connected. □

§3. Constructing deformations. Let f have a wandering domain U. In view of the above lemma, we can assume that $U_n = f^{\circ n}(U)$ is simply-connected and $f : U_n \to U_{n+1}$ is a conformal isomorphism for all $n \geq 0$. Given an L^∞ Beltrami differential μ defined on U, we can construct an f-invariant L^∞ Beltrami differential on $\widehat{\mathbb{C}}$ as follows. Use the forward and backward iterates of f to spread μ along the grand orbit

$$\text{GO}(U) = \{z \in \widehat{\mathbb{C}} : f^{\circ n}(z) \in U_m \text{ for some } n, m \geq 0\}.$$

On the complement $\widehat{\mathbb{C}} \smallsetminus \text{GO}(U)$, set $\mu = 0$. The resulting Beltrami differential is defined almost everywhere on $\widehat{\mathbb{C}}$, it satisfies $f^*\mu = \mu$ by the way it is defined, and $\|\mu\|_\infty < \infty$ since spreading $\mu|_U$ along $\text{GO}(U)$ by the iterates of the holomorphic map f does not change the dilatation. Now consider the deformation $\mu_t = t\mu$ for $|t| < \varepsilon$, where $\varepsilon > 0$ is small enough to guarantee $\|\mu_t\|_\infty < 1$ if $|t| < \varepsilon$. Note that since f is holomorphic, f^* acts as a linear rotation, so $f^*\mu_t = \mu_t$. Let $\varphi_t = \varphi^{\mu_t} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the normalized solution of the Beltrami equation $\overline{\partial}\varphi_t = \mu_t\, \partial\varphi_t$ which fixes $0, 1, \infty$. It is easy to see that $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$ is a rational map of degree d, and $t \mapsto f_t$ is holomorphic, with $f_0 = f$. The infinitesimal variation

$$w(z) = \frac{d}{dt}\Big|_{t=0} f_t(z)$$

defines a holomorphic vector field whose value at z lies in the tangent space $T_{f(z)}\widehat{\mathbb{C}}$. In other words, w can be thought of as a holomorphic section of the pull-back bundle $f^*(T\widehat{\mathbb{C}})$ which in turn can be identified with a tangent vector in $T_f\text{Rat}_d$. This is the so-called *infinitesimal deformation* of f induced by μ. We say that μ induces a *trivial* deformation if $w = 0$.

Another way of describing w is as follows: First consider the unique quasiconformal vector field solution to the $\overline{\partial}$-equation $\overline{\partial} v = \mu$ which vanishes at $0, 1, \infty$. This is precisely the infinitesimal variation $\frac{d}{dt}|_{t=0}\, \varphi_t(z)$ of the normalized solution of the Beltrami equation. It is not hard to check that $w = \delta_f v$, where

$$\delta_f v(z) = v(f(z)) - f'(z)v(z)$$

measures the deviation of v from being f-invariant. Note in particular that $\delta_f v$ is holomorphic even though v is only quasiconformal, and that $w = \delta_f v$ depends linearly on μ, a fact that is not immediately clear from the first description of w. It follows that μ induces a trivial deformation if and only if v is f-invariant.

It is easy to see that the triviality condition $\delta_f v = 0$ forces v to vanish on the Julia set $J(f)$. In fact, let $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0$ be a repelling cycle of f with multiplier λ. Then the condition $\delta_f v = 0$ implies $v(z_{j+1}) = f'(z_j)v(z_j)$ for all $j = 0, \ldots, n-1$, so that

$$\prod_{j=0}^{n-1} v(z_j) = \lambda \cdot \prod_{j=0}^{n-1} v(z_j).$$

Since $|\lambda| > 1$, it follows that $v(z_j) = 0$ for some, hence for all j. Now $J(f)$ is the closure of such cycles and v is continuous, so $v(z) = 0$ for all $z \in J(f)$.

§4. The proof. The above construction gives well-defined linear maps

$$B(U) \overset{i}{\hookrightarrow} B(\widehat{\mathbb{C}}, f) \xrightarrow{D} T_f \mathrm{Rat}_d \tag{1}$$

Here $B(U)$ is the space of L^∞ Beltrami differentials in U, $B(\widehat{\mathbb{C}}, f)$ is the space of f-invariant L^∞ Beltrami differentials on $\widehat{\mathbb{C}}$, and D is the linear operator $D\mu = w = \delta_f v$ constructed above.

