Notes for *Electromagnetics* (DEMAT, Spring 2021)

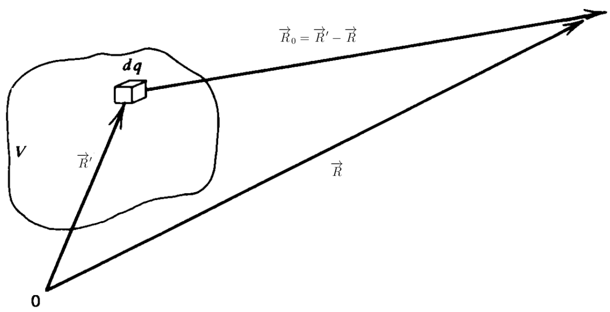
May 15, 2021

Contents

Notations and conventions

- $\mathbb{R}$ and $\mathbb{C}$ are the set of real and complex numbers, respectively.
- (x, y, z), (r, φ, z), and (R, θ, φ) are, respectively, the cartesian, cylindrical, and spherical coordinates, defined in Section 2.2.
- All functions, unless otherwise stated, are smooth, namely, all partial derivatives exist.
- In computing the electromagnetic fields or potentials at the *field* position $\overrightarrow{R}$, caused by sources situated at the volume spanned by *source* position vector $\overrightarrow{R}'$, we set

$$\overrightarrow{R}_0 = \overrightarrow{R}' - \overrightarrow{R}, \quad R_0 = \left|\overrightarrow{R}_0\right|, \quad R = \left|\overrightarrow{R}\right|, \quad R' = \left|\overrightarrow{R}'_0\right|, \quad \widehat{R}_0 = \frac{\overrightarrow{R}_0}{R_0}.$$



For example, the electrostatic field E at the point $\overrightarrow{R}$ caused by a charge density distribution ρ is given by

$$E(\overrightarrow{R}) = \frac{1}{4\pi\epsilon_0}\int \frac{\widehat{R}_0}{R_0^2}\rho(\overrightarrow{R}')dV.$$

- ∇, $\nabla\cdot$, $\nabla\times$, $\nabla^2 = \Delta$, $\Box$, $\Box\cdot$, $\Box^2$ are differential operators introduced in Section 2.4, formula (5.8), and Section 5.4.
- The tilde is used to denote the Fourier transform

$$\widetilde{f}(\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)\, e^{-\sqrt{-1}\omega t}dt,$$

$$f(t) = \mathcal{F}^{-1}\left(\widetilde{f}(\omega)\right) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widetilde{f}(\omega)\, e^{\sqrt{-1}\omega t}d\omega.$$

Chapter 1

What is this course about?

Motion is due to inertia (mass) and force, according to the Newton's laws of motion. So far, four fundamental forces have been identified in nature, and this course is about one of them: the **electromagnetic force**.[1] Putting gravity aside, the electromagnetic force is responsible for practically all motion phenomena one encounters in daily life above the nuclear scale. The study of this force, its creation and effects is called **electromagnetics**.

The electromagnetic force is attributed to **electric** and **magnetic fields** according to the **Lorentz law**

$$F = qE + qv \times B. \tag{1.1}$$

Here, F is the electromagnetic force (measured in Newton, N) applied on a point **electric charge** q (measured in Coulomb, C) moving with velocity v (measured in meter per second, m/s), E is the **electric field strength** (measured in volt per meter, V/m), B is the **magnetic flux density** (measured in Weber per meter squared, Wb/m^2), and $\times$ stands for the cross product of vectors in $\mathbb{R}^3$. At each point of space and each time, E, B are vector quantities. There are two other electric and magnetic fields: **Electric flux density** D (measured in Coulomb per meter squared, C/m^2) and **magnetic field strength** H (measured in Ampere per meter, A/m). The microscopic nature of the substance (or matter or medium) under study gives the **constitutive relations**:

$$D = \epsilon E, \quad B = \mu H.$$

Here, ϵ (**permittivity**, measured in Farad per meter, F/m) and μ (**permeability**, measured in Henry per meter, H/m) are tensors (3×3 matrices) in general, but they are just scalars for many different substances. For example, in vaccum:

$$\epsilon = \epsilon_0 = 8.85 \times 10^{-12} \ \frac{\mathrm{F}}{\mathrm{m}}, \quad \mu = \mu_0 = 4\pi \times 10^{-7} \ \frac{\mathrm{H}}{\mathrm{m}}.$$

Electric and magnetic fields are themselves caused by electric charges and their motion. (Magnetic charges have not yet been detected in nature.) The motion of charge creates **current** (measured in Ampere, A). Electric charge like mass is a fundamental property of matter, and at the moment we want to assume:

[1]Other fundamental forces are the strong nuclear force, the weak nuclear force, and the gravitational force [Fey, Volume II, Chapter 1].

- There are two kinds of charge, positive and negative. Neglecting nuclear phenomena, the negative (respectively, positive) charge of a matter is due to its excess (respectively, loss) of electrons. All electrons have the same negative charge of 1.6×10^{-19} Coulombs.

- Two charges of the same (respectively, opposite) signs repel (respectively, attract) each other.

- It is not possible to create or annihilate charges, but they just move from one place to another. This is called the **principle of the conservation of charge**.

Although the electric and magnetic fields generated by (moving) charges are given by complicated formulas, the space variation of these fields can be concisely given in the language of vector calculus: All classical, as opposed to quantum, electromagnetic phenomena are governed by **Maxwell's equations**

$$\nabla \cdot D = \rho, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = J + \frac{\partial D}{\partial t}, \tag{1.2}$$

accompanied by the **continuity equation**:

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0. \tag{1.3}$$

Here, ρ is the **volume electric charge density** (namely, the amount of charge per unit volume, measured in coulomb per meter cubed), J is the **volume electric current density** (namely, the amount of charge passing per unit time through per unit area of the surface perpendicular to the flow direction, measured in ampere per meter squared), $\cdot$ stands for the dot product of vectors, and ∇ (nabla) is the differential operator $\hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z$. (Vector calculus is reviewed in Chapter 2. Refer to Example 7 for the derivation of (1.3).) These equation are supposed to be satisfied at every point in space at all times. They express how the electric and magnetic field interact with themselves and with electric charge density and current. These equations are behind the analysis and design of all (classical) electromagnetic devices such as electric circuits, antennas, microwave waveguides, optical tools, electricity generators and transmission lines, etc. In this course, we learn about the meaning of these equations and the way they are used for the analysis and design of electromagnetic devices. Mathematics and physics are interwoven in this study. For pedagogical reasons, we develop our study at several levels in increasing order of difficulty:

- *Electrostatics* (Chapter 3): Motionless electric charges. Here, there are no magnetic effects, and E, D are time-independent and enough to develop the theory.

$$\nabla \cdot D = \rho, \quad \nabla \times E = 0, \quad B = 0, \quad H = 0.$$

- *Magnetostatics* (Chapter 4): Electric charges move, but we have steady (namely, time-independent) electric current. Here, again, E, D, B, H are time-independent.

$$\nabla \cdot D = \rho, \quad \nabla \cdot B = 0, \quad \nabla \times E = 0, \quad \nabla \times H = J.$$

- *Electrodynamics* (Chapter 5): General case. Time-varying electric and magnetic fields are coupled (namely, each one produce the other), and under certain conditions produce electromagnetic waves that radiate energy from the source to other points of space.

$$\nabla \cdot D = \rho, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = J + \frac{\partial D}{\partial t}.$$

Since motion[2] is a relativistic notion, special relativity considerations should be taken into account in electromagnetics: What we call an electric field in one reference frame must be interpreted as a magnetic field in another. As we will see in Chapter 5, this implies that electricity and magnetism are two sides of the same coin: **electromagnetism**.

Remark 1. The international system of units (SI) has seven fundamental units: meter (m) for length, second (s) for time, kilogram (kg) for mass, ampere (A) for electric current, kelvin (K) for temperature, mole (mole) for amount of substance, and candela (cd) for luminous intensity. All the other units can be determined in terms of these seven. Specially in electromagnetics, coulomb is $\mathrm{A} \cdot \mathrm{s}$, volt is joule per coulomb (namely $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} \cdot \mathrm{A}^{-1}$), farad is coulomb per volt, weber is volt $\cdot$ s, and henry is weber per ampere.

Exercise: Compute $1/\sqrt{\epsilon_0 \mu_0}$. Find its unit according to the information given in Remark 1. Is it a familiar quantity?

In Maxwell's equation, one is tempted to look at ρ, J as causes (source variables), and E, D, B, H as effects (field variables). However, the formation of electromagnetic fields affects the distribution of charges, hence changes the field variables. This shows that cause and effect are interwoven in a real electromagnetic problem. In fact, there are few, idealized, carefully-designed electromagnetic problems which can be solved completely. In practical engineering problems, one uses simulator softwares (like HFSS, FEKO, Momentum, CST Studio Suite, etc.) to solve electromagnetic problems numerically. However, the intuition that is gained by solving a simple, idealized electromagnetic problem is excessively valuable for physicists and electrical engineers. We encounter many such problems in this course.

Exercise: We will learn in Chapter 4 that electric currents flowing in wires produce magnetic fields. On the other hand, equation (1.1) shows that magnetic fields exert force on moving charges. Having these two facts in mind, find the definition of ampere in SI system of units.

Exercise: Find the name of the underlying numerical mathematics method used in one of the simulator softwares mentioned above.

At the end, I highly suggest reading the Feynman's masterful introduction to electromagnetics [Fey, Volume II, Chapter 1].

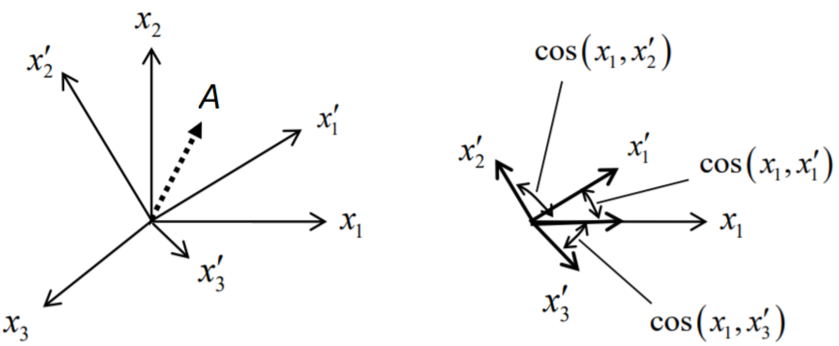
[2]Motion as referred to the classification of the subject into electrostatics, magnetostatics, and electrodynamics, or the velocity vector v in the Lorentz law force.

Chapter 2

Vector calculus

2.1 Vectors

Those physical quantities (like mass, length, time, electric charge, energy) which does not change under the rotation of the coordinate system (observer) with respect to it the quantity is measured are called **scalars**. **Vectors** are those quantities (like force, velocity, momentum, electric and magnetic field intensities) which under the rotation of the coordinate system change exactly the same way as the position vector changes. More precisely, a vector A in each cartesian coordinate system xyz is given by a 3-tuple of real numbers (A_1, A_2, A_3), which is related to the 3-tuple (A'_1, A'_2, A'_3) in the rotated cartesian system $x'y'z'$ by the following linear equations[1]:



$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos(x_1, x'_1) & \cos(x_1, x'_2) & \cos(x_1, x'_3) \\ \cos(x_2, x'_1) & \cos(x_2, x'_2) & \cos(x_2, x'_3) \\ \cos(x_3, x'_1) & \cos(x_3, x'_2) & \cos(x_3, x'_3) \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix}$$

Exercise: Write the equations of the change of components of a vector with two components when the coordinate system rotates by the angle φ in the counterclockwise orientation. (*Answer.* $x = \cos\varphi x' - \sin\varphi y'$, $y = \sin\varphi x' + \cos\varphi y'$.)

The best way to represent a vector is by an arrow in space, which clearly shows the analogy of vectors with displacement. Also, we use a scale such that one unit of our

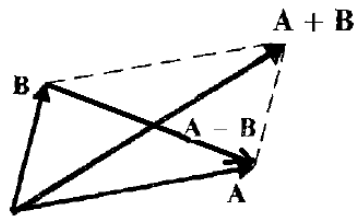
[1]Scalar and vectors are *tensors* of rank 0 and 1, respectively. Refer to [Fey, Volume I, Chapter 11], [Fey, Volume II, Chapter 31], [AW, Sections 3.4, 4.1], or [Tai, Sections 1.3–4] for a complete discussion.

vector corresponds to a certain convenient length. The magnitude and direction of the vector A is denoted by $|A|$ and $\hat{A}$, respectively. Two vector are the same if they have the same magnitude and direction, or equivalently, if they have the same components in one (hence all) coordinate systems. The vector with all three components zero is denoted by 0. Vectors of unit magnitude are called **unit vectors**.

We have several operations on vectors:

- The **scalar product** aA of a scalar a and a vector A is a vector whose magnitude equals $|a||A|$, and its direction is the direction of A if $a > 0$, and the reverse of the direction of A if $a < 0$. Clearly, $0A = 0$, $1A = A$, and $a(bA) = (ab)A$.

- The **addition** and **subtraction** of two vectors is a vector obtained by the parallelogram law:



Clearly,

$$A + 0 = 0,$$

$$A + B = B + A, \quad A + (B + C) = (A + B) + C,$$

$$a(A + B) = aA + aB, \quad (a + b)A = aA + bA.$$

With respect to a cartesian coordinate system, one can decompose a vector into its components:

$$A = A_x\widehat{x} + A_y\widehat{y} + A_z\widehat{z}.$$

Here, $\widehat{x}$ is the dimension-less vector of unit magnitude in the direction of x-axis; similarly for $\widehat{y}, \widehat{z}$. In terms of components, we have

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2},$$

$$\hat{A} = \frac{A}{|A|} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\widehat{x} + \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\widehat{y} + \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\widehat{z},$$

$$aA = aA_x\widehat{x} + aA_y\widehat{y} + aA_z\widehat{z},$$

$$A \pm B = (A_x \pm B_x)\widehat{x} + (A_y \pm B_y)\widehat{y} + (A_z \pm B_z)\widehat{z},$$

where $B = B_x\widehat{x} + B_y\widehat{y} + B_z\widehat{z}$.

- The **dot** (or **inner**) **product** $A \cdot B$ of two vectors A, B is the scalar given by the magnitude of A times the magnitude of B times the cosine of the angle between the direction of A and B. Equivalently,

$$A \cdot B = A_xB_x + A_yB_y + A_zB_z. \tag{2.1}$$

Clearly,
$$|A| = \sqrt{A \cdot A},$$
and two vectors are orthogonal exactly when their dot product vanishes.

Exercise: Using (2.1), show that $A \cdot B$ is a scalar. (*Hint.* Assume that A, B are in the xoy-plane, and then rotate the xoy-system to another $x'oy'$-system. Note that any rotation can be written as the composition of three rotations around the axes of coordinate systems, namely Euler angles.)

Exercise: Using the first definition of dot product, prove that
$$\begin{aligned} A \cdot B &= B \cdot A, \\ A \cdot (B + C) &= A \cdot B + A \cdot C. \end{aligned} \tag{2.2}$$
Then, use (2.2) to deduce (2.1).

Exercise: Assuming (2.2), prove the law of cosines in triangles. (*Hint.* $|A - B|^2 = (A - B) \cdot (A - B) = A \cdot A - 2A \cdot B + B \cdot B = \cdots$.)

- The **cross product** $A \times B$ of two vectors A, B is the vector whose magnitude is the magnitude of A times the magnitude of B times the sine of the (smaller) angle between the direction of A and B (this equals the area of the parallelogram spanned by A, B), it is perpendicular to both A, B, and its direction follows that of the thumb of the right hand when the other fingers rotate from A to B (through the smaller angle). Equivalently,
$$\begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\widehat{x} - (A_x B_z - A_z B_x)\widehat{y} + (A_x B_y - A_y B_x)\widehat{z}. \tag{2.3}$$

  Exercise: Using (2.3), show that $A \times B$ is a vector.

  Exercise: Prove that
$$\begin{aligned} A \times B &= -B \times A, \\ A \times (B + C) &= A \times B + A \times C. \end{aligned} \tag{2.4}$$

  Exercise: Assuming (2.4), deduce (2.3) from the first definition of the cross product.

- The **scalar triple product** of three vectors A, B, C:
$$A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{2.5}$$

  Exercise: Prove that in the scalar triple product, $\cdot$ and $\times$ may be freely interchanged so long as A, B, C remain in cyclic order, namely
$$\begin{aligned} A \cdot (B \times C) &= (A \times B) \cdot C \\ &= B \cdot (C \times A) = C \cdot (A \times B). \end{aligned}$$

  Exercise: Show that $|A \cdot (B \times C)|$ equals the volume of the parallelepiped spanned by A, B, C.

- The **vector triple product** of three vectors A, B, C:

$$A \times (B \times C) = (C \cdot A)B - (B \cdot A)C.$$

  This is called the "CAB minus BAC" rule.

  Exercise: Prove that

$$(A \times B) \times C = (A \cdot C)B - (B \cdot C)A.$$

Exercise: Prove that

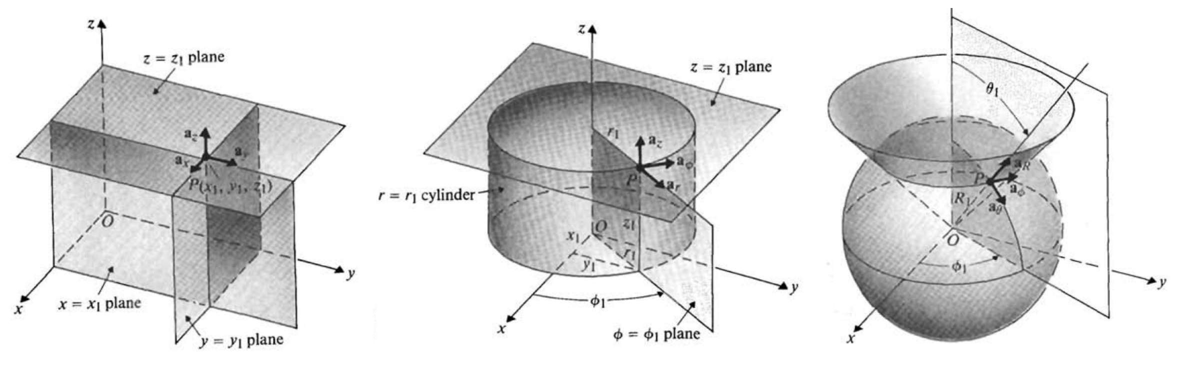
$$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C),$$

$$(A \times B) \times (C \times D) = (A \cdot C \times D)B - (B \cdot C \times D)A = (A \cdot B \times D)C - (A \cdot B \times C)D.$$

Exercise: (a) Assume three vectors A, B, C such that A is nonzero. Prove that if $A \cdot B = A \cdot C$ and $A \times B = A \times C$, then $B = C$. (b) Let A be a nonzero vector. Determine a formula for vector X is $A \cdot X$ and $A \times X$ is known. (*Hint.* (a) Assume $A = \widehat{x}$. (b) Write $X = aA + bA \times (A \times X)$, and then determine scalars a, b.)

2.2 Cylindrical and spherical coordinate systems

Since the laws of electromagnetism is written in the vector analysis language (recall Maxwell's equations), they are invariant under the translation and rotation of coordinate systems. However, in order to numerically solve a specific problem, one coordinate system might be preferable to the others, depending on the symmetry of the geometry of the situation. We start by explaining the notion of a **right-handed, orthogonal, curvilinear coordinate system** (u_1, u_2, u_3), most important examples being the cartesian (x, y, z), cylindrical (r, φ, z), and spherical (R, θ, φ) coordinate systems. (φ and θ are called **polar** and **azimuth angles**, respectively.)



Here,

- *Coordinate system* means that there is a one-to-one correspondence between the points of space and 3-tuples of real numbers (u_1, u_2, u_3), each u_i ranging on some specific set of values. For example, all points of the space, except the points on the z-axis, can be given unique cylindrical coordinates with

$$0 < r < \infty, \quad 0 \le \varphi < 2\pi, \quad -\infty < z < \infty;$$

also unique spherical coordinates with

$$0 < R < \infty, \quad 0 < \theta < \pi, \quad 0 \le \varphi < 2\pi.$$

At each point (u_1, u_2, u_3), let $\widehat{u}_1$ be the dimension-less vector of unit magnitude, orthogonal to the surface of constant u_1, which points to the direction where u_1 increases; similarly, define $\widehat{u}_2, \widehat{u}_3$.

- *Orthogonal* means that at each point, the unit vectors $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$ are mutually orthogonal to each other. *Right-handed* (in the presence of orthogonality) means $\widehat{u}_1 \times \widehat{u}_2 = \widehat{u}_3$. If so, then $\widehat{u}_2 \times \widehat{u}_3 = \widehat{u}_1$, $\widehat{u}_3 \times \widehat{u}_1 = \widehat{u}_2$.
- *Curvilinear* means that the unit vectors $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$ might change when the point changes.

The good news is that assuming such a system, any vector can be represented (at any point) by three real numbers, called its **components**:

$$\begin{aligned} A &= A_1\widehat{u}_1 + A_2\widehat{u}_2 + A_3\widehat{u}_3 \\ &= (A \cdot \widehat{u}_1)\widehat{u}_1 + (A \cdot \widehat{u}_2)\widehat{u}_2 + (A \cdot \widehat{u}_3)\widehat{u}_3. \end{aligned}$$

Especially, the **position vector** $\overrightarrow{R}$ is given by:

$$\overrightarrow{R} = x\widehat{x} + y\widehat{y} + z\widehat{z} = r\widehat{r} + z\widehat{z} = R\widehat{R}.$$

We have the following relations among different coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi,$$

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \tan^{-1}\frac{y}{x},$$

$$x = R\sin\theta\cos\varphi, \quad y = R\sin\theta\sin\varphi, \quad z = R\cos\theta,$$

$$R = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \quad \varphi = \tan^{-1}\frac{y}{x}.$$

Note that:

$$\widehat{r} = \frac{\partial \overrightarrow{R}}{\partial r} = \cos\varphi\ \widehat{x} + \sin\varphi\ \widehat{y},$$

$$\widehat{\varphi} = -\sin\varphi\ \widehat{x} + \cos\varphi\ \widehat{y} = \frac{\partial \widehat{r}}{\partial \varphi},$$

$$\widehat{R} = \frac{\partial \overrightarrow{R}}{\partial R} = \sin\theta\cos\varphi\ \widehat{x} + \sin\theta\sin\varphi\ \widehat{y} + \cos\theta\ \widehat{z},$$

$$\widehat{\theta} = \cos\theta\cos\varphi\ \widehat{x} + \cos\theta\sin\varphi\ \widehat{y} - \sin\theta\ \widehat{z} = \frac{\partial \widehat{R}}{\partial \theta}.$$

Exercise: The formulas (2.1), (2.3), (2.5) remain valid for every right-handed, orthogonal, curvilinear system.

Exercise: Write $\widehat{x}$ is terms of $\widehat{R}, \widehat{\theta}, \widehat{\varphi}$. (*Hint.* $\widehat{x} = (\widehat{x} \cdot \widehat{R})\widehat{R} + (\widehat{x} \cdot \widehat{\theta})\widehat{\theta} + (\widehat{x} \cdot \widehat{\varphi})\widehat{\varphi}$.)

