THE LEBESGUE DIFFERENTIATION THEOREM VIA NONOVERLAPPING INTERVAL COVERS

Abstract

A short proof is given for the Lebesgue Differentiation Theorem using a variation of the Heine-Borel covering property, without reliance on sophisticated approaches such as Vitali covers and the rising sun lemma.

In this paper we use a variation of the Heine-Borel covering property to prove the theorem due to Lebesgue that every monotone function \( f : [a, b] \to \mathbb{R} \) is differentiable almost everywhere. The approach is more accessible than typical treatments that use Vitali covers, the rising sun lemma or other methods [1, 2, 3, 4, 5]. Throughout \( \lambda \) represents Lebesgue measure on the real line.

A family of nondegenerate compact intervals \( C \) is a right adapted interval cover of a set \( E \subseteq \mathbb{R} \) if for each \( x \in E \) there is an interval \( [L(x), R(x)] \in C \) such that \( L(x) < x < R(x) \) and \( [s, R(x)] \in C \) for all \( s \in [L(x), x] \). The term left adapted interval cover is defined similarly, and we refer to either of these as an adapted interval cover. We say that a family of compact intervals is nonoverlapping if the interiors of the intervals are pairwise disjoint.

Covering Lemma. If \( C \) is an adapted interval cover of a compact set \( K \subseteq \mathbb{R} \), then there is a finite collection of nonoverlapping intervals in \( C \) that covers \( K \).

Proof. Without loss of generality, suppose that \( C \) is right adapted. Let \( a = \min K \) and \( b = \max K \) and let \( A \) be the set of all \( t \in [a, b] \) such that \( C \) has a finite nonoverlapping subcover of \( [a, t] \cap K \). Then \( a \in A \), so \( A \) is nonempty. Let \( \beta = \sup A \). We first show that \( \beta \in K \). Otherwise, \( \beta \) lies in a component \( (c, d) \) of \( [a, b] \setminus K \) and there is a finite collection \( D \) of nonoverlapping intervals in \( C \) that covers \( [a, \beta] \cap K \). Then \( D \) can be modified by deleting extraneous

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intervals to the right of $c$ and adding $[d, R(d)]$. This contradicts $\beta < d$, so $\beta \in K$.

Now let $t \in (L(\beta), \beta] \cap A$ and choose any finite nonoverlapping collection $D$ of intervals in $C$ that covers $[a, t] \cap K$. If $[r, s]$ is the right-most interval of $D$ that contains $t$, then either $s \geq b$ in which case $b \in A$ as desired or $s \leq \beta$ and $D$ can be modified to include $[s, R(\beta)]$. Then $\min\{R(\beta), b\} \in A$, which is impossible unless $b = \beta \in A$. \hfill $\square$

The key to the proof of the main theorem is a growth lemma for monotone functions in terms of Dini derivates. As usual, the upper right-hand Dini derivate is given by

$$D^+ f(x) = \inf_{\alpha > 0} \sup_{0 < h < \alpha} \frac{f(x + h) - f(x)}{h},$$

and the other derivates $D_+$, $D_-$ and $D_-$ are defined similarly. It is well known that the derivates of a monotone function are measurable.

**Growth Lemma.** Suppose that $f$ is strictly increasing on $[a, b]$. Let $C$ be the set of points in $(a, b)$ at which $f$ is continuous and let $E$ be a Borel subset of $C$.

(a) For any Dini derivate $D$, if $Df(x) > q$ on $E$, then $\lambda(f(E)) \geq q\lambda(E)$.

(b) For any Dini derivate $D$, if $Df(x) < p$ on $E$, then $\lambda(f(E)) \leq p\lambda(E)$.