Lemma. *$B(U)$ contains an infinite-dimensional subspace $N(U)$ of compactly supported Beltrami differentials with the following property: If $\mu \in N(U)$ satisfies $\mu = \overline{\partial} v$ for some quasiconformal vector field v with $v|_{\partial U} = 0$, then $\mu = 0$.*

Assuming this for a moment, let us see how this implies the theorem. Consider the above subspace $N(U)$ for a simply-connected wandering domain U and restrict the diagram (1) to this subspace. If $D(\mu) = 0$ for some $\mu \in N(U)$, or in other words if μ induces a trivial deformation, that means the normalized solution v to $\overline{\partial} v = \mu$ is f-invariant. Hence $v = 0$ on $J(f)$ and in particular on the boundary of U. By the property of $N(U)$, $\mu = 0$. This means that the infinite-dimensional subspace $N(U)$ injects into $T_f \mathrm{Rat}_d$ whose dimension is $2d + 1$. The contradiction shows that no wandering domain can exist.

It remains to prove the Lemma. Let us first consider the corresponding problem for the unit disk $\mathbb{D}$. Let $\widehat{N}(\mathbb{D}) \subset B(\mathbb{D})$ be the linear span of the Beltrami differentials $\mu_k(z) = \overline{z}^k \frac{d\overline{z}}{dz}$ for $k \geq 0$. The vector field

$$V_k(z) = \begin{cases} \dfrac{1}{k+1}\, \overline{z}^{k+1} \dfrac{\partial}{\partial z} & |z| < 1 \\[2ex] \dfrac{1}{k+1}\, z^{-(k+1)} \dfrac{\partial}{\partial z} & |z| \geq 1 \end{cases}$$

solves the equation $\overline{\partial} V_k = \mu_k$ on $\mathbb{D}$. Let $\mu = \overline{\partial} v \in \widehat{N}(\mathbb{D})$ and $v|_{\partial \mathbb{D}} = 0$, and take the appropriate linear combination V of the V_k which solves $\overline{\partial} V = \mu$. Then $V - v$ is holomorphic in $\mathbb{D}$ and coincides with V on the boundary $\partial\mathbb{D}$. This is impossible if $V|_{\partial\mathbb{D}}$ has any negative power of z in it. Hence $\mu = 0$. To get the compact support condition, let $N(\mathbb{D}) \subset B(U)$ consist of all Beltrami differentials which coincide with an element of $\widehat{N}(\mathbb{D})$ on the disk $|z| < 1/2$ and are zero on $1/2 \leq |z| < 1$. If $\mu = \overline{\partial} v \in N(\mathbb{D})$ and $v|_{\partial\mathbb{D}} = 0$, then v has to be zero on the annulus $1/2 < |z| < 1$ since it is holomorphic there. In particular, it is zero on $|z| = 1/2$. Now the same argument applied to the disk $|z| < 1/2$ shows $\mu = 0$.

For the general case, consider a conformal isomorphism $\psi : \mathbb{D} \xrightarrow{\cong} U$ with the inverse $\phi = \psi^{-1}$ and define $N(U) = \phi^*(N(\mathbb{D}))$. Let $v = v(z)\frac{\partial}{\partial z}$ be a quasiconformal vector field such that $\mu = \overline{\partial} v \in N(U)$ and $v|_{\partial U} = 0$. Then $\phi_*(v) = v(\psi(z))/\psi'(z)\,\frac{\partial}{\partial z}$ is a vector field on $\mathbb{D}$ which is holomorphic near the boundary $\partial\mathbb{D}$ and $v(\psi(z)) \to 0$ as $|z| \to 1$. By the reflection principle, $v(\psi(z))$ is identically zero near the boundary of $\mathbb{D}$. Since $\psi^*\mu = \overline{\partial}\phi_*(v) \in N(\mathbb{D})$, we must have $\psi^*\mu = 0$, which implies $\mu = 0$. $\square$

Remark. Sullivan's original argument [Ann. of Math. **122** (1985) 401-418] had to deal with two essential difficulties: (i) the possibility of U being non simply-connected, perhaps of infinite topological type; (ii) the possible complications near the boundary of U, for example when ∂U is not locally-connected. He addressed the former by using a direct limit argument, and the latter by using Carathéodory's theory of "prime ends." Both of these difficulties are surprisingly bypassed in the present proof.