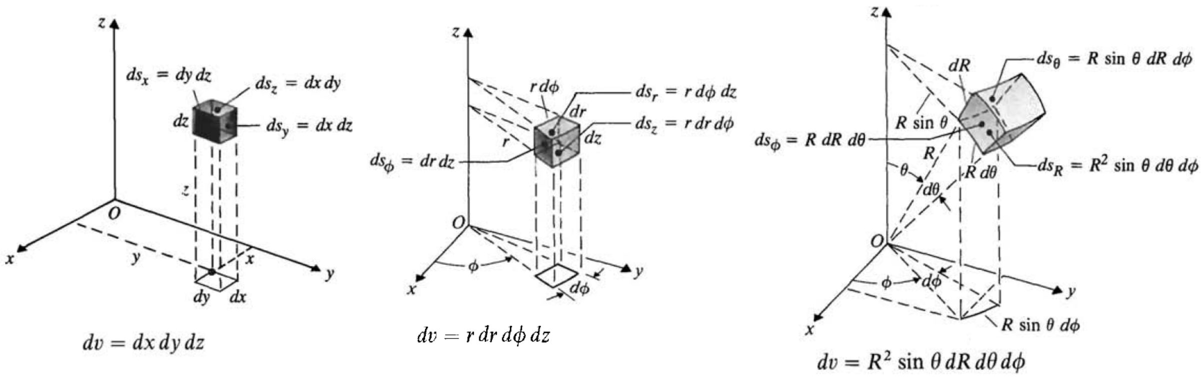
2.3 Infinitesimals and their integral

Suppose a point P, given by the placement (or position) vector $\overrightarrow{R}$, with coordinates (u_1, u_2, u_3) given in a right-handed, orthogonal system. When u_1 changes infinitesimally to $u_1 + du_1$, then $\overrightarrow{R}$ changes infinitesimally to $\overrightarrow{R} + h_1 du_1 \widehat{u}_1$, where h_1 is a function of u_1, u_2, u_3. Similarly h_2, h_3 can be defined. They are called **metric coefficients**. If (u_1, u_2, u_3) changes infinitesimally to $(u_1 + du_1, u_2 + du_2, u_3 + du_3)$, then $\overrightarrow{R}$ changes infinitesimally to $\overrightarrow{R} + d\overrightarrow{R}$ given by

$$d\overrightarrow{R} = h_1 du_1 \widehat{u}_1 + h_2 du_2 \widehat{u}_2 + h_3 du_3 \widehat{u}_3, \tag{2.6}$$

This latter quantity is called the **infinitesimal displacement**, and the notation $\overrightarrow{dl}$ is also used for it. For example,

$$\begin{aligned}\overrightarrow{dl} &= dx\widehat{x} + dy\widehat{y} + dz\widehat{z} \\ &= dr\widehat{r} + rd\varphi\widehat{\varphi} + dz\widehat{z} \\ &= dR\widehat{R} + Rd\theta\widehat{\theta} + R\sin\theta d\varphi\widehat{\varphi}.\end{aligned}$$



The magnitude of $\overrightarrow{dl}$, called the **infinitesimal length**, is given by

$$dl = \sqrt{(h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2}.$$

The volume of the parallelepiped spanned by $h_1 du_1 \widehat{u}_1, h_2 du_2 \widehat{u}_2, h_3 du_3 \widehat{u}_3$, called the **infinitesimal volume**, is given by

$$dV = h_1 h_2 h_3 du_1 du_2 du_3.$$

When u_1 is fixed, and u_2, u_3 vary infinitesimally, the area spanned by $h_2 du_2 \widehat{u}_2, h_3 du_3 \widehat{u}_3$, called the **infinitesimal area**, is given by

$$dS_1 = h_2 h_3 du_2 du_3.$$

The corresponding **infinitesimal surface**

$$\overrightarrow{dS}_1 = h_2 du_2 \widehat{u}_2 \times h_3 du_3 \widehat{u}_3 = h_2 h_3 du_2 du_3 \widehat{u}_3, \tag{2.7}$$

has magnitude dS_1, and is perpendicular to the constant u_1 surface; similarly for $\overrightarrow{dS}_2, \overrightarrow{dS}_3$.

In electromagnetics, we need to integrate different infinitesimals:

- Line integrals of:

$$f dl, \quad f\overrightarrow{dl}, \quad A dl, \quad A \cdot \overrightarrow{dl}, \quad A \times \overrightarrow{dl}.$$

- Surface integrals of:

$$f dS, \quad f\overrightarrow{dS}, \quad A dS, \quad A \cdot \overrightarrow{dS}, \quad A \times \overrightarrow{dS}.$$

- Volume integrals of:

$$f dV, \quad A dV.$$

Here, f (respectively, A) is a **scalar field** (respectively, **vector field**), namely, a scalar-valued (respectively, vector-valued) quantity which varies on space; for example, the temperature distribution of a room (respectively, the force applied to some particle). The meaning of these integrals should be clear from your experience with the basic differential and integral calculus. (Refer to [Tho, Chapter 16] or [Apo, Chapters 10-12] for a complete development.) For example, $\int_C f dl$ (C a curve) is defined as the limit of the Riemann sum $\sum f\Delta l$ as $\max \Delta l \to 0$, where $\Delta l = \sqrt{(\Delta x)^2 + \Delta y)^2 + \Delta z)^2}$ is the distance between two successive partition points of C. Also, $\int_S A \times \overrightarrow{dS}$ (S an oriented surface) is defined as the limit of the Riemann sum $\sum A \times \overrightarrow{\Delta S}$ as $\max |\overrightarrow{\Delta S}| \to 0$. Here, the surface S being **oriented** means that a smooth distribution of unit normal vectors $\widehat{n}$ is defined on S.[2] Then, $\overrightarrow{\Delta S}$ has the same direction as $\widehat{n}$, and its magnitude is the area of a small partition patch of S. In computing these integrals, one should fulfill the following steps in order to reduce the integral into (sums of) single, double, or triple integrals

$$\int f(x)dx, \quad \iint f(x,y)dxdy, \quad \iiint f(x,y,z)dxdydz.$$

1. Choose an appropriate coordinate system. In this course, these are cartesian, cylindrical, or spherical coordinates.

2. Write the integrand in terms of fundamental infinitesimals: $dx, dy, dz, dr, d\varphi, dR, d\theta$.

3. Parametrize the curve C (with one parameter), surface S (with two parameters), or volume V (with three parameters) where integration is taken on.

Example 2. Let us compute the integral $I := \int A \times \overrightarrow{dS}$ for the vector field $A = e^{-R}\widehat{R}$ on that part of the cone $z = \sqrt{x^2 + y^2}$ which lies in the first octant. The cone is given by $\theta = \pi/4$ in spherical coordinates. The integrand is

$$A \times \overrightarrow{dS} = e^{-R}\widehat{R} \times R\sin\theta dR d\varphi \widehat{\theta} = Re^{-R}\sin\theta dR d\varphi(-\sin\varphi\widehat{x} + \cos\varphi\widehat{y}).$$

The integration over $0 \le R \le \infty$, $0 \le \varphi \le \pi/2$ gives $I = (-\widehat{x} + \widehat{y})/\sqrt{2}$. ■

[2] There are surfaces (most notably, the Möbius band) with no such distribution, but we do not consider them in this course.

Remark 3. In this course, we almost always have curves and surfaces given by constant coordinates in cartesian, cylinderical, or spherical coordinates, so (2.6) and (2.7) are used. In general, if we move along a curve parametrized by the position vector

$$\overrightarrow{R} = \overrightarrow{R}(t) = x(t)\widehat{x} + y(t)\widehat{y} + z(t)\widehat{z},$$

then the displacement infinitesimal is given by

$$\overrightarrow{dl} = \frac{\partial \overrightarrow{R}}{\partial t} dt = \left(\frac{dx}{dt}\widehat{x} + \frac{dy}{dt}\widehat{y} + \frac{dz}{dt}\widehat{z} \right) dt.$$

If we have a surface paved by the position vector

$$\overrightarrow{R} = \overrightarrow{R}(u, v) = x(u, v)\widehat{x} + y(u, v)\widehat{y} + z(u, v)\widehat{z},$$

then the infinitesimal surface is given by

$$\overrightarrow{dS} = \frac{\partial \overrightarrow{R}}{\partial u} \times \frac{\partial \overrightarrow{R}}{\partial v} dudv = \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

Exercise: Find $\int A \cdot \overrightarrow{dl}$ for $A = x^2 y^3 \widehat{x} + \widehat{y} + z\widehat{z}$ along the curve made by the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z > 0$, paved counterclockwise when viewed from above ($z = +\infty$).

2.4 Differentials of scalar and vector fields

Remember from elementary calculus that the derivative $f'(x) = df/dx$ of a function $f : \mathbb{R} \to \mathbb{R}$ measures the rate of change of f with respect to its variable x. This section introduces differential operators which measure different aspects of space variations of scalar $f : \mathbb{R}^3 \to \mathbb{R}$ as well as vector fields $A : \mathbb{R}^3 \to \mathbb{R}^3$.

2.4.1 Gradient of a scalar field

Assume a scalar field f. We want to measure the rate of change of $f(P)$ as the point P moves infinitesimally to $P + \overrightarrow{dl}$. We describe two equivalent approaches:

- *Coordinate-dependent approach.* By Taylor's expansion

$$df = f_x dx + f_y dy + f_z dz = (\text{grad} f) \cdot \overrightarrow{dl},$$

  where

$$\text{grad} f := f_x \widehat{x} + f_y \widehat{y} + f_z \widehat{z}$$

  is called the **gradient** of f. One can pretend that grad$f$ is obtained by exerting the differential operator

$$\nabla := \widehat{x}\frac{\partial}{\partial x} + \widehat{y}\frac{\partial}{\partial y} + \widehat{z}\frac{\partial}{\partial z},$$

called **nabla** (or **del**), on f, hence, we have the notation

$$\text{grad} f = \nabla f.$$

The same sort of computations in an orthogonal coordinate system (u_1, u_2, u_3) shows that

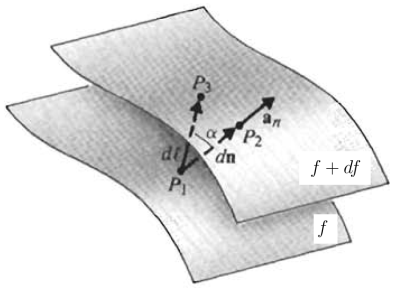
$$\nabla f = \frac{1}{h_1} f_{u_1} \widehat{u}_1 + \frac{1}{h_2} f_{u_2} \widehat{u}_2 + \frac{1}{h_3} f_{u_3} \widehat{u}_3.$$

Therefore,

$$\nabla \equiv \widehat{u}_1 \frac{\partial}{h_1 \partial u_1} + \widehat{u}_2 \frac{\partial}{h_2 \partial u_2} + \widehat{u}_3 \frac{\partial}{h_3 \partial u_3}. \tag{2.8}$$

Note that the equation $df = \text{grad} f \cdot \overrightarrow{dl}$ clearly shows that grad$f$ is a vector field.

- *Coordinate-free approach.* Draw the constant f surface passing through $P = P_1$. Fix df, and also draw the constant $f + df$ surface. Let P_2 and P_3 be points on the constant $f + df$ surface such that $\overrightarrow{P_1 P_2} = \overrightarrow{dn}$ is orthogonal to the constant f surface, and $\overrightarrow{P_1 P_3} = \overrightarrow{dl}$ is some arbitrary infinitesimal.



Since df is assumed to be fixed, the computations

$$\frac{df}{dl} = \frac{df}{dn}\frac{dn}{dl} = \frac{df}{dn} \cos\alpha \leq \frac{df}{dn},$$

$$df = \frac{df}{dl} dl = \frac{df}{dn} \hat{n} \cdot \widehat{dl}\; dl = \frac{df}{dn} \hat{n} \cdot \overrightarrow{dl}$$

shows that the vector

$$\nabla f = \frac{df}{dn} \hat{n}$$

represents both the magnitude and the direction of the maximum space rate of the change of f. Always remember: The gradient of f is orthogonal to the constant f surface.

Exercise 4. Prove that

$$\int_{P_1}^{P_2} \nabla f \cdot \overrightarrow{dl} = f(P_2) - f(P_1), \tag{2.9}$$

where the integral is taken over an arbitrary curve starting from point P_1 and ending in point P_2.

2.4.2 Divergence of a vector field

Assume a vector field A. Let S be an oriented surface, namely, there exists a smooth distribution of unit normal vector $\widehat{n}$ on S. The quantity

$$\int_S A\cdot \overrightarrow{dS},$$

where $\overrightarrow{dS}=\widehat{n}dS$, is called the **flux** of A passing through S. The **flux density** of A at a point can be measured by the scalar

$$\mathrm{div}A := \lim_{\Delta V\to 0}\frac{\oint_\sigma A\cdot \overrightarrow{dS}}{\Delta V}, \tag{2.10}$$

called the **divergence** of A.[3] Here, σ is a small closed, oriented surface (like a cube, cylinder, or sphere) around our point of interest, ΔV is the volume of the solid enclosed by σ, and $\widehat{n}$ (in $\overrightarrow{dS}=\widehat{n}dS$) is taken to be *outwards*.

Theorem 5. *In cartesian coordinates $A=A_x\widehat{x}+A_y\widehat{y}+A_z\widehat{z}$, we have*

$$\mathrm{div}A=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}. \tag{2.11}$$

More generally, in any orthogonal, curvilinear coordinates $A=A_1\widehat{u}_1+A_2\widehat{u}_2+A_3\widehat{u}_3$, we have

$$\mathrm{div}A=\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3A_1)+\frac{\partial}{\partial u_2}(h_1h_3A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right).$$

Proof. In the definition (2.10), let σ be a cube of side lengths dx, dy, dz, with faces parallel to yoz, xoz, xoy planes, and centered at our point of interest (x_0, y_0, z_0). The flux passing through the front face $x=x_0+dx/2$ equals

$$A(x_0+dx/2,y_0,z_0)\cdot dydz\widehat{x}=A_x(x_0+dx/2,y_0,z_0)dydz,$$

and the flux passing through the back face $x=x_0-dx/2$ equals

$$A(x_0-dx/2,y_0,z_0)\cdot(-dydz\widehat{x})=-A_x(x_0-dx/2,y_0,z_0)dydz.$$

These, add up to

$$A_x(x_0+dx/2,y_0,z_0)dydz-A_x(x_0-dx/2,y_0,z_0)dydz=\frac{\partial A_x}{\partial x}(x_0,y_0,z_0)dxdydz.$$

After dividing by $dV=dxdydz$, and taking the limit $\sigma\to 0$, we get the first summand $\partial A_x/\partial x$ in (2.11). Other summands appear similarly by integrating over the remaining four faces. The general coordinate system case is proved similarly. ■

[3] A justification for this appellation is given in [Lee, Page 422].

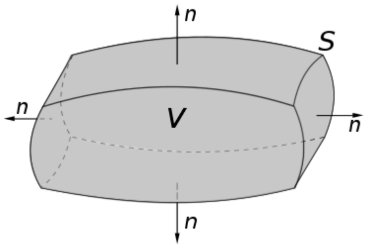
The formula (2.11) justifies the notation

$$\mathrm{div} A = \nabla \cdot A.$$

For any oriented, closed surface S enclosing volume V, we have the **divergence theorem**:

$$\oint_S A \cdot \overrightarrow{dS} = \int_V \nabla \cdot A \; dV. \tag{2.12}$$

Here, $\overrightarrow{dS}$ is pointing outwards. This formula can be proved by partitioning V into small volumes ΔV, and summing up all the definition equations $\oint A \cdot \overrightarrow{dS} = \mathrm{div} A \; dV$ [Che, Section 2.8].



Example 6. Let us verify the divergence theorem for the vector field

$$A := 4x\widehat{x} - 2y^2\widehat{y} + z^2\widehat{z},$$

on the region bounded by $x^2+y^2 = 4$, $z = 0$, and $z = 3$. The divergence of A is $4-4y+2z$, and the infinitesimal volume is $rdrd\varphi 4dz$, so

$$\int \nabla \cdot A \; dV = \int_{r=0}^{2} \int_{\varphi=0}^{2\pi} \int_{z=0}^{3} (4 - 4r\sin\varphi + 2z) r dr d\varphi dz = 84\pi.$$

The flux is the sum of three surface integrals

$$\int_{\text{upper lid}} A \cdot \overrightarrow{dS} = \int A \cdot dxdy\widehat{z} = \int z^2 dxdy = \int 9dxdy = 36\pi,$$

$$\int_{\text{lower lid}} A \cdot \overrightarrow{dS} = \int A \cdot dxdy\widehat{z} = \int z^2 dxdy = 0,$$

and

$$\int_{\text{peripherical cylinder}} A \cdot \overrightarrow{dS} = \int A \cdot 2d\varphi dz\widehat{r} = \int 2(4x\widehat{x} \cdot \widehat{r} - 2y^2\widehat{y} \cdot \widehat{r}) d\varphi dz$$
$$= \int_{\varphi=0}^{2\pi} \int_{z=0}^{3} 2(4r\cos^2\varphi - 2r^2\sin^3\varphi) d\varphi dz = 48 \int_0^{2\pi} (\cos^2\varphi - \sin^3\varphi) d\varphi = 48\pi.$$

Therefore, the total flux equals $36\pi + 48\pi = 84\pi$. ∎

Example 7. Assume a fixed volume $U \subseteq \mathbb{R}^3$, with boundary S. The amount of the electric charge inside U is given by $Q = \int_U \rho dV$, where ρ is the volume electric charge density. The time rate of the change of Q, according to the principle of the conservation of electric charges, is due to the passage of charges through the boundary, and is given by $-\oint_S J \cdot \overrightarrow{dS}$, where J is the volume electric current density. (By definition, J is a vector pointing to the flow of charge, and its magnitude is the amount of charge passing per unit time across per unit area of the surface perpendicular to the flow of charge.) Therefore,

$$\int_U \frac{\partial \rho}{\partial t} dV = \frac{d}{dt} \int_U \rho dV = -\oint_S J \cdot \overrightarrow{dS} = -\int_U \nabla \cdot J dV.$$

This proves the continuity equation $\nabla \cdot J = -\partial\rho/\partial t$, first appeared in (1.3). ■

Exercise: Find the divergence of the position vector field $A = \overrightarrow{R}$ in three cartesian, cylindrical, and spherical coordinate systems.

Exercise: Derive the formula for divergence in cylindrical coordinates in two ways: (a) Using (5). (b) Applying $\nabla \equiv \widehat{r}\frac{\partial}{\partial r} + \widehat{\varphi}\frac{\partial}{r\partial \varphi} + \widehat{z}\frac{\partial}{\partial z}$ to $A = A_r\widehat{r} + A_\varphi\widehat{\varphi} + A_z\widehat{z}$. (*Hint.* In (b), one needs to compute nine terms, which come from distributing THE three summands of ∇ over THE three summands of A. Two of them are

$$\left(\widehat{r}\frac{\partial}{\partial r}\right) \cdot (A_r\widehat{r}) = \widehat{r} \cdot \left(\frac{\partial A_r}{\partial r}\widehat{r} + A_r\frac{\partial \widehat{r}}{\partial r}\right) = \widehat{r} \cdot \left(\frac{\partial A_r}{\partial r}\widehat{r} + 0\right) = \frac{\partial A_r}{\partial r},$$

$$\left(\widehat{\varphi}\frac{\partial}{r\partial \varphi}\right) \cdot (A_r\widehat{r}) = \frac{\widehat{\varphi}}{r} \cdot \left(\frac{\partial A_r}{\partial \varphi}\widehat{r} + A_r\frac{\partial \widehat{r}}{\partial \varphi}\right) = \frac{\widehat{\varphi}}{r} \cdot \left(\frac{\partial A_r}{\partial \varphi}\widehat{r} + A_r\widehat{\varphi}\right) = \frac{A_r}{r}.$$

Continue!)

Exercise: Verify the divergence theorem for the position vector field $A = \overrightarrow{R}$ on the shell region enclosed by spherical surfaces at $R = a$ and $R = b$, $b > a$, centered at the origin.

Exercise: Show that the volume of the region enclosed by a closed, oriented surface S equals $\frac{1}{3}\oint_S \overrightarrow{R} \cdot \overrightarrow{dS}$.

Exercise: Prove that there is no smooth orientable closed surface which is everywhere tangent to the position vector.

2.4.3 Curl of a vector field

Assume a vector field A. Let C be an *oriented* curve, namely, one assumes a smooth distribution $\hat{t}$ of unit tangent vectors on C. (Any smooth curve has exactly two such distributions.) The quantity

$$\int_C A \cdot \overrightarrow{dl},$$

where $\overrightarrow{dl} = \hat{t}dl$, is called the **circulation** of A (or the **work** done by A) along C. The **circulation density** of A at a point can be measured by a vector curl$A$, called the **curl** (or **rotation**) of A, whose component along any unit vector $\hat{u}$ is given by[4]

$$(\mathrm{curl}A)\cdot\hat{u} := \lim_{\Delta S\to 0}\frac{\oint_c A\cdot\overrightarrow{dl}}{\Delta S}. \tag{2.13}$$

Here, c is a small, oriented, closed curve (like a square or circle) around our point of interest, c lies in the plane Π perpendicular to $\hat{u}$, ΔS is the area of the surface in Π enclosed by c, and the orientations of c and $\hat{u}$ are compatible according to the right-hand rule.

Theorem 8. *In cartesian coordinates $A = A_x\widehat{x} + A_y\widehat{y} + A_z\widehat{z}$, we have*

$$\mathrm{curl}A = \begin{vmatrix}\widehat{x} & \widehat{y} & \widehat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}, \tag{2.14}$$

to be expanded with respect to its first row. More generally, in any right-handed, orthogonal, curvilinear coordinates $A = A_1\widehat{u}_1 + A_2\widehat{u}_2 + A_3\widehat{u}_3$, we have

$$\mathrm{curl}A = \frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\widehat{u}_1 & h_2\widehat{u}_2 & h_3\widehat{u}_3\\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3}\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}. \tag{2.15}$$

Proof. In the definition (2.13), assume $\widehat{u} = \widehat{x}$, and let c be a square of side lengths dy, dz, in the $x = x_0$ plane, whose sides are parallel to y and z axes, and it is centered at our point of interest (x_0, y_0, z_0). The work done on the right side $y = x_0 + dx/2$ equals

$$A(x_0, y_0 + dy/2, z_0)\cdot dz\widehat{z} = A_z(x_0, y_0 + dy/2, z_0)dz,$$

and the work done on the left side $y = y_0 - dy/2$ equals

$$A(x_0, y_0 - dy/2, z_0)\cdot(-dz\widehat{z}) = -A_z(x_0, y_0 + dy/2, z_0)dz.$$

These, add up to

$$A_z(x_0, y_0 + dy/2, z_0)dz - A_z(x_0, y_0 - dy/2, z_0)dz = \frac{\partial A_z}{\partial y}(x_0, y_0, z_0)dydz.$$

After dividing by $dS = dydz$, and taking the limit $c \to 0$, we get the first summand $\partial A_z/\partial y$ in the x-component of $\nabla\times A$ in (2.14). The other summand $\partial A_y/\partial z$ appears by computing the work done on upper and lower sides $z = z_0 \pm dz/2$. The rest is similar. ■

[4]To prove that such a vector exists, one can first define curl$A$ by $\lim_{\Delta V\to 0}\frac{\oint \overrightarrow{dS}\times A}{\Delta V}$ (this definition itself comes from the identity (2.25), to be proved later.), and then show that this vector satisfies (2.13). This is done in [NB, Pages 19–20]. Another approach to the curl, taken in [Fey, Vomule II, Chapters 2–3], is to first prove that the expression (2.15) represents a vector field, and then follow the proof of Theorem 8 to show (2.13) for $\widehat{u} = \widehat{x}$; this proves (2.13) for every unit $\widehat{u}$.