**Proof.** Part (a): The proofs for $D^+$ and $D^-$ are similar and the other two cases are then consequences, so we proceed with $D^+$. Suppose that $D^+ f > q$ on a Borel set $E \subseteq C$. Since $f$ is strictly increasing, $f(E)$ is Borel measurable. Let $\varepsilon > 0$, and choose a compact set $K \subseteq E$ and an open set $U \supseteq f(E)$ such that $\lambda(E \setminus K) < \varepsilon$ and $\lambda(U \setminus f(E)) < \varepsilon$.

Construct a right adapted interval cover $C$ of $K$ as follows. For each $x \in K$, $f$ is continuous at $x$ so there is an open interval $I \subseteq (a, b)$ about $x$ such that $f(I) \subseteq U$. Choose a number $R(x) \in I$ satisfying $x < R(x)$ and

$$f(R(x)) - f(x) > q(R(x) - x).$$

Using continuity at $x$, choose $L(x) \in I$ such that $L(x) < x$ and

$$f(R(x)) - f(s) > q(R(x) - s)$$

whenever $L(x) \leq s \leq x$. Let

$$C = \{[s, R(x)] : x \in K, L(x) \leq s \leq x\}.$$
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Then \( C \) is a right adapted interval cover of \( K \), so there is a finite set of nonoverlapping intervals \( \{ [c_i, d_i] \}_{i=1}^n \) that covers \( K \) and associated points \( x_i \in K \) such that \( L(x_i) \leq c_i \leq x_i < d_i = R(x_i) \). The intervals \( [f(c_i), f(d_i)] \) are also nonoverlapping and lie in \( U \). Then

\[
\lambda(f(E)) > \lambda(U) - \varepsilon \geq \lambda(\bigcup_{i=1}^n [f(c_i), f(d_i)]) - \varepsilon = \sum_{i=1}^n (f(d_i) - f(c_i)) - \varepsilon
\]

\[
> \sum_{i=1}^n q(d_i - c_i) - \varepsilon \geq q\lambda(K) - \varepsilon > q\lambda(E) - \varepsilon(1 + q).
\]

Thus, \( \lambda(f(E)) \geq q\lambda(E) \).

Part (b): Let \( \varepsilon > 0 \) and choose a compact set \( K \subseteq f(E) \) and an open set \( U \supseteq E \) such that \( \lambda(f(E) \setminus K) < \varepsilon \) and \( \lambda(U \setminus E) < \varepsilon \). Now \( K \) must have the form

\[
K = [f(\alpha_0), f(\beta_0)] \setminus \bigcup_i (f(\alpha_i), f(\beta_i))
\]

for some finite or countable set of points \( \alpha_i, \beta_i \in E \), so

\[
f^{-1}(K) = [\alpha_0, \beta_0] \setminus \bigcup_{i \geq 1} (\alpha_i, \beta_i)
\]

which is a closed subset of \( E \). This permits us to apply the above technique to \( f^{-1}(K) \) and \( U \) to show that \( \lambda(f(E)) \leq p\lambda(E) \).

The proof of the main theorem now follows from two consequences of the growth theorem. In the setting of the lemma, if

\[
A = \{ x \in C : Df(x) = \infty \}
\]

for any Dini derivate \( D \), then for any positive real number \( q \),

\[
f(b) - f(a) \geq \lambda(f(A)) \geq q\lambda(A).
\]

Thus \( \lambda(A) = 0 \), so that the Dini derivate of \( f \) are finite a.e. Second, all sets of the form

\[
B = \{ x \in C : D_+f(x) < p < q < D_-f(x) \}
\]

satisfy \( q\lambda(B) \leq \lambda(f(B)) \leq p\lambda(B) \) so that \( \lambda(B) = 0 \). That is, \( D^-f \leq D_+f \) a.e. and similarly \( D^+f \leq D_-f \) a.e. The case of a general monotone function follows from the strictly increasing case in standard fashion. Thus we have the

**Lebesgue Differentiation Theorem.** If \( f : [a, b] \to \mathbb{R} \) is monotone, then \( f \) is differentiable almost everywhere.
References