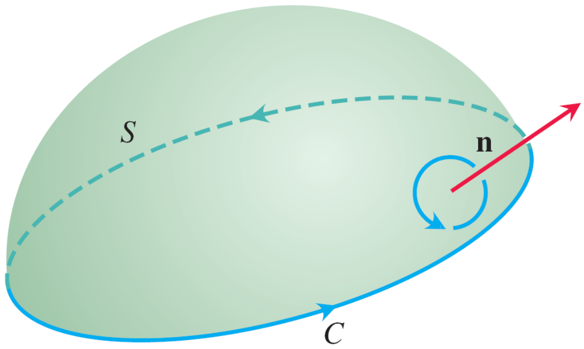
The formula (2.14) justifies the notation

$$\text{curl}A = \nabla \times A.$$

For any oriented, closed curve C (planar or not) enclosing surface S, we have the **Stokes' theorem**:

$$\oint_C A \cdot \overrightarrow{dl} = \int_S \nabla \times A \cdot \overrightarrow{dS}. \tag{2.16}$$

Here, the orientations of C and S are compatible according to the right-hand rule. This can be proved by partitioning S into small surfaces ΔS, and summing up all the definition equations $\oint A \cdot \overrightarrow{dl} = \text{curl}A \cdot \overrightarrow{dS}$ [Che, Section 2.10].



Example 9. Let us verify the Stokes' theorem for the vector field $A := \sin(\varphi/2)\widehat{\varphi}$ on the upper hemisphere surface $R = a$, $z > 0$. The curl of A, based on the formula (2.15), equals

$$\frac{1}{R^2 \sin\theta} \begin{vmatrix} \widehat{R} & R\widehat{\theta} & R\sin\theta\widehat{\varphi} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & R\sin\theta\sin(\varphi/2) \end{vmatrix} = \frac{R\cos\theta\sin(\varphi/2)\widehat{R} - R\sin\theta\sin(\varphi/2)\widehat{\theta}}{R^2\sin\theta}.$$

On the upper hemisphere S, we have $\overrightarrow{dS} = a^2 \sin\theta d\theta d\varphi \widehat{R}$, so

$$\int_S \nabla \times A \cdot \overrightarrow{dS} = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} a\cos\theta\sin(\varphi/2) d\theta d\varphi = 4a.$$

The boundary C of S is the circle $r = a$, so

$$\oint_C A \cdot \overrightarrow{dl} = \int_0^{2\pi} \sin(\varphi/2)\widehat{\varphi} \cdot a d\varphi \widehat{\varphi} = 4a.$$

■

Exercise 10. Compute the divergence and curl of the inverse square field $A = \widehat{R}/R^2$, at a point away from the origin.

Exercise: Verify the Stokes' theorem for the vector field $A = x^2y^3\widehat{x} + \widehat{y} + z\widehat{z}$ on the surface $x^2 + y^2 + z^2 = 4$, $z \geq 1$.

Exercise: Find the flux of $B = \nabla \times A$ passing through the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$, where $A = (y + \sqrt{z})\widehat{x} + e^{xyz}\widehat{y} + \cos(xz)\widehat{z}$.

Exercise: Compute the curl of vector fields $A = x\widehat{x}+y\widehat{y}+z\widehat{z}$ and $B = -y\widehat{x}+x\widehat{y}+z\widehat{z}$. Then, use an online vector field plotter to intuitively approve your computation.

Exercise: (a) Deduce Green's formula in the plane

$$\oint_C Pdx + Qdy = \int_S (-P_y + Q_x)dxdy$$

from the Stokes' theorem. Here, C is a closed curve in the xoy-plane, paved in the counterclockwise orientation, and it enclosed surfece S. (b) Show that the area of S is given by $\frac{1}{2}\oint_C xdy - ydx$.

2.4.4 Laplacian of a scalar field

The divergence of the gradient of a scalar field is called its **Laplacian**:

$$\nabla^2 f := \nabla \cdot \nabla f.$$

It is also denoted by Δf. In cartesian coordinates, we have

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

In a general orthogonal system (u_1, u_2, u_3), we have

$$\Delta f = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right). \tag{2.17}$$

The intuition about Laplacian comes from the following theorem:

Theorem 11. *The average $A_a(f;\rho)$ of a scalar field f on the sphere $|\overrightarrow{R} - a| = \rho$, and the average $B_a(\Delta f;\rho)$ of its Laplacian on the ball $|\overrightarrow{R} - a| \leq \rho$ are related by*

$$A_a(f;\rho) = f(a) + \int_0^\rho \frac{\rho}{3}B_a(\Delta f;\rho)d\rho.$$

Proof. After a shift, we may assume $a = 0$. By the divergence theorem,

$$\frac{4\pi\rho^3}{3}B_0(\Delta f;\rho) = \int_{R\leq\rho}\Delta f dV = \int_{R=\rho}\nabla f \cdot \overrightarrow{dS} = \int_0^\pi\int_0^{2\pi}\frac{\partial f}{\partial\rho}\rho^2\sin\theta d\theta d\varphi,$$

so

$$\frac{4\pi\rho}{3}B_0(\Delta f;\rho) = \int_0^\pi\int_0^{2\pi}\frac{\partial f}{\partial\rho}\sin\theta d\theta d\varphi. \tag{2.18}$$

On the other hand,

$$4\pi\rho^2A_0(f;\rho) = \int_{R=\rho} fdS = \int_0^\pi\int_0^{2\pi} f\rho^2\sin\theta d\theta d\varphi,$$

so

$$4\pi A_0(f;\rho) = \int_0^{\pi}\int_0^{2\pi} f \sin\theta d\theta d\varphi. \tag{2.19}$$

Putting (2.18), (2.19) together, we have

$$\frac{d}{d\rho}A_0(f;\rho) = \frac{\rho}{3}B_0(\Delta f;\rho).$$

We are done noting that $A_a(f;0) = f(a)$. ■

The scalar fields whose Laplacian vanish are called **harmonic functions**. By the previous theorem, harmonic functions satisfy the **mean value property**: *The average of any harmonic function on any sphere equals the value of the function at the center of the sphere.* (Conversely, any continuous function with the mean value property is a (smooth) harmonic function [Ahl, Page 242][AD, C.5.3].) The most important examples of harmonic functions are

$$\log r \text{ on } r \neq 0 \qquad \text{and} \qquad \frac{1}{R} \text{ on } R \neq 0.$$

(Also see Section 2.7.)

Exercise: Show that a non-constant harmonic function can not have minimum or maximum in a connected region except at its boundary. (*Hint.* Use the mean value property, continuity, and connectivity.)

Exercise: Prove that the following functions are harmonic:

(a)

$$\frac{\cos\theta}{R^2} = \frac{z}{R^3}, \frac{x}{R^3}, \frac{y}{R^3} \text{ on } R \neq 0,$$
$$\frac{3\cos^2\theta - 1}{R^3} = \frac{2z^2 - x^2 - y^2}{R^5} \text{ on } R \neq 0,$$
$$\frac{xy}{R^5}, \frac{xyz}{R^7} \text{ on } R \neq 0,$$

(b)

$$\varphi \text{ on } 0 < \varphi < 2\pi,$$
$$\log|\tan(\theta/2)| \text{ on } 0 < \theta < \pi,$$

(c)

$$e^{\lambda x}\cos(\lambda y), e^{\lambda x}\sin(\lambda y) \text{ on } \mathbb{R}^3, \text{ where } \lambda \text{ is a real parameter},$$
$$r^{\pm\lambda}\cos(\lambda\varphi), r^{\pm\lambda}\sin(\lambda\varphi) \text{ on } r \neq 0,$$

(d)

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{\sqrt{-1}n\varphi} = \frac{1-r^2}{1-2r\cos\varphi + r^2} \text{ on } \mathbb{R}^3 \setminus \{r=1, \varphi=0\}.$$

Exercise: Prove the Green's reciprocity theorem:

$$\oint_S (f\nabla g - g\nabla f)\cdot \overrightarrow{dS} = \int_V (f\Delta g - g\Delta f)dV,$$

for a volume V bounded by surface S.

Exercise 12. Fulfilling the following steps, prove that if $f(x, y, z)$ is a harmonic function, then so is

$$g(x, y, z) := \frac{1}{R} f\left(\frac{x}{R^2}, \frac{y}{R^2}, \frac{z}{R^2}\right),$$

where $R = \sqrt{x^2 + y^2 + z^2}$. g is called the **Kelvin transform** of f. (A direct proof of this fact by the chain rule needs a very long computation. Another clever proof can be found in [Axl, Page 63].)

(a) The transformation

$$(x, y, z) \mapsto (X, Y, Z), \quad X = \frac{x}{R^2}, \quad Y = \frac{y}{R^2}, \quad Z = \frac{z}{R^2},$$

celled the **inversion with respect to the unit sphere**, gives a left-handed, orthogonal, coordinate system.

(b) The metric coefficients are $\frac{1}{R^2}, \frac{1}{R^2}, \frac{1}{R^2}$.

(c) By (2.17),

$$\Delta f = R^6 \left(\frac{\partial}{\partial X}\left(\frac{1}{R^2}\frac{\partial f}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\frac{1}{R^2}\frac{\partial f}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\frac{1}{R^2}\frac{\partial f}{\partial Z}\right)\right).$$

(d) Set $g := f/R$. Then, using the computation

$$\begin{aligned}\left(\frac{1}{R^2} f_X\right)_X = \frac{1}{R^2} f_{XX} + \frac{2}{R} f_X \left(\frac{1}{R}\right)_X &= \frac{1}{R}\left(f_{XX}\frac{1}{R} + 2 f_X \left(\frac{1}{R}\right)_X\right) \\ &= \frac{1}{R}\left(g_{XX} - f\left(\frac{1}{R}\right)_{XX}\right),\end{aligned}$$

and the harmonicity of $\frac{1}{R}$, deduce that the Laplacians of f and g are related by $\Delta f = R^5 \Delta g$.

(e) Prove the fact. ■

Exercise: Using Exercise 12, prove that the functions $\frac{x}{R^3}, \frac{xy}{R^5}, \frac{xyz}{R^7}$ are harmonic.

2.4.5 Laplacian of a vector field

The **Laplacian** of a vector field A is defined by

$$\nabla^2 A := \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A). \tag{2.20}$$

It is also denoted by ΔA. It is straightforward to show that in cartesian coordinates $A = A_x \widehat{x} + A_y \widehat{y} + A_z \widehat{z}$, we have

$$\Delta A = (\Delta A_x)\widehat{x} + (\Delta A_y)\widehat{y} + (\Delta A_z)\widehat{z}.$$

2.5 Some differential and integral identities

$$\nabla \times \nabla f = 0, \quad \nabla \cdot \nabla \times A = 0, \tag{2.21}$$

$$\nabla(fg) = f\nabla g + g\nabla f,$$

$$\nabla \cdot (fA) = \nabla f \cdot A + f\nabla \cdot A, \tag{2.22}$$

$$\nabla \times (fA) = \nabla f \times A + f\nabla \times A, \tag{2.23}$$

$$\Delta(fg) = f\Delta g + g\Delta f + \nabla f \cdot \nabla g,$$

$$\nabla \cdot (A \times B) = \nabla \times A \cdot B - \nabla \times B \cdot A,$$

$$\oint f\overrightarrow{dS} = \int \nabla f \; dV \quad \text{(gradient theorem)} \tag{2.24}$$

$$\oint \overrightarrow{dS} \times A = \int \nabla \times A \; dV \quad \text{(curl theorem)} \tag{2.25}$$

$$\oint f\overrightarrow{dl} = \int \overrightarrow{dS} \times \nabla f \quad \text{(cross} - \text{gradient theorem)} \tag{2.26}$$

$$\oint \overrightarrow{dl} \times A = \int (\overrightarrow{dS} \times \nabla) \times A \quad \text{(cross} - \text{del} - \text{cross theorem)} \tag{2.27}$$

Exercise: (a) Prove (2.24) and (2.25) by applying (2.12) to vector fields fC and $A \times C$, where C is a constant vector. (b) Prove (2.26) and (2.27) by applying (2.16) to vector fields fC and $A \times C$, where C is a constant vector.

Exercise: Find formulas for $\nabla(A \cdot B)$ and $\nabla \times (A \times B)$ in [NB, Chapter 1].

Exercise: Prove that $\oint \overrightarrow{dS} = 0$.

Exercise: Prove that $\oint \overrightarrow{R} \times \overrightarrow{dl} = 2\int \overrightarrow{dS}$. (*Hint.* Assuming $\overrightarrow{dS} = dS_x\widehat{x} + dS_x\widehat{y} + dS_z\widehat{z}$, compute $(\overrightarrow{dS} \times \nabla) \times \overrightarrow{R} = -2\overrightarrow{dS}$, and then use (2.27). Another method is to argue geometrically and find an interpretation for $\oint_C \overrightarrow{R} \times \overrightarrow{dl}$ as the lateral surface vector, and then use the previous exercise.)

2.6 Scalar and vector potentials

Theorem 13 (Scalar potential). *Let A be a vector field on an open*[5] *subset $U \subseteq \mathbb{R}^3$, which has the property that any closed curve in U is the boundary of some surface in U.*[6] *Then, the followings are equivalent:*

(a) A is the gradient of some scalar field.

(b) The curl of A vanishes on U.

(c) The circulation of A along any closed curve in U vanishes.

---

[5] U being *open* means for any point $P \in U$, there exists a ball around P which is still contained in U.

[6] Namely, the first homology group of U vanishes.

Proof. (a)⇒(b) Immediate from the identity (2.21).

(b)⇒(c) Immediate from the Stokes' theorem and the topological assumption on U.

(c)⇒(a) We assert that $A = \nabla f$, where f is given by the line integral

$$f(P) = \int_{P_0}^{P} A \cdot \overrightarrow{dl},$$

along any path from a fixed point $P_0 \in U$ to P; it does not matter which path we use, because of our assumption (c). (Here we are assuming that U is connected, namely, any two points can be connected to each other. If not, apply the argument here to each *connected component* of U.) In computing

$$\frac{\partial f}{\partial x} = \frac{1}{dx}\left(\int_{P_0}^{P+dx\widehat{x}} A \cdot \overrightarrow{dl} - \int_{P_0}^{P} A \cdot \overrightarrow{dl}\right) = \frac{1}{dx}\int_{P}^{P+dx\widehat{x}} A \cdot \overrightarrow{dl},$$

let us take the straight-line path from P to $P + dx\widehat{x}$. Then,

$$\frac{\partial f}{\partial x} = \frac{1}{dx}(A_x\widehat{x} + A_y\widehat{y} + A_z\widehat{z}) \cdot dx\widehat{x} = A_x.$$

Similarly, $\partial f/\partial y = A_y$ and $\partial f/\partial z = A_z$. ∎

Example 14. A simple computation shows that the vector field

$$A = 3y\widehat{x} + (3x - 2z)\widehat{y} - (2y + z)\widehat{z}$$

is curl-free in $\mathbb{R}^3$. Let us find a scalar field f such that $A = \nabla f$. Integrating $f_x = 3y$ with respect to x gives

$$f = 3yx + g(y, z),$$

where g is a (smooth) function of y, z. Then,

$$3x - 2z = f_y = 3x + g_y$$

implies $g_y = -2z$. Integrating this with respect to y gives

$$g = -2zy + h(z),$$

where h is a function of z. Then,

$$-2y - z = f_z = g_z = -2y + h'(z),$$

implies $h'(z) = -z$. Therefore, $h(z) = -z^2/2 + C$, where C is a constant. The whole analysis shows that

$$f = 3xy - 2yz - \frac{z^2}{2} + C$$

satisfies $A = \nabla f$. *Alternative method.* We compute the line integral

$$f(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} A \cdot d\overrightarrow{R}$$

along the straight-line path

$$\overrightarrow{R}(t) = tx\widehat{x} + ty\widehat{y} + tz\widehat{z}, \quad 0 \le t \le 1,$$

from $(0,0,0)$ to (x,y,z). Since

$$\begin{aligned} A \cdot d\overrightarrow{R} &= 3Y dX + (3X - 2Z)dY - (2Y + Z)dZ, \qquad X = tx,\ Y = ty,\ Z = tz \\ &= (3xy + (3x - 2z)y - (2y + z)z)tdt \\ &= (6xy - 4yz - z^2)tdt, \end{aligned}$$

we have

$$f = \int_0^1 (6xy - 4yz - z^2)tdt = 3xy - 2yz - \frac{z^2}{2}.$$

This is the same as before. ∎

Exercise: Show that the vector field

$$A := (e^x \cos y + yz)\widehat{x} + (xz - e^x \sin y)\widehat{y} + (xy + z)\widehat{z}$$

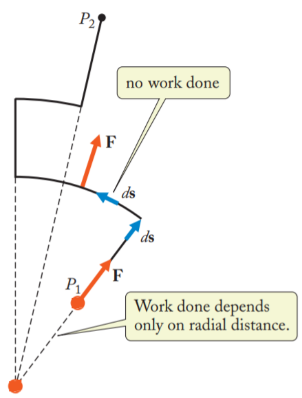
is curl-free in $\mathbb{R}^3$, and find a scalar field f such that $A = \nabla f$.

Exercise 15. Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary (smooth) function.

(a) Prove that the divergence and curl of the vector field $A := g(R)\widehat{R}$ is respectively given by $g_R + \frac{2}{R}g$ and 0.

(b) Prove that the gradient and laplacian of the scalar field $f := g(R)$ is respectively given by $g_R\widehat{R}$ and $g_{RR} + \frac{2}{R}g_R$.

(c) Use the following figure to intuitively justify the fact that a radial (or central) vector fields $F := g(R)\widehat{R}$ is curl-free.



One can not drop the simply connected assumption from Theorem 13, as the following example shows:

Example 16. The vector field

$$A = \frac{-y}{x^2 + y^2}\widehat{x} + \frac{x}{x^2 + y^2}\widehat{y} = \frac{r\widehat{\varphi}}{r^2} = \frac{\widehat{\varphi}}{r}$$

is curl-free. However, it can not be written as the gradient of a scalar field in its maximal domain of definition $\mathbb{R}^3 \setminus z$-axis, because if it could, then the circulation of A along a circle of radius a in the xoy-plane around the origin would have been

$$0 = \oint A \cdot \overrightarrow{dl} = \int_0^{2\pi} \frac{\widehat{\varphi}}{a} \cdot a\widehat{\varphi} d\varphi = 2\pi.$$

However, on any open subset of the space which does not turn around the z-axis (namely, any subset of $\mathbb{R}^3 \setminus \{r = 0 \text{ or } \varphi = \varphi_0\}$ for some φ_0), the polar angle φ is a well-defined (single-valued) function, and $A = \nabla\varphi$. ■

Theorem 17 (Vector potential). *Let A be a vector field on an open subset $U \subseteq \mathbb{R}^3$, which has the property that any closed surface in U is the boundary of some open subset of U.*[7] *Then, the followings are equivalent:*

(a) A is the curl of some vector field.

(b) The divergence of A vanishes on U.

(c) The flux of A across any closed surface vanishes.

Proof. We only prove the special case $U = \mathbb{R}^3$. (The general case is proved in differential topology, under the name of the de Rham theorem [Lee, 18.14]: The vanishing of the second singular homology group implies the vanishing of the second de Rham cohomology group.)

(a)⇒(b) Immediate from the identity (2.21).

(b)⇒(c) Immediate from the divergence theorem and the topological assumption on U.

(c)⇒(a) We assert that $A = \nabla \times B$, where B is given by

$$B(x, y, z) = \left(\int_{z_0}^{z} A_y(x, y, t)dt - \int_{y_0}^{y} A_z(x, t, z_0)dt \right) \widehat{x} - \left(\int_{z_0}^{z} A_x(x, y, t)dt \right) \widehat{y}, \quad (2.28)$$

and $P_0 = (x_0, y_0, z_0)$ is a fixed point in space. We compute

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = A_x(x, y, z),$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = A_y(x, y, z),$$

$$\begin{aligned}\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -\int_{z_0}^{z} \frac{\partial A_x}{\partial x}(x, y, t)dt - \int_{z_0}^{z} \frac{\partial A_y}{\partial y}(x, y, t)dt + A_z(x, y, z_0) \\ &= \int_{z_0}^{z} \frac{\partial A_z}{\partial z}(x, y, t)dt + A_z(x, y, z_0) = A_z(x, y, z).\end{aligned}$$

This completes the proof. The main idea in deriving the formula (2.28) is that by adding an expression like ∇f to B, one can assume that B has no z-component. The rest is straightforward. ■

[7] Namely, the second homology group of U vanishes.

One can not drop the topological assumption from Theorem 17, as the following example shows:

Example 18. The inverse square field

$$A = \frac{\widehat{R}}{R^2} = \frac{x\widehat{x} + y\widehat{y} + z\widehat{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

is well-defined and divergence-free on $\mathbb{R}^3 \setminus \{(0,0,0)\}$, but can not be written as the curl of a vector field, because if it could, then the flux of A across an sphere of radius a around the origin would have been

$$0 = \oint A \cdot \overrightarrow{dS} = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{\widehat{R}}{a^2} \cdot a^2 \sin\theta d\theta d\varphi \widehat{R} = 4\pi.$$

However, on any open subset of the space which does not turn around the z-axis (namely, any subset of $\mathbb{R}^3 \setminus \{r = 0 \text{ or } \varphi = \varphi_0\}$ for some φ_0), we can write $A = \nabla \times (\varphi \sin\theta \widehat{\theta})$. ■

Exercise: (a) Show that the inverse square field $B = \widehat{R}/R^2$ can be written as the curl of the vector field

$$A_+ = \frac{1-\cos\theta}{R\sin\theta}\widehat{\varphi} \quad \left(\text{respectively, } A_- = \frac{-1-\cos\theta}{R\sin\theta}\widehat{\varphi}\right)$$

on $U_+ = \mathbb{R}^3 \setminus \{(0,0,z)\ z \leq 0\}$ (respectively, $U_- = \mathbb{R}^3 \setminus \{(0,0,z)\ z \geq 0\}$). (b) Note that $\mathbb{R}^3 \setminus \{(0,0,0)\}$ can be covered by U_+, U_-, but the values of A_+, A_- do not match on the intersection $U_+ \cap U_-$; in fact, $A_+ - A_- = \nabla(2\theta)$.

2.7 Dirac delta function

The **Dirac delta** (or **unit impulse**) **function** $\delta(t)$ on the real line $\mathbb{R}$ can be defined in either of the following equivalent ways:

- $\delta(t)$ is characterized by two properties:

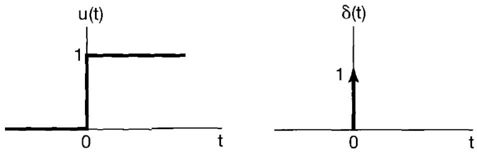
$$\int_{\mathbb{R}} \delta(t)dt = 1 \qquad \text{and} \qquad \delta(t) = 0 \text{ for } t \neq 0.$$

  No Riemann (or even Lebesgue) integrable function satisfy these properties.

- $\delta(t)$ can be thought as the limit of rectangular pulses

$$\delta_\epsilon(t) = \begin{cases} \epsilon^{-1}, & 0 < t < \epsilon, \\ 0, & \text{otherwise,} \end{cases}$$

  as $\epsilon \to 0+$.



- $\delta(t)$ could be thought of as the *formal* derivative[8] of the unit step function $u(t)$. More generally, if a function $f(t)$ is differentiable everywhere on the real lineexcept at finitely many discontinuity points $t_1, \ldots, t_n$, and that at each point t_k, $k = 1, \ldots, n$, the function jumps j_k units, then the summand

$$\sum_{k=1}^{n} j_k \delta(t - t_k),$$

appears in the first derivative $f'(t)$. For example, if $f(t) = |t|$, then

$$f'(t) = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \end{cases} \qquad \text{and} \qquad f''(t) = 2\delta(t).$$

- $\delta(t)$ can be characterized by the **sifting** (or **sampling**) **property**:

$$f(t)\delta(t-a) \equiv f(a)\delta(t-a),$$

where a is a real constant, and f is any smooth function. The integral version of the sampling property is

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau.$$

This expresses a function as linear combinations (superpositions) of impulses.

Similarly, the Dirac delta function on the three-dimensional space $\delta(\overrightarrow{R}) = \delta(x, y, z)$ is characterized by properties:

$$\int_{\mathbb{R}^3} \delta(x, y, z)dV = 1 \qquad \text{and} \qquad \delta(x, y, z) = 0 \text{ for } (x, y, z) \neq 0.$$

This shows that the volume charge density (measured in coulomb per meter cubed) due to a point charge of q coulombs situated at the point (x_0, y_0, z_0) is given by

$$\rho = q\delta(x - x_0, y - y_0, z - z_0).$$

In terms of the one-dimensional Dirac functions, we have

$$\delta(x, y, z) = \delta(x)\delta(y)\delta(z).$$

Exercise: Show that the three-dimensional delta function in spherical coordinates is given by $(R^2 \sin\theta)^{-1}\delta(R)\delta(\theta)\delta(\varphi)$.

Exercise: Justify that the volume charge density of an infinitely-thin rod situated along the z-axis can be expressed by $\lambda(z)\delta(x)\delta(y)$, where λ is the linear charge density measured in coulomb per unit length.

[8]Technical terminology is *distributional* (or *weak*) derivative.

Theorem 19. *(a)* $-\Delta\frac{1}{R} = \nabla \cdot \frac{\widehat{R}}{R^2} = 4\pi\delta(\overrightarrow{R})$.

(b) The outward flux of the inverse square field $R^{-2}\widehat{R}$ across any closed surface S is either 4π or 0 depending on whether S contains the origin or not.

Proof. (a) According to Exercise 15, away from the origin, we have

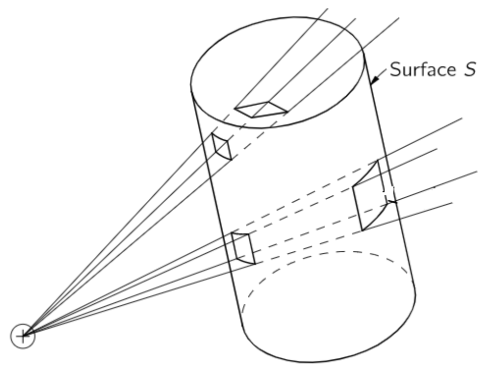
$$-\Delta\frac{1}{R} = \nabla \cdot \frac{\widehat{R}}{R^2} = \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) + \frac{2}{R}\frac{1}{R^2} = 0.$$

This shows that the inverse square field $\widehat{R}/R^2$ is divergence-free, except at the origin. If we want to make the divergence theorem valid on the whole space, then for any ball of radius $a > 0$ around the origin, we must have

$$\int_{\text{ball } R\leq a}\left(\nabla \cdot \frac{\widehat{R}}{R^2}\right)dV = \int_{\text{sphere } R=a}\frac{\widehat{R}}{R^2}\cdot\overrightarrow{dS} = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{\widehat{R}}{a^2}\cdot a^2\sin\theta d\theta d\varphi\widehat{R} = 4\pi.$$

Because of this, one says that the weak (or distributional) divergence of the inverse square field equals 4π times the three-dimensional Dirac delta function.

(b) This is (a) expressed in words. However, there is a beautiful, elementary, geometric argument for this fact that we now present. Let us first assume that S does not contain the origin. The fundamental fact is that the flux of the inverse square field passing through an infinitesimal surface $\overrightarrow{dS} = dS\widehat{n}$ does not depend on the distance of $\overrightarrow{dS}$ to the origin or its orientation, but only on the **solid angle** represented by $\overrightarrow{dS}$, namely, on the area of the projection of $\overrightarrow{dS}$ onto the unit sphere. The total flux across S is zero because the flux corresponding to two pieces of S cut out by rays from the origin and forming a small solid angle cancel each other. Next, assume that S contains the origin. By the



same argument, the flux across S equals the flux across the unit sphere, which clearly equals its area, 4π. ■

Theorem 20. *For any constant k, we have*

$$\left(\Delta + k^2\right)\left(\frac{e^{\pm\sqrt{-1}kR}}{R}\right) = -4\pi\delta(\overrightarrow{R}).$$

Proof. Let us try to find a radial scalar field $G = G(R)$ which satisfies $(\Delta + k^2)\, G = -4\pi\delta(\overrightarrow{R})$. For this property to hold, it is necessary and sufficient that for every $a > 0$, we have

$$-4\pi = \int_{\text{ball } R\leq a} \left(\Delta + k^2\right) G dV.$$

By the divergence theorem,

$$\begin{aligned} -4\pi &= \int_{\text{sphere } R=a} \nabla G \cdot \overrightarrow{dS} + k^2 \int_{R\leq a} G dV \\ &= \int_{R=a} \frac{\partial G}{\partial R} dS + k^2 \int_{R\leq a} G dV = 4\pi a^2 \frac{dG}{dR}(a) + 4\pi k^2 \int_0^a R^2 G dR, \end{aligned}$$

or equivalently,

$$a^2 \frac{dG}{dR}(a) + k^2 \int_0^a R^2 G dR = -1.$$

After differentiation,

$$a^2 \frac{d^2 G}{da^2} + 2a \frac{dG}{da} + k^2 a^2 G = 0, \quad \lim_{a\to 0} a^2 G'(a) = -1.$$

One can easily check that this latter differential equation has the general solution

$$G(a) = c_1 \frac{e^{\sqrt{-1}ka}}{a} + c_2 \frac{e^{-\sqrt{-1}ka}}{a},$$

where c_1, c_2 are constants. The initial condition $\lim a^2 G'(a) = -1$ is satisfied exactly when $c_1 + c_2 = 1$. Arguing backwards shows that $G = R^{-1} e^{\pm\sqrt{-1}kR}$ satisfies $(\Delta + k^2)G = -4\pi\delta$. ■

Exercise: Justify

$$\delta(-t) = \delta(t), \quad \delta(2t) = \frac{1}{2}\delta(t), \quad t^a \delta(t) = 0,$$

where a is a positive real number.

Exercise: Justify

$$\delta\left(t^2 - 1\right) = \frac{1}{2}\delta(t+1) + \frac{1}{2}\delta(t-1).$$

(*Hint.* For part (b), either use the definition $\delta(t) = \lim_{\epsilon\to 0+} \delta_\epsilon(t)$, and approximation $\sqrt{1 \pm \epsilon} = 1 \pm \epsilon/2$, or differentiate $u\left(t^2 - 1\right) = u(-t-1) + u(t-1)$ by chain rule.)

Exercise: Justify

$$\int_{-\infty}^{\infty} \exp(\sqrt{-1}xt) dx = 2\pi\delta(t).$$

Exercise: What is the flux of the vector field $A = (x\widehat{x} + y\widehat{y} + z\widehat{z})/(x^2 + y^2 + z^2)^{3/2}$ passing through the sphere $(x-1)^2 + y^2 + z^2 = 2$? (*Answer.* 4π.)

Exercise: Prove the two-dimensional analogue of Theorem 19: $\Delta \log r = \nabla \cdot \frac{\widehat{r}}{r}$ equals 2π times the two-dimensional Dirac delta function $\delta(x, y)$.

2.8 Helmholtz-Hodge decomposition theorem

Theorem 21. *Let $U \subseteq \mathbb{R}^3$ be an open subset with smooth boundary S.*

(a) A $C^2(\overline{U})$ vector field A can be written as the sum of a curl-free, and a divergence-free vector field. More precisely,

$$A = -\nabla f + \nabla \times B, \tag{2.29}$$

where, using notations introduced on page 3,

$$f(\overrightarrow{R}) = \int_U \frac{\nabla' \cdot A(\overrightarrow{R}')}{4\pi R_0} dV - \oint_S \frac{A(\overrightarrow{R}') \cdot \overrightarrow{dS}}{4\pi R_0},$$

$$B(\overrightarrow{R}) = \int_U \frac{\nabla' \times A(\overrightarrow{R}')}{4\pi R_0} dV + \oint_S \frac{A(\overrightarrow{R}') \times \overrightarrow{dS}}{4\pi R_0},$$

and

$$\nabla' = \widehat{x}\frac{\partial}{\partial x'} + \widehat{y}\frac{\partial}{\partial y'} + \widehat{z}\frac{\partial}{\partial z'}$$

is the nabla operator taken with respect to the source (not the field) position vector $\overrightarrow{R}' = x'\widehat{x} + y'\widehat{y} + z'\widehat{z}$.

(b) A $C^2(\overline{U})$ vector field is uniquely determined with its curl and divergence on U, and its value on S.

(b′) Let U have the property that any closed curve in U is the boundary of some surface in it. Then, a $C^2(\overline{U})$ vector field is uniquely determined with its curl and divergence on U, and its normal component on S.

(b″) A $C^2(\mathbb{R}^3)$ vector field A is uniquely determined with its divergence and curl if it decays no slower than $1/R$ at infinity, namely, if $RA \to 0$ as $R \to \infty$. If so, then

$$A = -\nabla \int_U \frac{\nabla' \cdot A}{4\pi R_0} dV + \nabla \times \int_U \frac{\nabla' \times A}{4\pi R_0} dV.$$

Proof. (a) According to Theorem 19, $4\pi A(\overrightarrow{R})$ equals

$$-\int_U \Delta\left(\frac{1}{R_0}\right) A(\overrightarrow{R}')dV = -\int_U \Delta\left(\frac{A(\overrightarrow{R}')}{R_0}\right) dV = -\Delta \int_U \frac{A(\overrightarrow{R}')}{R_0} dV.$$

This latter, according to (2.20), can be written as

$$-\nabla f + \nabla \times B,$$

where

$$f = \nabla \cdot \int_U \frac{A(\overrightarrow{R}')}{R_0} dV, \quad B = \nabla \times \int_U \frac{A(\overrightarrow{R}')}{R_0} dV.$$

Using the identity (2.22) and the divergence theorem, we have

$$f = \int_U \nabla \cdot \left(\frac{A(\overrightarrow{R}')}{R_0} \right) dV = \int_U A(\overrightarrow{R}') \cdot \nabla \left(\frac{1}{R_0} \right) dV = -\int_U A(\overrightarrow{R}') \cdot \nabla' \left(\frac{1}{R_0} \right) dV$$
$$= -\int_U \nabla' \cdot \left(\frac{A(\overrightarrow{R}')}{R_0} \right) dV + \int_U \frac{\nabla' \cdot A(\overrightarrow{R}')}{R_0} dV = -\oint_S \frac{A(\overrightarrow{R}')}{R_0} \cdot \overrightarrow{dS} + \int_U \frac{\nabla' \cdot A(\overrightarrow{R}')}{R_0} dV.$$

Similarly, using the identity (2.23) and the divergence theorm,

$$B = \int_U \nabla \times \left(\frac{A(\overrightarrow{R}')}{R_0} \right) dV' = \int_U \nabla \left(\frac{1}{R_0} \right) \times A(\overrightarrow{R}') dV = -\int_U \nabla' \left(\frac{1}{R_0} \right) \times A(\overrightarrow{R}') dV$$
$$= -\int_U \nabla' \times \left(\frac{A(\overrightarrow{R}')}{R_0} \right) dV + \int_U \frac{\nabla' \times A(\overrightarrow{R}')}{R_0} dV = \oint_S \frac{A(\overrightarrow{R}') \times \overrightarrow{dS}}{R_0} + \int_U \frac{\nabla' \times A(\overrightarrow{R}')}{R_0} dV.$$

(b) Immediate from (a).

(b′) Since the problem is linear, assuming $\nabla \cdot A = 0$, $\nabla \times A = 0$, $A_n := A \cdot \widehat{n} = 0$, we need to show that A vanishes everywhere on U. Since A is curl-free and the first homology of U vanishes, one can find a scalar field f such that $A = \nabla f$. Since $0 = \nabla f \cdot n$, we have

$$0 = \oint_S (f \nabla f) \cdot \overrightarrow{dS} = \int_U \nabla \cdot (f \nabla f) dV = \int_U |\nabla f|^2 dV + \int_V f(\Delta f) dV = \int_U |\nabla f|^2 dV.$$

Therefore, $\nabla f \equiv 0$ on U.

(b″) Apply the representation formula in (a) to a ball U of radius R around the origin. When R grows large, the surface integral terms approach zero. ■

Exercise: This exercise gives a direct proof of Theorem 21.(c) using Fourier transform. Fourier transform of a vector field B is given by the vector field $\widetilde{B}$ defined by

$$\widetilde{B}(\overrightarrow{k}) = \int_{\mathbb{R}^3} B(\overrightarrow{R}) e^{\sqrt{-1}\overrightarrow{k} \cdot \overrightarrow{R}} dV(\overrightarrow{R}),$$

with the inverse given by

$$B(\overrightarrow{R}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \widetilde{B}(\overrightarrow{k}) e^{-\sqrt{-1}\overrightarrow{k} \cdot \overrightarrow{R}} dV(\overrightarrow{k}).$$

Similarly, the Fourier transform of scalar fields is defined.

(a) Show that applying the Fourier transform to (2.29) gives $\widetilde{A} = -\sqrt{-1}\overrightarrow{k} \cdot \widetilde{f} + \sqrt{-1}\overrightarrow{k} \times \widetilde{B}$.

(b) Show that $\widetilde{f} = \sqrt{-1}\frac{\overrightarrow{k} \cdot \widetilde{A}}{\overrightarrow{k} \cdot \overrightarrow{k}}$ and $\widetilde{B} = \sqrt{-1}\frac{\overrightarrow{k} \times \widetilde{A}}{\overrightarrow{k} \cdot \overrightarrow{k}}$ satisfy this latter equation.

(c) Give formulas for f and B using inverse Fourier transform.

2.9 Transport theorems

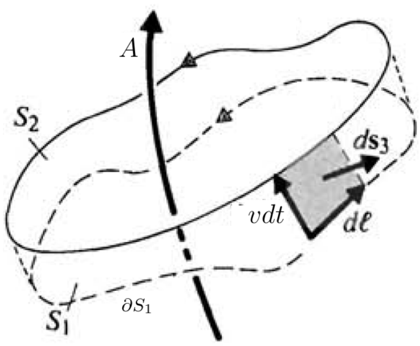
If the curve, surface, or volume that we are taking integral over changes through time, then we will have the following transport theorems:

$$\frac{d}{dt}\int_{C(t)} A\cdot\overrightarrow{dl} = \int_{C(t)}\left(\frac{\partial A}{\partial t} - v\times(\nabla\times A)\right)\cdot\overrightarrow{dl}, \tag{2.30}$$

$$\frac{d}{dt}\int_{S(t)} A\cdot\overrightarrow{dS} = \int_{S(t)}\left(\frac{\partial A}{\partial t} + v(\nabla\cdot A) - \nabla\times(v\times A)\right)\cdot\overrightarrow{dS}, \tag{2.31}$$

$$\frac{d}{dt}\int_{V(t)} f dV = \int_{V(t)}\left(\frac{\partial f}{\partial t} + \nabla\cdot(fv)\right)dV, \tag{2.32}$$

where v is the velocity vector field of the evolution of $C(t)$, $S(t)$, or $V(t)$. We only prove the second identity; others can be proved similarly. Let S_1, S_2, S_3 respectively denote the surfaces $S(t)$, $S(t+dt)$, and the lateral surface swept by the displacement vector vdt as $S(t)$ evolves into $S(t+dt)$. Let V denote the region bounded by S_1, S_2, S_3. In the following computations, by A, $\frac{\partial A}{\partial t}$, we mean $A(t)$, $\frac{\partial A}{\partial t}(t)$, respectively. We compute the



left hand side of (2.31) as follows:

$$\int_{S(t+dt)} A(t+dt)\cdot\overrightarrow{dS} = \int_{S_2}\left(A + \frac{\partial A}{\partial t}dt\right)\cdot\overrightarrow{dS} = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{S_1}\left(A + \frac{\partial A}{\partial t}dt\right)\cdot\overrightarrow{dS} = \int_{S_1} A\cdot\overrightarrow{dS} + dt\int_{S_1}\frac{\partial A}{\partial t}\cdot\overrightarrow{dS},$$

$$\begin{aligned} I_2 = \int_{-S_1+S_2+S_3}\left(A + \frac{\partial A}{\partial t}dt\right)\cdot\overrightarrow{dS} &= \int_V \nabla\cdot\left(A + \frac{\partial A}{\partial t}dt\right)dV \\ &= \int_{S_1}\nabla\cdot\left(A + \frac{\partial A}{\partial t}dt\right)vdt\cdot\overrightarrow{dS} = dt\int_{S_1}(\nabla\cdot A)v\cdot\overrightarrow{dS}, \end{aligned}$$

$$\begin{aligned} I_3 = -\int_{S_3}\left(A + \frac{\partial A}{\partial t}dt\right)\cdot\overrightarrow{dS} = -\int_{\partial S_1}\left(A + \frac{\partial A}{\partial t}dt\right)\cdot\overrightarrow{dl}\times vdt &= -dt\int_{\partial S_1} A\cdot\overrightarrow{dl}\times v \\ &= -dt\int_{\partial S_1} v\times A\cdot\overrightarrow{dl} = -dt\int_{S_1}\nabla\times(v\times A)\cdot\overrightarrow{dS}. \end{aligned}$$

All the three terms in the right hand side of (2.31) have been revealed.

Exercise: Prove (2.30) and (2.32).

Exercise: In (2.31), assuming $S(t)$ to be closed and setting $f := \nabla \cdot A$, find another proof for (2.32).

Exercise: Either prove directly, or deduce the following elementary transport theorem for a time-dependent real-valued function $f(x,t)$ of a real variable x:

$$\frac{d}{dt}\int_{u(t)}^{v(t)} f(x,t)dx = \int_{u(t)}^{v(t)} \frac{\partial f}{\partial t}(x,t)dx + v'(t)f(v(t),t) - u'(t)f(u(t),t).$$

2.10 Visualizing scalar and vector fields

A vector field can be visualized by drawing vectors at many points in space, each of which gives the field strength and direction at that point. Alternatively, one can draw **field lines**: A collection of smooth curves that are everywhere tangent to the vector field, and the number of lines per unit area at the surface perpendicular to the direction of the field should be proportional to the magnitude of the field. Therefore, the number of field lines passing a surface is proportional to the flux. Therefore, unless for divergence-free regions, new lines sometimes start up in order to keep the number up to the strength of the field.

Exercise: Draw the field lines for the constant field, inverse square field, and the position vector field.

A scalar field can be visualized by **level surfaces** (or **isosurfaces**), namely surfaces such that the scalar field is constant on them. When the scalar field is a potential function (respectively, temperature), these are called equipotential (respectively, isothermal) surfaces.

2.11 Differential forms

The left column in Figure 1 shows how Maxwell was writing his equations [Max]. Later on, Gibbs and Heaviside developed vector calculus (presented in this chapter) to write Maxwell's equations in a concise way, the right column in the latter figure. Elie Cartan developed a calculus generalizing the vector calculus presented in the chapter. In this theory, scalar and vector fields have a common generalization called *differential forms* ω; The gradient, divergence, and curl have a common generalization called the *exterior derivative* d [Lee, Page 426]; and curves, surfaces, and volumes have common generalization called (smooth) manifolds M. In this theory, the identities (2.9), (2.16), (2.12), together with the fundamental theorem of (undergraduate) calculus $\int_a^b f'(x)dx = f(b) - f(a)$, are generalized to

$$\oint_{\partial M} \omega = \int_M d\omega.$$

Also, the null identities (2.21) have the common form

$$dd\omega = 0.$$

All three transport theorems of Section 2.9 can be generalized to

$$\frac{d}{dt}\int_{M(t)} \omega = \int_{M(t)} \frac{\partial \omega}{\partial t} + \int_{M(t)} v \lrcorner d\omega + \int_{\partial M(t)} v \lrcorner \omega,$$

where $\lrcorner$ is the operation of interior product [Fla]. This theory is developed in [Lee, Chapters 14–16]. Electromagnetics can also be developed in this language [Lin].

$a = \frac{dH}{dy} - \frac{dG}{dz}$ $b = \frac{dF}{dz} - \frac{dH}{dx}$ $c = \frac{dG}{dx} - \frac{dF}{dy}$	(A)	$\mathbf{B} = \nabla \times \mathbf{A}$
$P = c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{dF}{dt} - \frac{d\psi}{dx}$ $Q = a\frac{dz}{dt} - c\frac{dx}{dt} - \frac{dG}{dt} - \frac{d\psi}{dy}$ $R = b\frac{dx}{dt} - a\frac{dy}{dt} - \frac{dH}{dt} - \frac{d\psi}{dz}$	(B)	$\mathbf{E} = \mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$
$X = vc - wb$ $Y = wa - uc$ $Z = ub - va$	(C)	$\mathbf{F} = \mathbf{J} \times \mathbf{B}$
$a = \alpha + 4\pi A$ $b = \beta + 4\pi B$ $c = \gamma + 4\pi C$	(D)	$\mathbf{B} = \mu_o \mathbf{H} + \mathbf{M}$
$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}$ $4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx}$ $4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy}$	(E)	$\mathbf{J} = \nabla \times \mathbf{H}$
$\mathfrak{D} = \frac{1}{4\pi} K \mathfrak{E}$	(F)	$\mathbf{D} = \epsilon \mathbf{E}$
$\mathfrak{K} = C \mathfrak{E}$	(G)	$\mathbf{J}_c = \sigma \mathbf{E}$
$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}$	(H)	$\mathbf{J} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$
$u = p + \frac{df}{dt}$ $v = q + \frac{dg}{dt}$ $w = r + \frac{dh}{dt}$	(H*)	$\mathbf{J} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$
$\mathfrak{C} = (C + \frac{1}{4\pi} K \frac{d}{dt}) \mathfrak{E}$	(I)	$\mathbf{J} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$
$u = CP + \frac{1}{4\pi} K \frac{dP}{dt}$ $v = CQ + \frac{1}{4\pi} K \frac{dQ}{dt}$ $w = CR + \frac{1}{4\pi} K \frac{dR}{dt}$	(I*)	$\mathbf{J} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$
$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$	(J)	$\varrho = \nabla \cdot \mathbf{D}$
$\sigma = lf + mg + nh + l'f' + m'g' + n'h'$	(K)	$\varrho_s = \mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2)$
$\mathfrak{B} = \mu \mathfrak{H}$	(L)	$\mathbf{B} = \mu \mathbf{H}$

Figure 1: The original set of equations as labeled by Maxwell in his treatise [Max], with their interpretation in modern vector calculus notation of Gibbs and Heaviside. The simplest equations were also written in vector form. Taken from [Lin, Page 3].

Chapter 3

Electrostatics

3.1 Axioms

Since the situations described by the full Maxwell's equations (1.2) can be very complicated (these equations already contain all of the special relativity effects), we start by analyzing the simplest situation: We are in the vacuum, and all the charges are fixed in space, namely, ρ does not depend on time, and $J = 0$. This is called **electrostatics**, and is governed by the differential axioms

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \nabla \times B = 0.$$

According to the Helmholtz-Hodge theorem, there are no magnetic effects (B vanishes everywhere), and the electric phenomena are governed by

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times E = 0. \tag{3.1}$$

These differential equations, according to the divergence and Stokes theorems, can be equivalently expressed by the following integral equations:

$$\oint_S E \cdot \overrightarrow{dS} = \frac{Q}{\epsilon_0}, \quad \oint_C E \cdot \overrightarrow{dl} = 0, \tag{3.2}$$

where S is an arbitrary closed surface containing total charge Q inside, and C is an arbitrary closed curve.

We first want to justify that: *The electrostatic field produced by the volume charge density ρ is given by*

$$E(\overrightarrow{R}) = \frac{1}{4\pi\epsilon_0} \int \frac{\overrightarrow{R} - \overrightarrow{R}'}{\left|\overrightarrow{R} - \overrightarrow{R}'\right|^3} \rho(\overrightarrow{R}') dV,$$

or, more concisely,

$$E = \frac{1}{4\pi\epsilon_0} \int \frac{\widehat{R}_0}{R_0^2} \rho dV. \tag{3.3}$$

Here are some theoretical/experimental justifications for this formula:

- It is immediate from the Helmholtz-Hodge theorem (Theorem 21) assuming some growth conditions at infinity:

$$E = -\nabla \int_V \frac{\rho(\overrightarrow{R}')/\epsilon_0}{4\pi R_0} dV = -\frac{1}{4\pi\epsilon_0}\int_V \nabla\left(\frac{1}{R_0}\right)\rho dV = \frac{1}{4\pi\epsilon_0}\int \frac{\widehat{R}_0}{R_0^2}\rho dV.$$

This computation also shows that the electrostatic field (3.3) can be written as $E = -\nabla\Phi$, where the **electrostatic potential** V is given by

$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho dV}{R_0}. \tag{3.4}$$

The inclusion of the negative sign in $E = -\nabla\Phi$ is to make the potential decreasing as we move along the field lines.

- Assuming the uniqueness of solutions to the electrostatic axioms (3.1 or 3.2), from the linearity of these equations, we discover a fundamental fact in field theory: *If a charge distribution ρ_j, $j = 1, 2$ produces field E_j, then the charge distribution $\rho_1 + \rho_2$ produces the field $E_1 + E_2$.* This is called the **superposition principle**, and its validity goes beyond electrostatics. Because of this principle, to show (3.3), it suffices to show that the electrostatic field due to a point charge of q coulombs situated at the origin is given by $E = \frac{q}{4\pi\epsilon_0}\frac{\widehat{R}}{R^2}$. By symmetry, $E = E_R(R)\widehat{R}$. Plugging this into the integral axiom $\oint E \cdot \overrightarrow{dS} = Q/\epsilon_0$, applied to a sphere of radius R centered at the origin, gives

$$E_R \times 4\pi R^2 = \frac{q}{\epsilon_0},$$

and we are done.

Exercise: The electrostatic potential of a point charge q situated at the origin is given by $\Phi = \frac{1}{4\pi\epsilon_0}\frac{1}{R}$.

Exercise: Where did the axiom $\nabla \times E = 0$ appear in the justification above? (Note that every radial field $E_R(R)\widehat{R}$ already satisfies this axiom.)

- Experiments done by Coulomb/Cavendish show that the electric force between two motion-less point charges q, q' is inversely proportional to the square of the distance between them, applies on the line connecting them, is proportional to each charge, and is repulsive (respectively, attractive) if the charges are like (respectively, unlike).[1] Therefore, according to the Lorentz law (1.1), the electrostatic field at the point $\overrightarrow{R}'$ generated by a point charge q situated at the point $\overrightarrow{R}$ is given by

$$E(\overrightarrow{R}) = (\text{The Coulomb force applied on the unit charge at } \overrightarrow{R}') = (\text{constant}) \times \frac{q}{R^2}\widehat{R}_0.$$

The constant is determined to be 9×10^9 in SI, which for some historical reasons is written as $1/(4\pi\epsilon_0)$. This formula combined with the superposition principle (also confirmed by experiments) gives (3.3).

[1] If charges move, then this force needs to be corrected by the special relativity considerations [Fey, Volume I, Chapter 28].

Remark 22. Sometimes we are assuming that charges are distributed on surfaces or curves. Although in such situations the volume charge density ρ is infinity, but the formulas (3.3) and (3.4) are still valid if one replaces ρdV by σdS or λdl, where σ (respectively, λ) is the surface (respectively, linear) charge density, measured in coulomb per meter squared (respectively, coulomb per meter).

Here are some examples to use the formulas (3.3) and (3.4) in action:

1. The electrostatic field at the point $\overrightarrow{R} = r\widehat{r} + z\widehat{z}$, produced by a uniform linear charge density λ on a rod situated along the z-axis is given by

$$E = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}} \frac{r\widehat{r} + z\widehat{z} - z'\widehat{z}}{|r\widehat{r} + z\widehat{z} - z'\widehat{z}|^3}\lambda dz' = \frac{\lambda}{4\pi\epsilon_0}\int_{\mathbb{R}} \frac{r\widehat{r} + (z - z')\widehat{z}}{(r^2 + (z - z')^2)^{\frac{3}{2}}}dz' = \frac{\lambda r\widehat{r}}{4\pi\epsilon_0}\int_{\mathbb{R}} \frac{dz'}{(r^2 + z'^2)^{\frac{3}{2}}},$$

which, after the change of variable $z' = r\tan\alpha$, equals

$$\frac{\lambda}{2\pi\epsilon_0}\frac{\widehat{r}}{r}. \tag{3.5}$$

Here is an easier way to derive this: By symmetry, $E = E_r(r)\widehat{r}$. Plugging this into the integral axiom $\oint E \cdot \overrightarrow{dS} = Q/\epsilon_0$ applied to a cylinder of height h and radius r along the z-axis gives

$$E_r \times 2\pi r h = \frac{\lambda h}{\epsilon_0}.$$

Since $\frac{\widehat{r}}{r} = \nabla \log r$, the electrostatic potential is given by

$$-\frac{\lambda}{2\pi\epsilon_0}\log r. \tag{3.6}$$

Remark 23. The field (3.5) blows up at the points with large z and bounded r. This violates our growth-at-infinity assumption that was used to derive (3.3). A more precise treatment would be to first consider a finite rod, and then study the fields as the length of the bar becomes large. This is done [NB, Example 2.2], and ends up to the same result (3.5). In the same spirit, the integral in (3.4) for the infinite rod turns out to be infinite:

$$\frac{1}{4\pi\epsilon_0}\int \frac{\lambda dz'}{|r\widehat{r} + z\widehat{z} - z'\widehat{z}|} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{dz'}{(r^2 + z'^2)^{\frac{1}{2}}} = \infty.$$

However, one can extract the "principal part (3.6)" from this improper integral by first integrating over a finite rod, and then make the length large, as the following computations show:

$$\int_{-L}^{L} \frac{dz'}{(r^2 + z'^2)^{\frac{1}{2}}} = 2\int_0^L \frac{dz'}{\sqrt{r^2 + z'^2}} = \left[\log \frac{1 + \frac{z'}{\sqrt{r^2+z'^2}}}{1 - \frac{z'}{\sqrt{r^2+z'^2}}}\right]_{z'=0}^{z'=L} = \log \frac{1 + \frac{L}{\sqrt{r^2+L^2}}}{1 - \frac{L}{\sqrt{r^2+L^2}}}$$
$$= \log\left(\frac{4L^2 + O(L)}{r^2}\right) = \log\left(4L^2 + O(L)\right) - \frac{1}{2}\log r.$$

2. The electrostatic field at the point $\overrightarrow{R} = x\widehat{x}+y\widehat{y}+z\widehat{z}$ produced by the constant surface charge density σ situated on the xoy-plane is given by

$$E = \frac{1}{4\pi\epsilon_0}\int \frac{x\widehat{x}+y\widehat{y}+z\widehat{z}-(x'\widehat{x}+y'\widehat{y})}{|x\widehat{x}+y\widehat{y}+z\widehat{z}-(x'\widehat{x}+y'\widehat{y}+z'\widehat{z})|^3}\sigma dx'dy'$$
$$= \frac{\sigma}{4\pi\epsilon_0}\int \frac{(x-x')\widehat{x}+(y-y')\widehat{y}+z\widehat{z}}{((x-x')^2+(y-y')^2+z^2)^{\frac{3}{2}}}dx'dy' = \frac{\sigma}{4\pi\epsilon_0}\int \frac{-x'\widehat{x}-y'\widehat{y}+z\widehat{z}}{(x'^2+y'^2+z^2)^{\frac{3}{2}}}dx'dy'$$
$$= \frac{\sigma z\widehat{z}}{4\pi\epsilon_0}\int \frac{r'dr'd\varphi'}{(r'^2+z^2)^{\frac{3}{2}}},$$

which, after the change of variable $r' = z\tan\alpha$, equals

$$\pm\frac{\sigma}{2\epsilon_0}\widehat{z}, \tag{3.7}$$

depending on whether $z > 0$ or $z < 0$. Here is an easier way to derive this: By symmetry, $E = E_z(z)\widehat{z}$, where $E_z(z)$ is an odd function of z. Plugging this into the integral axiom $\oint E\cdot\overrightarrow{dS} = Q/\epsilon_0$ applied to a cylinder of height $2h$ and radius a with bases at $z = \pm h$ gives

$$2E_z(h)\times\pi a^2 = \frac{\sigma\pi a^2}{\epsilon_0}.$$

Since $\widehat{z} = \nabla z$, the electrostatic potential is given by

$$\mp\frac{\sigma}{2\epsilon_0}z. \tag{3.8}$$

Remark 24. The same comments as in Remark 23 also hold here. The field (3.7) does not satisfy the "faster that $1/R$" decay assumption that was used to derive (3.3). A more precise treatment would be to first consider a finite disc, and then study the fields as the radius of the disc becomes large. This is done [NB, Example 2.4], and ends up to the same result (3.7). In the same spirit, the integral in (3.4) for the infinite surface turns out to be infinite:

$$\frac{1}{4\pi\epsilon_0}\int\frac{\sigma dx'dy'}{|r\widehat{r}+z\widehat{z}-z'\widehat{z}|} = \frac{\sigma}{4\pi\epsilon_0}\int\frac{rdrd\varphi}{(r^2+(z-z')^2)^{\frac{1}{2}}} = \frac{\sigma}{2\epsilon_0}\int_0^\infty\frac{rdr}{(r^2+(z-z')^2)^{\frac{1}{2}}} = \infty.$$

However, one can extract the "principal part (4.8)" from this improper integral by first integrating over a finite disc, and then make the radius large. Do this as an exercise.

Exercise: Compute the electrostatic field due to a charge Q distributed uniformly on a ball of radius a around the origin.

Exercise: Find the electrostatic field due to two parallel infinite sheets with equal and opposite uniform charge densities $\pm\sigma$. (*Answer.* σ/ϵ_0 between plates, and zero outside.)

Example 25. Let us find the motion-less charge distribution which causes the field

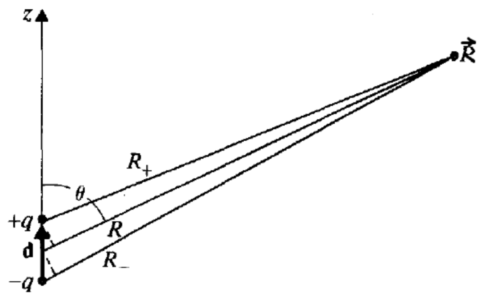
$$E = \begin{cases} az\widehat{z}, & 0\le z\le l,\\ b\widehat{z}, & \text{otherwise,}\end{cases}$$

where a, b, l are constants, and $0 \neq b \neq al$. According to the differential axiom $\nabla \cdot E = \rho/\epsilon_0$, we have $\rho = a\epsilon_0$ when $0 < z < l$, and $\rho = 0$ when $z < 0$ or $z > l$. At $z = 0$, E_z jumps from b to 0, hence $\rho = -b\epsilon_0\delta(z)$. This means that we have the surface charge distribution $\sigma = -b\epsilon_0$ at $z = 0$. Similarly, we have the surface charge distribution $\sigma = (b - al)\epsilon_0$ at $z = l$.

Exercise: Assume two thin spherical shells of radii $a < b$ around the origin, with uniformly distributed total charges q, Q. Find the electric field using the integral axiom (3.2), and then find the electric potential with the formula $\Phi(P) = -\int_{P_0}^{P} E \cdot \overrightarrow{dl}$.

3.2 Electric dipole

An **electric dipole** is two point charges $\pm q$, separated by some a distance d. The quantity q times the displacement vector from $-q$ to q is called the **electric dipole moment** and denoted by p. Let us place charges $\pm q$ at $\pm\frac{d}{2}\widehat{z}$, and find the approximate form of the electrostatic field produced by this dipole at points far away from the origin.



We compute

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right) \approx \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R - \frac{d}{2}\cos\theta} - \frac{1}{R + \frac{d}{2}\cos\theta}\right) \approx \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2}$$

$$= \frac{qd}{4\pi\epsilon_0}\frac{x}{R^3} = \frac{p \cdot \widehat{R}}{4\pi\epsilon_0}\frac{1}{R^2}, \quad (3.9)$$

$$E \approx -\nabla\left(\frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2}\right) = -\frac{qd}{4\pi\epsilon_0}\left(-2\frac{\cos\theta}{R^3}\widehat{R} - \frac{\sin\theta}{R^3}\widehat{\theta}\right) = \frac{qd}{4\pi\epsilon_0}\frac{1}{R^3}\left(2\cos\theta\widehat{R} + \sin\theta\widehat{\theta}\right)$$

$$= \frac{qd}{4\pi\epsilon_0}\frac{1}{R^3}\left(3\cos\theta\widehat{R} + \left(\sin\theta\widehat{\theta} - \cos\theta\widehat{R}\right)\right) = \frac{3(p\cdot\widehat{R})\widehat{R} - p}{4\pi\epsilon_0}\frac{1}{R^3}.$$

The important point here is that: *The electrostatic potential and field of an electric dipole decays like R^{-2} and R^{-3} with respect to distance, compared to the R^{-1} and R^{-2} variations for a point charge.*

Exercise: Show that the far-field of the electric dipole is divergence and curl free away from the dipole's center.

Exercise: Find another proof for (3.9) by reading [Fey, Volume II, Section 6.4].

3.3 Far-field multipole expansion

Let us analyze the electrostatic potential caused by a charge distribution ρ bounded in some part of the space, at points far away. To do this, we use the binomial formula

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots, \quad |z|<1,$$

to expand

$$\frac{1}{R_0} = \frac{1}{\left|\vec{R}-\vec{R}'\right|} = \frac{1}{R}\left(1 - 2\frac{\vec{R}\cdot\vec{R}'}{R^2} + \left(\frac{R'}{R}\right)^2\right)^{-\frac{1}{2}} \tag{3.10}$$

in the integrand of (3.4). In the far-field region $\frac{R'}{R}<1$, we have

$$\begin{aligned}\frac{1}{R_0} &= \frac{1}{R}\left(1 - \frac{1}{2}\left(-2\frac{\vec{R}\cdot\vec{R}'}{R^2} + \left(\frac{R'}{R}\right)^2\right) + \frac{3}{8}\left(-2\frac{\vec{R}\cdot\vec{R}'}{R^2} + \left(\frac{R'}{R}\right)^2\right)^2 - + \cdots\right) \\ &= \frac{1}{R}\left(1 + \widehat{R}\cdot\frac{\vec{R}'}{R} + \frac{1}{2}\left(3\left(\widehat{R}\cdot\frac{\vec{R}'}{R}\right)^2 - \left(\frac{R'}{R}\right)^2\right) + \cdots\right).\end{aligned}$$

This gives the **multi-pole expansion**

$$\Phi = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots, \tag{3.11}$$

where

$$\Phi^{(0)} = (\text{monopole term}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}, \quad Q = (\text{total charge}) = \int dq,$$

$$\Phi^{(1)} = (\text{dipole term}) = \frac{1}{4\pi\epsilon_0}\frac{p\cdot\widehat{R}}{R^2}, \quad p = (\text{dipole moment}) = \int \vec{R}' dq,$$

$$\begin{aligned}\Phi^{(2)} = (\text{quadrupole term}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{2R^3}\int\left(3\left(\widehat{R}\cdot\vec{R}'\right)^2 - R'^2\right)dq \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{2R^5}\int\left(3\left(\vec{R}\cdot\vec{R}'\right)^2 - R^2R'^2\right)dq.\end{aligned}$$

The main point in this expansion is: The successive terms decay by a factor of order $1/R$, and a term becomes dominant if all the previous ones vanish. For example, if the total charge $Q=\int dq$ of the distribution vanishes, then the dipole terms becomes dominant.

In cartesian coordinates $\vec{R} = x_1\widehat{x}_1 + x_2\widehat{x}_2 + x_3\widehat{x}_3$, $\vec{R}' = x_1'\widehat{x}_1' + x_2'\widehat{x}_2' + x_3'\widehat{x}_3'$, the quadruple term can be written as

$$\Phi^{(2)} = \frac{1}{4\pi\epsilon_0}\frac{1}{2R^5}\sum_{i,j=1}^{3} Q_{ij}(3x_jx_j - R^2\delta_{ij}),$$

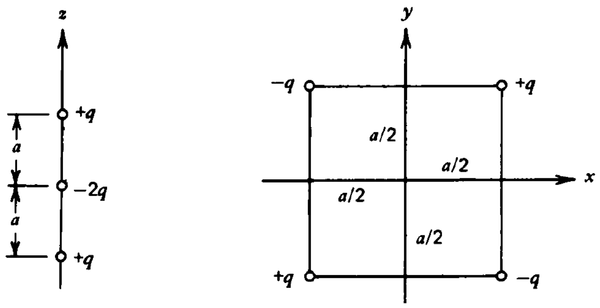
where

$$Q_{ij} = \text{(quadrupole matrix)} = \int x_i' x_j' dq, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

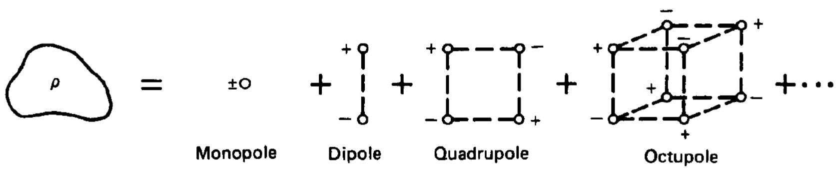
Exercise: Show that if the total charge of the distribution vanishes, then the dipole moment does not depend on the choice of the origin, namely, the start point of the charge position vector $\overrightarrow{R}'$.

Exercise: Compute the dipole moment of the following charge distributions: (a) A cylindrical rod of length l, placed along the z axis $0 \leq z \leq l$, with the volume charge density $\rho = a(z - l/2)$. (b) A spherical shell, centered at the origin, with the surface charge density $\sigma = \sigma_0 \cos\theta$. (*Answer.* (a) $\widehat{z}\frac{l^2 a}{12} \times$ volume; (b) $\widehat{z}\sigma_0 \times$ volume.)

Exercise: For the following charge distributions, show that the quadruple term is the dominant term, and compute it. (*Answer.* (a) $\frac{qa^2}{4\pi\epsilon_0}\frac{3\cos^2\theta - 1}{R^3}$; (b) $\frac{qa^2}{4\pi\epsilon_0}\frac{3xy}{R^5}$.)



One intuition behind the multipole expansion is to approximate an arbitrary charge distribution by a combination of a point charge, a point dipole, a point quadrupole, and so forth, as shown in the following figure. This viewpoint can be applied to approximate quantities other than the electrostatic potential.



Remark 26. One can find explicit formulas for higher multipole terms using the generating function of Legendre polynomials. One need only to expand (3.10) using (3.12):

$$\frac{1}{R_0} = \frac{1}{R}\sum_{n\geq 0}\left(\frac{R'}{R}\right)^n P_n(\cos\psi), \quad \cos\psi = \widehat{R}\cdot\widehat{R}', \quad R \gg R'.$$

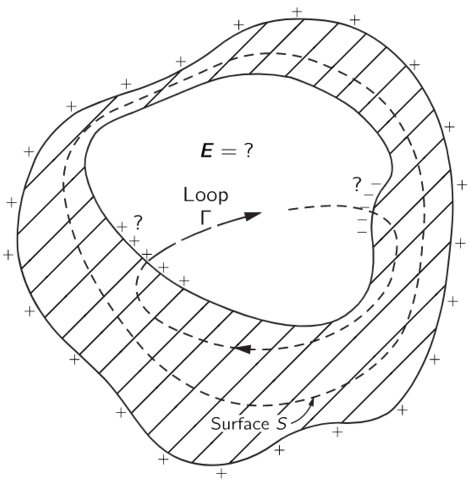
3.4 Conductors

From now on, we want to include the effect of matters in electrostatics. A **conductor** is a matter with a sufficient supply of free electrons, namely, loose electrons in the outmost shells of the atoms. Most metals are so, although, the conductivity properties of materials change with temperature, pressure, etc. Materials with negligible content of free electrons are called **dielectrics** or **insulators**. (We later introduce a quantity called conductivity through the equation $J = \sigma E$, and then conductors are those material with high conductivity.) When a conductor is placed in an external electrostatic field, its free electrons content moves as a whole body in the opposite direction of the field until it makes the total field inside zero. This transient phenomenon takes a short time of the order 10^{-19} seconds. After the quiliberium:

- According to the differential axiom $\rho = \epsilon_0 \nabla \cdot E$, all the charges will be distributed on the surface.

- Since the electric field is zero inside, the whole conductor is an equipotential body. Therefore, on the conductor's surface, the electrostatic field is normal to the surface. A simple application of the integral axiom $\oint E \cdot \overrightarrow{dS} = Q/\epsilon_0$ shows that the magnitude of this normal field equals σ/ϵ_0, if the conductor is placed in vacuum.

Exercise: Consider a conductor with a vacuum hollow (cavity) inside. Prove that there is no electric field in the cavity, and all the polarization charges accumulate on the outer surface of the conductor. (*Hint.* Use both axioms (3.2).)



3.5 Electrostatic boundary conditions

Reference [NB, Section 3.3].

3.6 Electrostatic boundary value problems

In electrostatics, if the charge distribution is given, then the field intensity can be computed using the integral formula (3.3). There are situations where the exact charge distribution is not known. For example, if we place several conductors and point charges (or other known charge distributions) in vaccum, then the equilibrium surface charge distribution on the conductors is not known apriori: This charge distribution depends on the electric field ($\sigma = \epsilon_0 E_n$), and the electric field itself depends on the charge distribution. Since each conductor is an equipotential body, and there is no charge in the vaccum region between conductors and known charges, it is a good idea to eliminate the electric field in the axioms $\nabla \cdot E = 0$, $E = -\nabla\Phi$, and work directly with Φ:

$$\Delta\Phi = \nabla \cdot \nabla\Phi = -\nabla \cdot E = 0.$$

The second-order partial differential equation $\Delta\Phi = 0$ is called the **Laplace equation**: *In charge-free regions, the electrostatic potential is a harmonic function.* This equation together with the constant-Φ condition on the conductor surfaces is called a **electrostatics boundary value problem**, and this section is devoted to it.

Remark 27. A more general equation than the Laplace equation is the **Poisson equation** $\Delta\Phi = -\rho/\epsilon_0$, obtained by eliminating the electric field between the axioms $\nabla \cdot E = \rho/\epsilon_0$, $E = -\nabla\Phi$.

Theorem 28 (Existence and uniqueness theorem for the Dirichlet and Neumann problems)**.** *Let $U \subseteq \mathbb{R}^3$ be a connected, open subset with smooth boundary S. Let Φ_0 be a smooth function on S.*

*(a) The problem of finding a smooth function Φ on $\overline{U}$ which satisfies the Laplace equation $\Delta\Phi = 0$ on U, and the **Dirichlet boundary condition** $\Phi = \Phi_0$ on S has a unique solution. (If U is unbounded, then we assume that Φ decays no slower that $1/R$ at infinity.)*

*(b) The problem of finding a smooth function Φ on $\overline{U}$ which satisfies the Laplace equation $\Delta\Phi = 0$ on U, and the **Neumann boundary condition** $\partial\Phi/\partial n = \Phi_0$ on S has a solution if and only if $\oint_S \Phi_0 dS = 0$. If so, then the solution is unique up to additive constants. (If U is unbounded, then we assume that Φ decays no slower that $1/R$ at infinity.)*

Proof. We only prove the uniqueness here. The existence is a very deep result, with different proofs: [Fol, 3.40, 7.33][CH, Volume II, Chapter 4][Eva, 6.3.2][Jos, 23.15][Hel, Med, GM][Tay, Chapter 5][Ahl, Chapter 6][Con, Chapters 10, 19, 21]. Note that the necessary condition in (b) comes from

$$\oint_S \Phi_0 dS = \oint_S \frac{\partial\Phi}{\partial n} dS = \oint_S \nabla\Phi \cdot \overrightarrow{dS} = \int_U \Delta\Phi dV = 0.$$

(a) We repeat the argument used in the proof of Theorem 21.(b′). Since the problem is linear, assuming $\Delta\Phi = 0$, $\Phi|_S = 0$, we need to show that $\Phi \equiv 0$ on U. The computation

$$0 = \oint_S (\Phi\nabla\Phi) \cdot \overrightarrow{dS} = \int_U \nabla \cdot (\Phi\nabla\Phi) dV = \int_U |\nabla\Phi|^2 dV + \int_U \Phi(\Delta\Phi) dV = \int_U |\nabla\Phi|^2 dV,$$

shows that $\nabla\Phi \equiv 0$ on U. Since U is connected and $\Phi|_S \equiv 0$, it follows that $\Phi \equiv 0$ on U.

(b) The argument in (a) works. Note that $\Phi\nabla\Phi \cdot \overrightarrow{dS} = \Phi\frac{\partial\Phi}{\partial n} dS$. ■

3.6.1 Method of separation of variables

The main use of the uniqueness theorem is that, if by any means (guess, conformal transformations, probabilistic methods, etc.), we find *a* solution, that is *the* solution. In order to efficiently fulfill this strategy, we need a good supply of harmonic functions. This is addressed in Theorems 29, 31.

Theorem 29 (One-variable harmonic functions)**.** *The harmonic functions on the three-dimensional space which only depend on one variable in cartesian, cylindrical, or spherical coordinates are*

$$ax + b, \quad ay + b, \quad az + b,$$
$$a \log r + b, \quad a\varphi + b,$$
$$\frac{a}{R} + b, \quad a \log \tan(\theta/2) + b,$$

where a, b are constants.

Proof. If $\Phi = \Phi(\theta)$ is harmonic, then

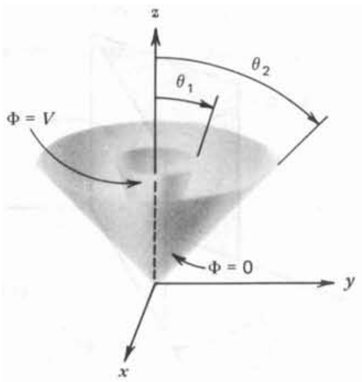
$$0 = \Delta\Phi = \frac{1}{R^2 \sin\theta} \left(\frac{\partial}{\partial\theta} (\sin\theta\Phi') \right),$$

hence,

$$\Phi = b + \int \frac{a d\theta}{\sin\theta} = b + a \log \tan(\theta/2).$$

The other cases can be treated similarly. ■

Example 30. Consider a capacitor consisting of two conductor cones, with axes along the z-axis, apexes at the origin, and with angles $\theta_1 < \theta_2$. Suppose that these cones are kept at potentials $0, \Phi_0$, respectively. By symmetry, we guess that the potential Φ between cones



only depends on θ. One can easily find constant a, b such that $\Phi = a \log \tan(\theta/2) + b$ satisfies the boundary conditions $\Phi(\theta_1) = 0, \Phi(\theta_2) = \Phi_0$:

$$\Phi = \Phi_0 \frac{\log \frac{\tan(\theta/2)}{\tan(\theta_1/2)}}{\log \frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}}.$$

This is *the* electrostatic potential, according to the uniqueness theorem. ■

Exercise: Find the electrostatic field E in the previous example, as well as the surface charge distribution on conductors. (*Hint.* Use the equations $E = -\nabla\Phi$, $\sigma = \epsilon_0 E \cdot \widehat{n}$.)

Theorem 31 (Two-variable harmonic functions)**.** *The following functions are harmonic:*

$$(ax + a')\,(by + b')\,,$$

$$(a\cos\alpha x + a'\sin\alpha x)\left(be^{\alpha y} + b'e^{-\alpha y}\right),$$

$$\left(ae^{\alpha x} + a'e^{-\alpha x}\right)(b\cos\alpha y + b'\sin\alpha y)\,,$$

$$r^{\pm n}\cos n\varphi, \quad r^{\pm n}\sin n\varphi, \quad (n = 1, 2, 3, \ldots)$$

$$R^n P_n(\cos\theta), \quad R^{-n-1}P_n(\cos\theta), \quad (n = 1, 2, 3, \ldots)$$

where a, a', b, b', α are constants, and

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2 - 1)^n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^j \frac{(2n - 2j)!}{2^n j!(n - j)!(n - 2j)!}t^{n-2j}$$

is the ***Legendre polynomial*** *of degree n. Other ways to characterize the Legendre polynomials are:*

- *The generating function:*

$$\sum_{n=0}^{\infty} P_n(t)z^n = \frac{1}{\sqrt{1 - 2zt + z^2}}. \tag{3.12}$$

- *$P_0(t), P_1(t), P_2(t), \ldots$, up to multiplicative constants, are the output of the Gram-Schmidt orthogonalization procedure applied to $1, t, t^2, \ldots$, with the inner product given by $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$. We have*

$$\int_{-1}^{1} P_n(t)P_m(t)dt = \frac{2}{2n + 1}\delta_{nm}.$$

Proof. We show how these functions have been found. Let us try to find non-constant, harmonic functions of the form $\Phi = \Phi(x, y) = A(x)B(y)$. We have

$$0 = \Delta\Phi = A''B + AB'',$$

so

$$\frac{A''}{A} + \frac{B''}{B} = 0.$$

This is a function of x plus a function of y equals zero. Since we are looking for non-constant harmonic functions, we must have

$$\frac{A''}{A} = (\text{constant}) = \lambda^2 = -\frac{B''}{B}.$$

(There is no loss of generality in showing an arbitrary constant by λ^2, $\lambda \in \mathbb{C}$. It will soon become clear why we have chosen such a representation.) Depending on whether $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$, we get the solutions in the first three lines of the statement of the theorem.

Next, let us try to find non-constant, harmonic functions of the form $\Phi = \Phi(r, \varphi) = A(r)B(\varphi)$. We have

$$0 = \Delta\Phi = \frac{1}{r}\left(\frac{\partial}{\partial r}(rA'B) + \frac{\partial}{\partial\varphi}\left(\frac{1}{r}AB'\right)\right) = \frac{1}{r}\left((rA')'B + \frac{A}{r}B''\right),$$

so

$$r\frac{(rA')'}{A} + \frac{B''}{B} = 0.$$

This is a function of r plus a function of φ equals zero. Since we are looking for non-constant harmonic functions, we must have

$$r\frac{(rA')'}{A} = (\text{constant}) = \lambda^2 = -\frac{B''}{B}.$$

(There is no loss of generality in showing an arbitrary constant by λ^2, $\lambda \in \mathbb{C}$. It will soon become clear why we have chosen such a representation.) The resulting ordinary differential equations

$$r^2A'' + rA' - \lambda^2 A = 0, \quad B'' + \lambda^2 B = 0$$

have general solutions

$$A = ar^{\lambda} + br^{-\lambda}, \quad B = c\cos\lambda\varphi + d\sin\lambda\varphi.$$

Reversing the whole argument shows that

$$r^{\pm\lambda}\cos\lambda\varphi, \quad r^{\pm\lambda}\sin\lambda\varphi$$

are harmonic functions for any complex constant λ. Why we have chosen λ to be a positive integer in the statement of the theorem? Because in the simplest examples that we want to analyze in this chapter, the geometry of our situations contains the full range of φ, namely, $[0, 2\pi]$; and, in order to our smooth function $B(\varphi)$ to satisfy the natural periodic condition $B(\varphi+2\pi) = B(\varphi)$, λ must be an integer. On the other hand, λ ranging over negative integers gives no essentially new harmonic function not given by λ ranging over positive integers.

The treatment of the spherical coordinates is similar. Plugging $\Phi = \Phi(R, \theta) = A(R)B(\theta)$ into $\Delta\Phi = 0$ gives

$$0 = \frac{1}{R^2\sin\theta}\left(\frac{\partial}{\partial R}\left(R^2\sin\theta A'B\right) + \frac{\partial}{\partial\theta}(\sin\theta AB')\right) = \frac{\sin\theta\,(R^2A')'B + A(\sin\theta B')'}{R^2\sin\theta},$$

so

$$\frac{(R^2A')'}{A} + \frac{1}{\sin\theta}\frac{(\sin\theta B')'}{B} = 0.$$

This is a function of R plus a function of θ equals zero. Since we are looking for non-constant harmonic functions, we must have

$$\frac{(R^2A')'}{A} = (\text{constant}) = \lambda(\lambda+1) = -\frac{1}{\sin\theta}\frac{(\sin\theta B')'}{B}.$$

(There is no loss of generality in showing an arbitrary constant by $\lambda(\lambda+1)$, $\lambda \in \mathbb{C}$. It will soon become clear why we have chosen such a representation.) The resulting ordinary differential equations are

$$R^2A'' + 2RA' - \lambda(\lambda+1)A = 0, \quad B'' + \cot\theta B' + \lambda(\lambda+1)B = 0.$$

The first equation has the general solution

$$A = aR^{\lambda} + bR^{-\lambda-1},$$

and the second equation, after the change of variable $t = \cos\theta$ becomes

$$(1-t^2)\frac{d^2B}{dt^2} - 2t\frac{dB}{dt} + \lambda(\lambda+1)B = 0.$$

This is called the **Legendre differential equation**. Since the functions

$$\frac{-2t}{1-t^2} \quad \text{and} \quad \frac{\lambda(\lambda+1)}{1-t^2}$$

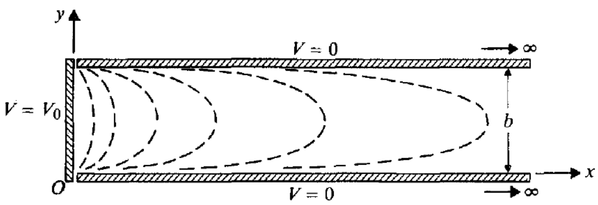
are given by power series (real analytic functions) with the radius of convergence 1 around the point $t = 0$, a classical theorem [Apo, Volume II, Page 169][Sim, Page 180] guarantees that the equation can be solved by plugging a power series representation $P(t) := \sum_{j=0}^{\infty} a_j t^j$. Doing so, we infer that a_0, a_1 are arbitrary, and

$$a_{j+2} = \frac{(j-\lambda)(j+\lambda+1)}{(j+1)(j+2)}a_j \quad \text{for } j = 0, 1, 2, \ldots.$$

This formula shows that when λ is an integer, then the Legendre equation has a polynomial solution. To find all these polynomial solutions, it suffices to assume that λ is nonnegative, because a negative integer $\lambda = -n$ gives rise to the same $\lambda(\lambda+1)$ if one uses the nonnegative integer $\lambda = n-1$. So, assume the nonnegative integer $\lambda = n$. Then the polynomial solution, up to a multiplicative constant, is given by

$$P_n(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^j\frac{(2n-2j)!}{2^n j!(n-j)!(n-2j)!}t^{n-2j} = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n.$$

(This and the other properties of the Legendre polynomials is investigated in [Sim, Chapter 8][AW, Chapter 15].) It can be shown that if λ is not an integer, then any nonzero solution of the Legendre differential equation blows up at $t = \pm 1$, namely, $\theta = 0, \pi$. Since in the simplest examples that we want to analyze in this chapter, the geometry of our situations contains the full range of θ, namely, $[0, \pi]$, we only analyzed the integer λ case. ■



Example 32. Two grounded, semi-infinite, parallel-plane conductors are separated by distance b, and a third conductor perpendicular to and insulated from both is maintained at a constant potential V_0. We want to find the electrostatic potential $\Phi = \Phi(x, y)$ in the region between. As the first step, let us find all nonzero, two-variable, separable $A(x)B(y)$ harmonic functions satisfying the zero boundary conditions at $y = 0$ and $y = b$. The general form of such functions is

$$\sin\left(\frac{n\pi}{b}y\right)\left(Ce^{-\frac{n\pi}{b}x} + De^{\frac{n\pi}{b}x}\right),$$

with constants C, D. We choose $D = 0$ to avoid the potential from blowing up as $x \to \infty$. It remains to find constant C_n such that

$$\Phi(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{b}y\right) e^{-\frac{n\pi}{b}x},$$

satisfies the remaining boundary condition

$$V_0 = \Phi(0, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{b}y\right).$$

This can be done by Fourier analysis methods, as we now elaborate. Multiplying both sides of this latter equation by $\sin\left(\frac{m\pi}{b}y\right)$, $m = 1, 2, \ldots$, and then integrating $\int_0^b dy$, because of the orthogonality relations

$$\int_0^b \sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{b}y\right) dy = \frac{b}{2}\delta_{mn},$$

we have

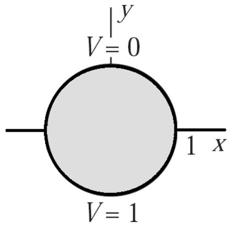
$$C_m = \frac{2}{b}\int_0^b V_0 \sin\left(\frac{m\pi}{b}y\right) dy = \frac{2V_0}{b}\frac{b}{m\pi}\left[-\cos\left(\frac{m\pi}{b}y\right)\right]_{y=0}^{y=b} = \begin{cases} \frac{4V_0}{m\pi}, & m \text{ odd,} \\ 0, & m \text{ even.} \end{cases}$$

The whole analysis shows that

$$\Phi = \frac{4V_0}{\pi}\left(\sin\left(\frac{\pi}{b}y\right)e^{-\frac{\pi}{b}x} + \frac{1}{3}\sin\left(\frac{3\pi}{b}y\right)e^{-\frac{3\pi}{b}x} + \frac{1}{5}\sin\left(\frac{3\pi}{b}y\right)e^{-\frac{5\pi}{b}x} + \cdots\right)$$

is our desired electrostatic potential. ■

Example 33. Let us solve the following electrostatic problem. The idea is to find con-



stant a_n, b_n such that

$$\Phi = \Phi(r,\varphi) = \sum_{n=0}^{\infty} r^n \left(a_n \cos n\varphi + b_n \sin n\varphi\right)$$

solves our boundary value problem. We must have

$$\sum_{n=0}^{\infty} a_n \cos n\varphi + b_n \sin n\varphi = \Phi(1,\varphi) = \begin{cases} 0, & \text{if } 0 < \varphi < \pi, \\ 1, & \text{if } \pi < \varphi < 2\pi. \end{cases} \tag{3.13}$$

This can be done by Fourier analysis methods, as we now elaborate. Integrating $\int_0^{2\pi} d\varphi$ gives

$$2\pi a_0 = \pi \Rightarrow a_0 = \frac{1}{2}.$$

Multiplying both sides of (3.13) by $\cos m\varphi$ (respectively, $\sin m\varphi$), $m = 1, 2, \ldots$, and then integrating $\int_0^{2\pi} d\varphi$, because of the orthogonality relations

$$\int_0^{2\pi} \cos n\varphi \cos m\varphi d\varphi = \int_0^{2\pi} \sin n\varphi \sin m\varphi d\varphi = \pi\delta_{mn},$$

we have

$$a_m = \frac{1}{\pi}\int_{\pi}^{2\pi} \cos m\varphi d\varphi = 0,$$

$$b_m = \frac{1}{\pi}\int_{\pi}^{2\pi} \sin m\varphi d\varphi = \frac{1}{m\pi}\left[-\cos m\varphi\right]_{\varphi=\pi}^{\varphi=2\pi} = \begin{cases} -\frac{2}{m\pi}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases}$$

The whole analysis shows that

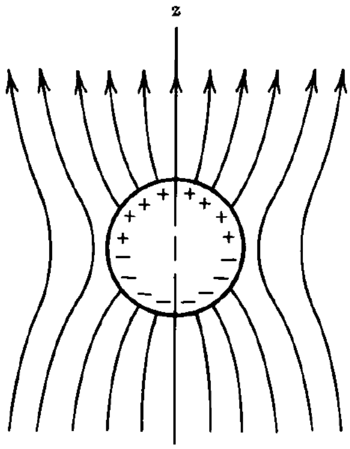
$$\Phi = \frac{1}{2} - \frac{2}{\pi}\left(r\sin\varphi + \frac{r^3}{3}\sin 3\varphi + \frac{r^5}{5}\sin 5\varphi + \cdots\right)$$

is our desired electrostatic potential. ■

Example 34. Suppose we put an uncharged conducting sphere of radius ρ at the origin, inside an initially uniform electric field $E = E_0\widehat{z}$. To find the potential outside the sphere, we try to constant a, b, c, a_n, b_n such that the harmonic function

$$\Phi(R,\theta) = a + \frac{b}{R} + c\log\tan(\theta/2) + \sum_{n=1}^{\infty}\left(a_n R^n + b_n R^{-n-1}\right)P_n(\cos\theta)$$

satisfies our boundary conditions:



- $\Phi(\rho, \theta) = \text{(constant)}$ for every θ.
- $\Phi(R, \theta) \approx -E_0 z + \text{(constant)} = -E_0 R\cos\theta + \text{(constant)}$ for every θ and large R.
- $0 = \text{(total charge)} = -\epsilon_0 \oint_{\text{sphere } R=\rho} \frac{\partial \Phi}{\partial R}(\rho, \theta) dS$.

The second condition gives

$$0 = c = a_2 = a_3 = \cdots, \quad a_1 = -E_0,$$

so

$$\Phi = a + \frac{b}{R} + \left(-E_0 R + \frac{b_1}{R^2}\right)\cos\theta + \sum_{n=2}^{\infty} \frac{b_n}{R^{n+1}} P_n(\cos\theta).$$

The first condition gives

$$b_1 = E_0 \rho^3, \quad 0 = b_2 = b_3 = \cdots,$$

so

$$\Phi = a + \frac{b}{R} - E_0\left(R - \frac{\rho^3}{R^2}\right)\cos\theta.$$

The last condition gives

$$0 = -\epsilon_0 \int \left(\frac{-b}{\rho^2} - 3E_0\cos\theta\right)\rho^2 d\varphi d\theta,$$

hence, $b = 0$. The whole analysis shows that

$$\Phi = a - E_0\left(R - \frac{\rho^3}{R^2}\right)\cos\theta$$

is our desired electrostatic potential. ■

Theorem 35. *The exact electrostatic potential due to a spherical shell of radius a around the origin, carrying a surface charge distribution $\sigma = \sigma_0 \cos\theta$ is given by*

$$\Phi = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{p\cdot\widehat{R}}{R^2}, & R \geq a, \\ \frac{\sigma_0}{3\epsilon_0} z, & R \leq a, \end{cases}$$

where $p = \frac{4}{3}\pi a^3 \sigma_0 \widehat{z}$ is the dipole moment of the distribution.

Proof. [NB, Examples 2.17, 3.6], [Fey, Volume II, Section 6.4]. ■

Remark 36. One can easily find a plenty of three-variable harmonic functions in cartesian coordinates. For example, the function

$$\left(ae^{\sqrt{-1}\alpha x}+a'e^{-\sqrt{-1}\alpha x}\right)\left(be^{\sqrt{-1}\beta y}+b'e^{-\sqrt{-1}\beta y}\right)\left(ce^{\sqrt{-1}\gamma z}+c'e^{-\sqrt{-1}\gamma z}\right),$$

is harmonic for any choice $a, a', b, b', c, c', \alpha, \beta, \gamma$ of constants satisfying $\alpha^2+\beta^2+\gamma^2=0$. However, to find three-variable harmonic functions in cylindrical and spherical coordinates, one needs the Bessel and associated Legendre functions. We are not going to pursue this, but the details can be found [AW, Chapter 9].

3.6.2 Method of images

We explain this method by several examples.

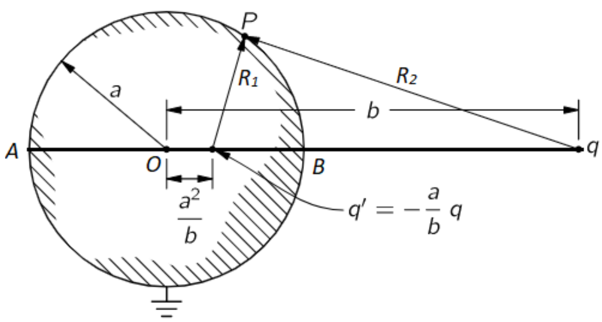
Example 37. Consider a charge q at the point $(0,0,d)$ on the z-axis, above an infinite plane conductor on the xoy-plane of potential zero. The problem is to find the electrostatic potential Φ on the whole space $\mathbb{R}^3$. By the uniqueness theorem, Φ vanishes on $z<0$. The electrostatic boundary value problem on $z>0$ is equivalent to the problem of finding the electrostatic potential of a system of two charges, q at $(0,0,d)$ and $-q$ at $(0,0,-d)$. By the uniqueness theorem,

$$\Phi(x,y,z)=\begin{cases}\frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right), & \text{if } z\geq 0,\\ 0, & \text{if } z<0.\end{cases}$$

■

Exercise: Show that the total charge induced on the conductor in Example above equals $-q$. (*Hint.* Integrate $\sigma=\epsilon_0 E_z=-\epsilon_0\partial\Phi/\partial z$.)

Example 38. Consider a point charge q at distance b from the center of a conducting sphere of radius a and potential 0. The problem is to find the electrostatic potential Φ



on the whole space. By the uniqueness theorem, Φ vanishes inside the sphere. To find the electrostatic potential outside the sphere, we try to construct an equivalent boundary value problem. We use the mathematical fact: The locus of the points in space whose distances from two fixed points are in a constant ratio is a sphere whose center lies on the line connecting the two fixed points. *Proof of the fact.* If the two fixed points are $(0, 0, \pm d)$, then the desired locus is the set of points (x, y, z) such that

$$\sqrt{x^2 + y^2 + (z - d)^2} = C\sqrt{x^2 + y^2 + (z + d)^2}.$$

This can be written as

$$x^2 + y^2 + \left(z + \frac{C^2 + 1}{C^2 - 1}d\right)^2 = \left(\frac{2Cd}{C^2 - 1}\right)^2,$$

which is a sphere. In the special case $C = 1$, the locus is the bisecting plane $z = 0$. Q.E.D. The idea is to put a certain charge q' (to be determined), unlike q, on the line segment connecting the charge q and the center of the sphere, such that the sphere becomes a surface of constant potential zero. This happen if for any point P on the sphere of distances $R_1(P)$ and $R_2(P)$ to the charges q' and q, we have $R_1/R_2 = -q'/q$. Let A and B be the two points of the intersection of the sphere with the line connecting q' and q. If x is the distance of q' to the center of the sphere, we must have

$$\frac{a + x}{b + a} = \frac{R_1(A)}{R_2(A)} = -\frac{q'}{q} = \frac{R_1(B)}{R_2(B)} = \frac{a - x}{b - a}.$$

This gives

$$x = \frac{a^2}{b}, \quad q' = -\frac{a}{b}q.$$

All this is shown in the picture above. ■

Exercise: Find the electrostatic potential when a point charge q is put near a conductor wedge of 90 degrees. (*Hint.* Three imaginary charges are needed.)

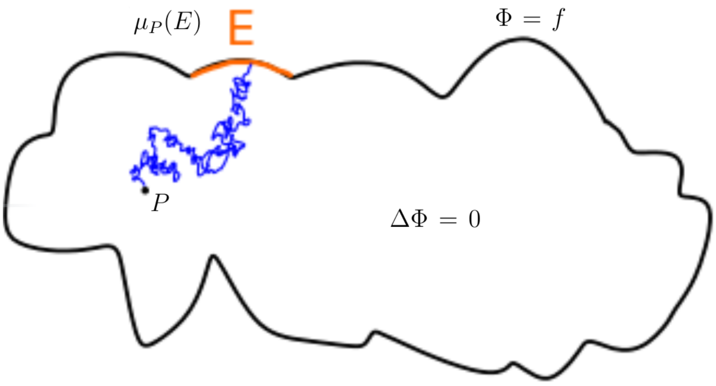
3.6.3 Method of Brownian motion

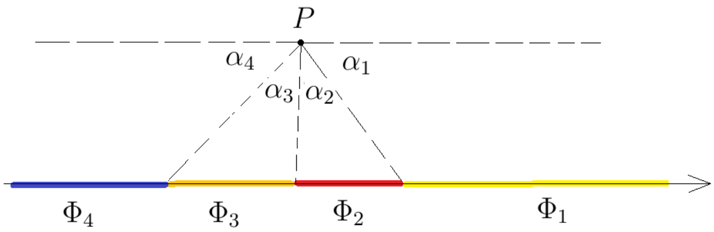
Suppose the Laplace equation $\Delta\Phi = 0$ in an open subset $U \subseteq \mathbb{R}^N$, together with the Dirichlet boundary condition $\Phi = f$ on the boundary S of U. There is a probabilistic way to solve this problem [Tay, Chapter 11][GM]: *If, for any point $P \in U$ and any (measurable) subset $E \subseteq S$, $\mu_P(E)$ denotes the probability that the Brownian motion (random walk, Wiener process) started at P, first crosses the boundary through E, then the unique solution to the Dirichlet problem is given by*

$$\Phi(P) = \int_S f d\mu_P,$$

where the integration is in the sense of Lebesgue.

In some situations, it is possible to guess the probability measure μ, called the **harmonic measure**, and this gives a clever way to solve the Dirichlet problem. Here is an example. We want to a harmonic function $\Phi(x, y)$ on $y > 0$, whose values on the x-axis is given by the piecewise step function shown in the figure. The probability that a Brownian motion started at P, first exists the x-axis through the yellow part is given



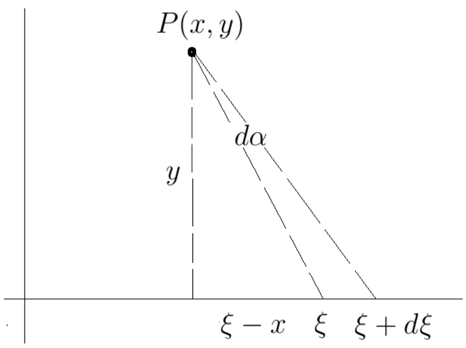


by the angle α_1 divided by π, and likewise for the other parts. Therefore, the solution to our problem is given by the finite sum

$$\sum_{j=1}^{4} \frac{\alpha_j}{\pi} \Phi_j.$$

Exercise: Use the following figure to prove that the solution to the Laplace equation $\Phi_{xx} + \Phi_{yy} = 0$ on $y > 0$ with the boundary condition $\Phi|_{y=0} = f$ is given by

$$\Phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - \xi)^2 + y^2} f(\xi) d\xi.$$



3.6.4 Method of conformal mappings

Methods of complex analysis can be used to solved two-dimensional electrostatic problems. We need the following facts:

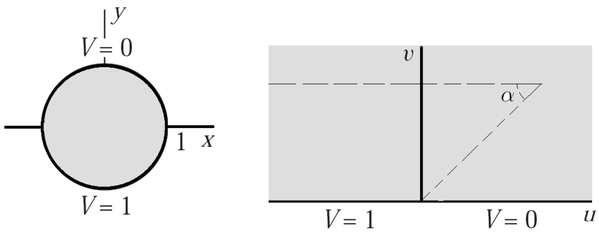
- Let $z = x + \sqrt{-1}y$, $w = u + \sqrt{-1}v$ be two complex variables such that w depends in a differentiable way on z, in the sense that the expression $(w(z+h) - w(z))/h$ has a finite limit as the complex variable h tends 0. Then, $u(x, y)$ (as well as $v(x, y)$) is a harmonic function, the transformation $(x, y) \mapsto (u, v)$ is conformal (namely, preserves angles and orientation), and the harmonic functions of (x, y) are transformed to harmonic functions of u, v.
- Any simply connected domain in the z-plane[2], except the whole plane, can be conformally transformed to the upper-half plane. This is called the Riemann's mapping theorem, and is true whether or not the boundary of the domains are smooth or not. There are explicit conformal maps, discovered by Schwartz and Christoffel, which transform a polygonal domain to the upper half-plane.
- The unit disc in the z-plane is transformed to the upper-half plane by the mapping

$$z = \frac{\sqrt{-1} - w}{\sqrt{-1} + w}, \quad w = \sqrt{-1}\frac{1 - z}{1 + z}. \tag{3.14}$$

  Here is why: $|z| < 1$ corresponds to $|w - \sqrt{-1}| < |w + \sqrt{-1}| <$, and the upper-half plane is the locus of points which are closed to $\sqrt{-1}$ than $-\sqrt{-1}$.

Here is a simple example. Many more can be found in [Chu, Chapters 9–12].

Example 39. Consider the two-dimensional inner Dirichlet problem illustrated in left part of the following figure. Under the transformation (3.14),



$$u + \sqrt{-1}v = \sqrt{-1}\frac{1 - (x + \sqrt{-1}y)}{1 + (x + \sqrt{-1}y)} = \frac{2y}{(1 + x)^2 + y^2} + \sqrt{-1}\frac{1 - x^2 - y^2}{(1 + x)^2 + y^2},$$

we need to solve the problem in the right part, which based on our knowledge from Section 3.6.3 is solved by:

$$\Phi(u, v) = \frac{\alpha}{\pi} = \frac{1}{\pi}\cot^{-1}\frac{u}{v}.$$

Transforming back into the z-plane, we get

$$\Phi(x, y) = \frac{1}{\pi}\cot^{-1}\frac{2y}{1 - x^2 - y^2}.$$

■

[2]Namely, an open subset $D \subseteq \mathbb{C}$ such that both D and its complement with respect to the extended plane $\mathbb{C} \cup \{\infty\}$ are connected.

Exercise: Compare the solutions of Example 39 with Example 33.

Exercise: Find another solution for Example 32 using the conformal mapping

$$w = \sin\left(\sqrt{-1}\left(\frac{\pi}{b}z - \sqrt{-1}\frac{\pi}{2}\right)\right),$$

to map the region there to the upper-half plane.

The appendix gives a table of conformal mapping, taken from [Chu, Appendix 2].

Chapter 4

Magnetostatics

4.1 Axioms

The next simplest level is to assume that: We are in the vacuum, charges are moving but with steady (namely, time-independent) current. This is called **magnetostatics**, and is governed by the differential axioms

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad \nabla \times B = \mu_0 J.$$

In the previous chapter, we studies the first two axioms. In this chapter, we study the first fundamental consequences of the axioms

$$\nabla \cdot B = 0, \quad \nabla \times B = \mu_0 J. \tag{4.1}$$

These differential equations, according to the divergence and Stokes theorems, can be equivalently expressed by the following integral equations:

$$\oint_S B \cdot \overrightarrow{dS} = 0, \quad \oint_C B \cdot \overrightarrow{dl} = \mu_0 I, \tag{4.2}$$

where S is an arbitrary closed surface with boundary C, and I is the current passing through S. (All such currents are equal because $\oint J \cdot dS = -\frac{d}{dt}\int \rho dV = 0$.) Note that the direction of I and the orientation of C are consistent according to the right-hand rule.

We first want to justify that: *The magnetostatic field produced by the volume current density J is given by*

$$B = \frac{\mu_0}{4\pi} \int \frac{J dV \times \widehat{R}_0}{R_0^2}. \tag{4.3}$$

Here are some theoretical/experimental justifications for this formula:

- It is immediate from the Helmholtz-Hodge theorem (Theorem 21) assuming some growth conditions at infinity:

$$B = \nabla \times \int_V \frac{\mu_0 J}{4\pi R_0} dV = \frac{\mu_0}{4\pi} \int_V \nabla\left(\frac{1}{R_0}\right) \times J dV = \frac{\mu_0}{4\pi} \int \frac{J dV \times \widehat{R}_0}{R_0^2}.$$

This computation also shows that the magnetostatic field (4.3) can be written as $B = \nabla \times A$, where the **magnetostatic potential** A is given by

$$A = \frac{\mu_0}{4\pi} \int \frac{JdV}{R_0}. \tag{4.4}$$

(Another method to derive (4.4) is given in [NB, Section 8.5].)

- Experiments show that the magnetic force that a small element $I\overrightarrow{dl}$ of steady current applies on another element $I'\overrightarrow{dl}'$ of steady current is proportional to

$$II'\frac{\overrightarrow{dl}' \times (\overrightarrow{dl} \times \widehat{R}_0)}{R_0^2} = I'\overrightarrow{dl}' \times \left(\frac{I\overrightarrow{dl} \times \widehat{R}_0}{R_0^2}\right).$$

  Comparing this with the Lorentz law (1.1) show that the expression in the parenthesis is the magnetic field generated by the steady current element $I\overrightarrow{dl}$. This fact together with the superposition principle proves (4.3).

Remark 40. Sometimes we are assuming that our steady current is flowing along surfaces or curves. Although in such situations the volume current density J is infinity, but the formulas (4.3) and (4.4) are still valid if one replaces JdV by KdS or $I\overrightarrow{dl}$, where K is the surface current density (measured in ampere per meter), and I is the current.

Here are some examples to use the formulas (4.3) and (4.4) in action:

1. The magnetostatic field at the point $\overrightarrow{R} = r\widehat{r} + z\widehat{z}$, produced by a steady current I flowing along the z-axis is given by

$$B = \frac{\mu_0}{4\pi} \int \frac{Idz'\widehat{z} \times (r\widehat{r} + z\widehat{z} - z'\widehat{z})}{|r\widehat{r} + z\widehat{z} - z'\widehat{z}|^3} = \frac{\mu_0 I}{4\pi} \int \frac{r\widehat{\varphi}dz'}{(r^2 + (z - z')^2)^{\frac{3}{2}}} = \frac{\mu_0 I r\widehat{\varphi}}{4\pi} \int \frac{dz'}{(r^2 + z'^2)^{\frac{3}{2}}},$$

   which, after the change of variable $z' = r\tan\alpha$, equals

$$\frac{\mu_0 I}{2\pi}\frac{\widehat{\varphi}}{r}. \tag{4.5}$$

   Here is an easier way to derive this: By symmetry and the form of the integral (4.3), $B = B_\varphi(r)\widehat{\varphi}$. (That $B_r = B_z = 0$ is justified in [NB, Example 8.3] by only using the axioms (4.2).) Plugging this into the integral axiom $\oint B \cdot \overrightarrow{dl} = \mu_0 I$ applied to a circle of radius r, with center on the z-axis, and lying in a plane perpendicular to $\widehat{z}$ gives

$$B_\varphi \times 2\pi r = \mu_0 I.$$

   Since $\frac{\widehat{\varphi}}{r} = -\nabla \times (\widehat{z}\log r)$, the magnetostatic potential is given by

$$-\frac{\mu_0 I}{2\pi}\log r\widehat{z}. \tag{4.6}$$

   Exercise: Derive (4.6) by extracting the "principal part" from the integral (4.4).

2. The magnetostatic field at the point $\overrightarrow{R} = x\widehat{x}+y\widehat{y}+z\widehat{z}$ produced by the steady constant surface current density $K\widehat{y}$ flowing on the xoy-plane is given by

$$B = \frac{\mu_0}{4\pi}\int \frac{K dx' dy' \widehat{y} \times (x\widehat{x} + y\widehat{y} + z\widehat{z} - (x'\widehat{x} + y'\widehat{y}))}{|x\widehat{x} + y\widehat{y} + z\widehat{z} - (x'\widehat{x} + y'\widehat{y})|^3}$$
$$= \frac{\mu_0 K}{4\pi}\int \frac{-(x - x')\widehat{z} + z\widehat{x}}{((x - x')^2 + (y - y')^2 + z^2)^{\frac{3}{2}}} dx'dy' = \frac{\mu_0 K}{4\pi}\int \frac{x'\widehat{z} + z\widehat{x}}{(x'^2 + y'^2 + z^2)^{\frac{3}{2}}} dx'dy'$$
$$= \frac{\mu_0 K z\widehat{x}}{4\pi}\int \frac{r'dr'd\varphi'}{(r'^2 + z^2)^{\frac{3}{2}}},$$

which, after the change of variable $r' = z\tan\alpha$, equals

$$\pm\frac{\mu_0 K}{2}\widehat{x}, \tag{4.7}$$

depending on whether $z > 0$ or $z < 0$. Here is an easier way to derive this: By symmetry and the form of the integral (4.3), $B = B_x(z)\widehat{x}$, where $B_x(x)$ is an odd function of x. (That $B_y = B_z = 0$ is justified in [NB, Example 8.5] by only using the axioms (4.2).) Plugging this into the integral axiom $\oint B \cdot \overrightarrow{dl} = \mu_0 I$ applied to a rectangular loop of x-spread L and z-spread W gives

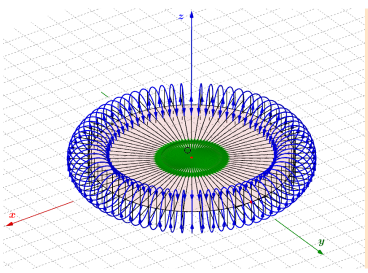
$$2B_x(z) \times L = \mu_0 K L.$$

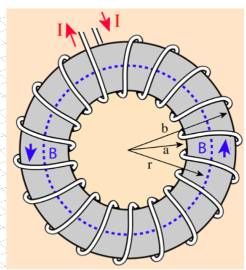
Since $\widehat{x} = -\nabla \times (z\widehat{y})$, the magnetostatic potential is given by

$$\mp\frac{\mu_0 K}{2} z\widehat{y}. \tag{4.8}$$

The integral axioms (4.2) and the computation of (4.3) in different examples says: *Magnetic field lines never start or stop at a point, but they always wrap in closed loops around electric current sources.*

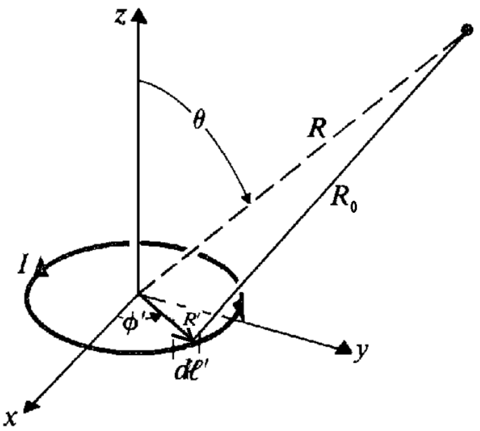
Exercise: Using the integral axiom $\oint B \cdot \overrightarrow{dl} = \mu_0 I$ and symmetry considerations, show that the magnetic field along the mid-circle of a torus wraped uniformly with wires carrying uniform current I is given by $B = \widehat{\varphi}\mu_0 n I$, where $n = \frac{NI}{2\pi r}$ is the number of wraps per unit length.





4.2 Magnetic dipole

A **magnetic dipole** is a closed loop of electric current I. The quantity I times the area of the loop is called the **magnetic dipole moment** and denoted by m. Let us place the loop at xoy-plane with center at the origin, and find the approximate form of the magnetostatic field produced by this dipole at points far away from the origin. At first,



we find an approximate formula for $1/R_0$ appearing in the integral (4.4).

$$\frac{1}{R_0}=\frac{1}{\left|\overrightarrow{R}-\overrightarrow{R'}\right|}=\frac{1}{|(R\sin\theta\cos\varphi\widehat{x}+R\sin\theta\sin\varphi\widehat{y}+R\cos\theta\widehat{z})-(a\cos\varphi'\widehat{x}+a\sin\varphi'\widehat{y})|}$$

$$=\left(R^2+a^2-2aR\sin\theta\cos(\varphi-\varphi')\right)^{-\frac{1}{2}}\approx R\left(1+\frac{a}{R}\sin\theta\cos(\varphi-\varphi')\right).$$

Then,

$$A=\frac{\mu_0}{4\pi}\int\frac{I\overrightarrow{dl}}{R_0}\approx\frac{\mu_0 Ia}{4\pi R}\int_0^{\pi}\left(1+\frac{a}{R}\sin\theta\cos(\varphi-\varphi')\right)(-\sin\varphi'\widehat{x}+\cos\varphi'\widehat{y})\,d\varphi'$$

$$=\frac{\mu_0 Ia^2\sin\theta}{4\pi R^2}\int_0^{2\pi}\cos(\varphi-\varphi')(-\sin\varphi'\widehat{x}+\cos\varphi'\widehat{y})\,d\varphi'=\frac{\mu_0 Ia^2\sin\theta}{4R^2}\widehat{\varphi}=\frac{\mu_0}{4\pi}\frac{m\times\widehat{R}}{R^2}, \tag{4.9}$$

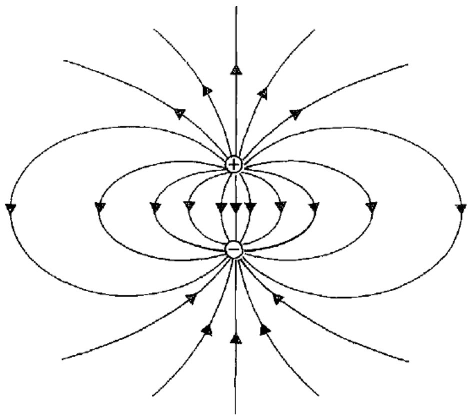
$$B\approx\nabla\times\left(\frac{\mu_0 Ia^2\sin\theta}{4R^2}\widehat{\varphi}\right)=\frac{\mu_0}{4\pi}\frac{|m|}{R^3}\left(2\cos\theta\widehat{R}+\sin\theta\widehat{\theta}\right)=\frac{\mu_0}{4\pi}\frac{3(m\cdot\widehat{R})\widehat{R}-m}{R^3}.$$

We recognize similarity with the electric dipole.

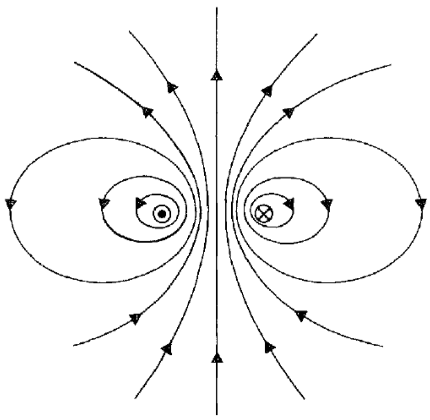
There is a more conceptual way to drive (4.9), which also show that the formula is valid for every other loop. According to the identity (2.26), we have

$$A=\frac{\mu_0 I}{4\pi}\oint_C\frac{\overrightarrow{dl}}{R_0}=\frac{\mu_0 I}{4\pi}\int_S\overrightarrow{dS}\times\nabla'\left(\frac{1}{R_0}\right)=\frac{\mu_0 I}{4\pi}\int_S\overrightarrow{dS}\times\frac{\widehat{R}_0}{R_0^2}$$

$$\approx\frac{\mu_0 I}{4\pi}\left(\int\overrightarrow{dS}\right)\times\frac{\widehat{R}}{R^2}=\frac{\mu_0}{4\pi}\frac{m\times\widehat{R}_0}{R_0^2}.$$



Electric dipole



Magnetic dipole

Chapter 5

Electrodynamics

Unlike the previous two chapters, this chapter assumes no restriction on the motion of charges or the time variation of currents. The study of electromagnetic phenomena in such generality is called **electrodynamics**. Experiments show that the divergence axioms of electrostatics and magnetostatics

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot B = 0$$

are still valid, but the curl axioms $\nabla \times E = 0$, $\nabla \times B = \mu_0 J$ need to be modified to

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}.$$

The first correction is due to Faraday (Section 5.1.1), and the second due to Maxwell (Section 5.1.2). The totality of the axioms

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

are called Maxwell's equations, and this chapter is devoted to their study.

5.1 Axioms

5.1.1 Electric induction

In electrodynamics, the electrostatic axiom $\nabla \times E = 0$ should be corrected to

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{5.1}$$

or its equivalent integral version

$$\oint_C E \cdot \overrightarrow{dl} = -\int_S \frac{\partial B}{\partial t} \cdot \overrightarrow{dS}, \tag{5.2}$$

where S is an arbitrary oriented surface with boundary C, and the orientation of S and C are consistent according to the right-hand rule.

Here we try to clarify the experimental origin of this axiom. Faraday discovered that when the magnetic flux passing through a wire circuit changes with time, an electric current is *induced* in the circuit, and the direction of the induced current is such that the magnetic field generated by it opposes the change of the original magnetic flux. (This latter fact was discovered by Lenz.) In general, the generation of a conduction current is attributed to a **electromotive force** (or **voltage**), defined as the amount of work done by the electric and magnetic fields to circulate an imaginary unit charge around the circuit. Faraday's experiments showed that: *When the magnetic flux passing through a wire circuit changes with time, the induced electromotive force equals the time rate of change of the magnetic flux.* Since the electromagnetic force experienced by a unit charge equals $E + v \times B$, the Faraday law can be expressed by

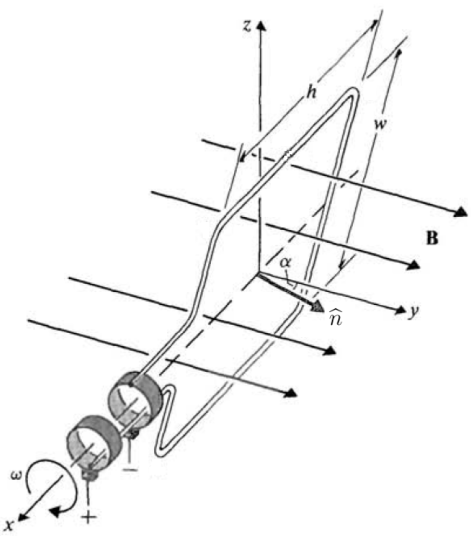
$$\oint_C (E + v \times B) \cdot \overrightarrow{dl} = -\frac{d}{dt}\int_S B \cdot \overrightarrow{dS}.$$

The minus sign comes from the Lenz law. This equation, using the transport theorem (2.31) and the axiom $\nabla \cdot B = 0$, reduces to (5.2). Note that the change of flux

$$\frac{d}{dt}\int_S B \cdot \overrightarrow{dS} = \int_S \frac{\partial B}{\partial t} \cdot \overrightarrow{dS} - \int_C v \times B \cdot \overrightarrow{dl}$$

might be because of the change in the magnetic field (the first term), or the change in the shape or orientation of the circuit (the second term).

Example 41. Let us compute the electromotive force induced in a $h \times w$ rectangular wire loop which is rotating with angular velocity ω in a uniform time-varying magnetic field $B = B(t)\widehat{y}$. We are assuming that the loop is in the xoz plane at $t = 0$. Since the



unit normal vector to the loop is $\widehat{n} = \widehat{y}\cos\alpha - \widehat{z}\sin\alpha$, $\alpha = \omega t$, the magnetic flux is given by

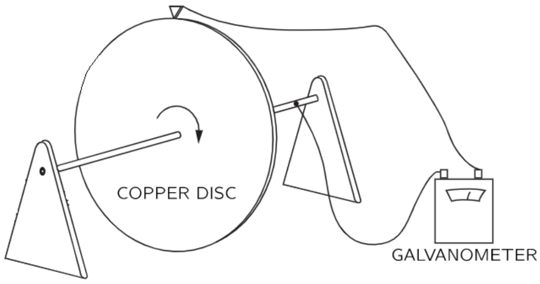
$$\int B \cdot \overrightarrow{dS} = \widehat{y}B(t) \cdot \widehat{n}\int dS = hwB(t)\cos\omega t.$$

Therefore, the electromotive force is given by

$$-\frac{d}{dt}\left(hwB(t)\cos\omega t\right) = -hwB'(t)\cos\omega t + hw\omega B(t)\sin\omega t.$$

Note that the first term is due to change of the magnetic field, and the second term is due to rotation. ■

Exercise: A conducting disk of radius a is rotating with a constant angular frequency ω about its central axis, and is placed in a uniform and constant magnetic field B which is parallel to its axis of rotation. Brush contacts are provided at the axis and on the rim



of the disk. Determine the electromotive force produced. (*Hint.* Compute $\int v \times B \cdot \overrightarrow{dl}$. The answer is $B\omega a^2/2$. It is helpful to read [NB, Page 360] or [Fey, Volume II, Section 17.2].)

5.1.2 Displacement current

Before Maxwell, the curl of the magnetic field had been given by

$$\nabla \times B = \mu_0 J. \tag{5.3}$$

Maxwell noticed that this equation is incompatible with the other axioms plus the continuity equation:

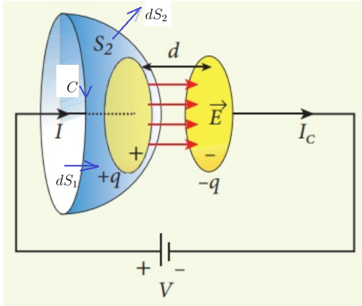
$$0 = \nabla \cdot \nabla \times B = \mu_0 \nabla \cdot J = -\mu_0 \frac{\partial \rho}{\partial t} = -\mu_0 \epsilon_0 \frac{\partial(\nabla \cdot E)}{\partial t} = \nabla \cdot \left(-\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right),$$

and that the contradiction disappears if one corrects (5.3) to

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}. \tag{5.4}$$

The added term $J_D := \epsilon_0 \frac{\partial E}{\partial t}$ has the dimension of the electric durrent density J, and is called the **displacement current density** because of the interpretation of it given in the following example.

Example 42. Suppose that a variable (say sinusoidal) voltage V is applied to a parallel-plate capacitor, the yellow part in the following figure. The capacitor gets the polarization



charge

$$q = \int \epsilon_0 E \cdot \overrightarrow{dS} = \epsilon_0 EA = \epsilon_0 \frac{V}{d} A = CV,$$

where $C = \epsilon_0 A/d$ is the capacitance of the capacitor. The current flowing through the wire of the circuit equals $I = dq/dt = CdV/dt$. However this (conduction) current can not pass through the capacitor. The circulation of B along the closed curve C shown in the figure is given by $\mu_0 I + \mu_0 I_D$ where $I = \int_S J \cdot \overrightarrow{dS}$ and $I_D = \int_S J_D \cdot \overrightarrow{dS}$ are the usual and displacement currents passing through any surface S which bounds C. When $S = S_1$, $I = C\frac{dV}{dt}$ and $I_D = 0$. When $S = S_2$, then $I = 0$ and

$$I_D = \epsilon_0 \int_{S_2} \frac{\partial E}{\partial t} \cdot \overrightarrow{dS} = \epsilon_0 \frac{d}{dt} \oint_{S_2 - S_1} E \cdot \overrightarrow{dS} = \frac{dq}{dt} = C\frac{dV}{dt}.$$

One sees that $I + I_D$ is the same for $S = S_1$ and $S = S_2$, as it must be. ∎

Remark 43. In electrodynamics, the electric field inside conductors might not vanish. The reason is that when the fields are changing, the charges in conductors might not have enough time to rearrange themselves to make the field zero. The only general statement is that electric fields in conductors produce currents.

5.1.3 Maxwell's equations, Wave equation

The fundamental axioms of electrodynamics in the vaccum are:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}.$$

These are called Maxwell's equations. They are accompanied by the continuity equation

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0,$$

(Example 7) and the Lorentz law

$$F = qE + qv \times B.$$

In order to solve the Maxwell's equation in $\mathbb{R}^3$, one bring the potential functions into the scene. Since $\nabla \cdot B = 0$, B can be expressed as

$$B = \nabla \times A. \tag{5.5}$$

Since

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times A) = -\nabla \times \frac{\partial A}{\partial t},$$

it follows that E can be expressed as

$$E = -\nabla\Phi - \frac{\partial A}{\partial t}.$$

Plugging these two representation into the two other Maxwell's equations gives

$$\mu_0 J - \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\nabla\Phi + \frac{\partial A}{\partial t}\right) = \mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t} = \nabla \times B = \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A,$$

$$\frac{\rho}{\epsilon_0} = \nabla \cdot E = -\nabla \cdot \left(\nabla\Phi + \frac{\partial A}{\partial t}\right) = -\nabla^2\Phi - \frac{\partial}{\partial t}(\nabla \cdot A).$$

We rewrite them as

$$\nabla^2 A - \mu_0\epsilon_0\frac{\partial^2 A}{\partial t^2} = -\mu_0 J + \nabla\left(\nabla \cdot A + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right),$$

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\epsilon_0}.$$

We have freedom in choosing the divergence of A (changing A to $A+\nabla f$ does not change (5.5), whatever scalar field f is), so we assume

$$\nabla \cdot A + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t} = 0. \tag{5.6}$$

This is called the **Lorentz gauge**. Inserting this into our previous equations gives

$$\Box^2 A = -\mu_0 J, \quad \Box^2\Phi = -\frac{\rho}{\epsilon_0}. \tag{5.7}$$

where the second-order differential operator

$$\Box^2 := \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{5.8}$$

is called **d'Alembertian**, and

$$c := \frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 3 \times 10^8 \,\frac{\text{meter}}{\text{second}}$$

coincides with the speed of light. The equation $\Box^2 u = f$ appears in all linear, oscillatory phenomena, and is called the **inhomogeneous wave equation** [Sim, Pages 303–4, 372–3].

Theorem 44. *(a) A solution of the wave equations (5.7) is given by*

$$\Phi\left(\overrightarrow{R},t\right)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho\left(\overrightarrow{R}',t-R_0/c\right)}{R_0}dV, \tag{5.9}$$

$$A\left(\overrightarrow{R},t\right)=\frac{\mu_0}{4\pi}\int\frac{J\left(\overrightarrow{R}',t-R_0/c\right)}{R_0}dV. \tag{5.10}$$

(b) A solution of the wave equations (5.7) in the time-harmonic case $e^{\sqrt{-1}\omega t}$ is given by

$$\widetilde{\Phi}\left(\overrightarrow{R}\right)=\frac{1}{4\pi\epsilon_0}\int\frac{e^{-\sqrt{-1}kR_0}}{R_0}\widetilde{\rho}\left(\overrightarrow{R}'\right)dV, \tag{5.11}$$

$$\widetilde{A}\left(\overrightarrow{R}\right)=\frac{\mu_0}{4\pi}\int\frac{e^{-\sqrt{-1}kR_0}}{R_0}\widetilde{J}\left(\overrightarrow{R}'\right)dV, \tag{5.12}$$

*where $k:=\frac{\omega}{c}$ is the so-called **wave number**, and the tilde is used to denote the phasors*[1] *or the Fourier transform*

$$\widetilde{\Phi}\left(\overrightarrow{R},\omega\right)=\mathcal{F}\left(\Phi\left(\overrightarrow{R},t\right)\right)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Phi\left(\overrightarrow{R},t\right)e^{-\sqrt{-1}\omega t}dt,$$

$$\Phi\left(\overrightarrow{R},t\right)=\mathcal{F}^{-1}\left(\widetilde{\Phi}\left(\overrightarrow{R},\omega\right)\right)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widetilde{\Phi}\left(\overrightarrow{R},\omega\right)e^{\sqrt{-1}\omega t}d\omega.$$

Proof. (a) Immediate from (b) by the Fourier transform. Direct arguments are given in [NB, Section 15.2] or [Fey, Volume II, Chapter 21].

(b) Under the Fourier transform, the wave equations (5.7) become

$$\left(\nabla^2+k^2\right)\widetilde{A}=-\mu_0\widetilde{J},\quad \left(\nabla^2+k^2\right)\widetilde{\Phi}=-\frac{\widetilde{\rho}}{\epsilon_0},$$

the so-called **Helmholtz equations**. Setting

$$G(\overrightarrow{R}):=\frac{1}{4\pi}\frac{e^{-\sqrt{-1}kR}}{R},$$

according to Theorem 20, we have $\left(\nabla^2+k^2\right)G(\overrightarrow{R})=-\delta$, therefore

$$\left(\nabla^2+k^2\right)\widetilde{\Phi}(\overrightarrow{R})=\frac{1}{\epsilon_0}\left(\nabla^2+k^2\right)\int\rho(\overrightarrow{R}')G(\overrightarrow{R}-\overrightarrow{R}')dV=$$
$$\frac{1}{\epsilon_0}\int\rho(\overrightarrow{R}')\left(\nabla^2+k^2\right)G(\overrightarrow{R}-\overrightarrow{R}')dV=-\frac{1}{\epsilon_0}\int\rho(\overrightarrow{R}')\delta(\overrightarrow{R}-\overrightarrow{R}')dV=-\frac{\rho(\overrightarrow{R})}{\epsilon_0}.$$

We have shown $\left(\nabla^2+k^2\right)\widetilde{\Phi}=-\rho/\epsilon_0$. Decomposing into cartesian components proves $\left(\nabla^2+k^2\right)\widetilde{A}=-\mu_0\widetilde{J}$. ■

The formulas (5.9), (5.10) are called **retarded potentials**, because of the time delay R_0/c: To compute the potentials at time t, we must use the values of sources at earlier times $t-R_0/c$, and that the influences propagate with the speed of light. When $R_0\ll c$ (or $\omega=0$ in formulas (5.11), (5.12)), electrodynamics reduces to electrostatics or magnetostatics.

[1] [Che, Sections 7.7.1–2].

5.2 Flow of energy

Consider the following computation:

$$
\begin{aligned}
-\nabla\cdot\left(E\times\frac{B}{\mu_0}\right) = -(\nabla\times E)\cdot\frac{B}{\mu_0}+\left(\nabla\times\frac{B}{\mu_0}\right)\cdot E &= \frac{\partial B}{\partial t}\cdot\frac{B}{\mu_0}+\left(J+\epsilon_0\frac{\partial E}{\partial t}\right)\cdot E\\
&= \frac{\partial}{\partial t}\left(\frac{|B|^2}{2\mu_0}+\frac{\epsilon_0|E|^2}{2}\right)+J\cdot E.
\end{aligned}
$$

We have

- $u := \frac{|B|^2}{2\mu_0}+\frac{\epsilon_0|E|^2}{2}$ is the amount of energy saved per unit volume by the magnetic and electric fields [Jac, Section 6.7].
- $J\cdot E$, depending on the situation, is the amount of energy dissipated per unit volume per second in conductors (if J is the conduction current density given by conductivity σ times E), or the energy used per unit volume per second to accelerate charges (if J is the convection current density given by volume charge density ρ times the velocity), or minus the amount of energy produced by a source per unit volume per second. The reason is that the amount of mechanical work done by the electromagnetic field on the infinitesimal charge $dq = \rho dV$ per unit time is given by

$$
\frac{(\rho E+\rho v\times B)\cdot\overrightarrow{dl}\,dV}{dt} = E\cdot(\rho v)dV + 0 = J\cdot E dV.
$$

Therefore, by the law of conservation of energy,

$$
\int -\nabla\cdot\left(E\times\frac{B}{\mu_0}\right)dV = \oint_S E\times\frac{B}{\mu_0}\cdot\overrightarrow{dS}
$$

is the amount of electromagnetic energy that passes per second through per unit area. The vector field

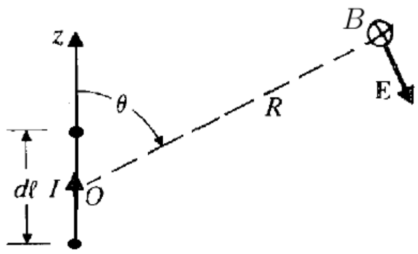
$$
P = E\times\frac{B}{\mu_0}, \tag{5.13}
$$

is called the **Poynting vector**. In the time-harmonic case,

$$
\begin{aligned}
P = \mathrm{Re}\left(\widetilde{E}e^{\sqrt{-1}\omega t}\right)\times\mathrm{Re}\left(\frac{\widetilde{B}}{\mu_0}e^{\sqrt{-1}\omega t}\right) &= \frac{\left(\widetilde{E}e^{\sqrt{-1}\omega t}+\widetilde{E}^*e^{-\sqrt{-1}\omega t}\right)\left(\widetilde{B}e^{\sqrt{-1}\omega t}+\widetilde{B}^*e^{-\sqrt{-1}\omega t}\right)}{4\mu_0}\\
&= \frac{1}{2\mu_0}\mathrm{Re}\left(\widetilde{E}\times\widetilde{B}e^{2\sqrt{-1}\omega t}+\widetilde{E}\times\widetilde{B}^*\right),
\end{aligned}
$$

so the average of P is given by

$$
\langle P\rangle = \frac{1}{2}\mathrm{Re}\left(\widetilde{E}\times\frac{\widetilde{B}^*}{\mu_0}\right). \tag{5.14}
$$



5.3 Electromagnetic radiation

To explain the possibility of energy transmit by electromagnetic waves, we will analyze the simplest model: **Hertzian dipole**. It consists of an infinitesimal conducting wire of length dl which terminates in two small conductive spheres (capacitive end-loading), and carries a uniform, sinusoidal current $i = \mathrm{Re}(I \exp \sqrt{-1}\omega t)$. The retarded potential (5.12) is given by

$$A = \widehat{z}\frac{\mu_0 I dl}{4\pi}\frac{e^{-\sqrt{-1}kR}}{R},$$

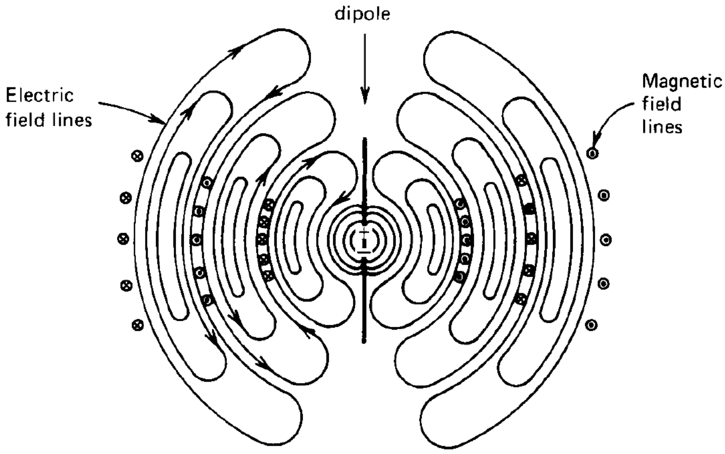
where $k = \omega/c = 2\pi/\lambda$ is the wave number. Therefore, the electric and magnetic fields and their far-field approximations ($kR = 2\pi R/\lambda \gg 1$) are given by

$$B = \nabla \times A = -\widehat{\varphi}\frac{\mu_0 I dl}{4\pi}k^2 \sin\theta \left(\frac{1}{\sqrt{-1}kR} + \frac{1}{\left(\sqrt{-1}kR\right)^2}\right) e^{-\sqrt{-1}kR}$$
$$\approx \widehat{\varphi}\sqrt{-1}\frac{\mu_0 I dl}{4\pi}k \sin\theta \frac{e^{-\sqrt{-1}kR}}{R},$$

$$E = \frac{\nabla \times B}{\sqrt{-1}\omega\mu_0\epsilon_0} = -\frac{I dl \eta_0 k^2}{4\pi}e^{-\sqrt{-1}kR} \times$$
$$\left(\widehat{R}2\cos\theta\left(\frac{1}{(\sqrt{-1}kR)^2} + \frac{1}{\left(\sqrt{-1}kR\right)^3}\right) + \widehat{\theta}\sin\theta\left(\frac{1}{\sqrt{-1}kR} + \frac{1}{(\sqrt{-1}kR)^2} + \frac{1}{\left(\sqrt{-1}kR\right)^3}\right)\right)$$
$$\approx \widehat{\theta}\sqrt{-1}\frac{I dl}{4\pi}k\eta_0 \sin\theta \frac{e^{-\sqrt{-1}kR}}{R},$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$.

The most important observation here is that: *The the far-field approximations of electric and magnetic fields are orthogonal to each other, with the same phase, and their magnitudes vary inversely with the distance from the source, namely $1/R$. Therefore, the time-averaged Poynting vector (5.14) is nonzero, and varies like $1/R^2$, hence its integral over large-radius spheres is finite, which means that energy is radiated from the Herzian dipole.* This is in contrast with our cases in electrostatics and magnetostatics. (A static charge has no magnetic field and its electric field varies like $1/R^2$. A steady current produces electric and magnetic fields both vary like $1/R^2$. In both cases the integral of the Poynting vector (5.13) over large-radius spheres is negligible, which means no energy is transmitted from these structures.)



A masterful, elementary description of the formation of electromagnetic waves is given in [Fey, Volume II, Section 18.4]: "How the electromagnetic energy is radiated? The answer is: by the combined effects of the Faraday law $\nabla \times E = -\partial B/\partial t$, and the new term of Maxwell $\nabla \times B = -\mu_0\epsilon_0 \partial E/\partial t$. They cannot help maintaining themselves. Suppose the magnetic field were to disappear. There would be a changing magnetic field which would produce an electric field. If this electric field tries to go away, the changing electric field would create a magnetic field back again. So by a perpetual interplay–by the swishing back and forth from one field to the other–they must go on forever. It is impossible for them to disappear. They maintain themselves in a kind of a dance–one making the other, the second making the first–propagating onward through space."

5.4 Special relativity considerations

A reasonable, general principle in physics– suggested by Einstein's special theory of relativity– is that the laws of physics should give the same results in all inertial reference frames, namely, with respect to observers moving with constant velocity (= no acceleration) with respect to each other. (In the realm of mechanics, this had been suggested by Newton.) Here are some situations that such considerations appear in electromagnetics:

- The Lorentz force $qE + qv \times B$ experienced by a point charge q depends on the inertial frame chosen. To be more specific, assume that a unit point charge, in constant velocity with respect to the laboratory table, is in the magnetic field B of a steady current. The charge experiences the force $v \times B$. With respect to a reference installed on the charge, there is no $v \times B$, so it is reasonable to assume that the magnetic field B in the former frame has transformed to the electric field $E = v \times B$ in the latter. We give an explanation for this transformation in this chapter. (A direct, brilliant, elementary explanation is given in [Fey, Volume II, Section 13.6].)

- Maxwell discovered that the speed of the propagation of electromagnetic waves in vacuum equals $1/\sqrt{\mu_0\epsilon_0} \approx 3 \times 10^8 \frac{\text{m}}{\text{s}}$, which coincides with the speed of light measured different experiments. This miraculous coincidence between experiments on charges (ϵ_0), currents (μ_0), and light left him no doubt that light is an electromagnetic wave. However, this discovery posed a big question: The speed of the electromagnetic wave is $1/\sqrt{\mu_0\epsilon_0}$ in all inertial frames, and this contradict with the Galilean law of addition of velocities, which is itself based on the classical, fundamental perceptions of the *Euclidean space* and *absolute time.*

Some notations used throughout this section. We assume two inertial reference frames O, O', with parallel axes, whose origins coincided at the initial time $t = 0$, but afterwards O' is moving with constant speed v along the z-axis. The spatial cartesian coordinates and time of a common **event** (or **phenomenon**) that O and O' measure are denoted by (x, y, z, t) and (x', y', z', t'), respectively. We also set

$$\beta := \frac{v}{c}, \quad \gamma := \frac{1}{\sqrt{1-\beta^2}}.$$

According to the classical mentality, the measurements in O and O' are related by the **Galilean transformation**:

$$(x', y', z', t') = (x, y, z - vt, t)\,.$$

According to this transformation, the speed of a light signal propagating in the z direction, measured in O and O' should be related by

$$c = c' + v.$$

However, the electromagnetic wave equation describing the propagation of light in vacuum (Section 5.1.3) says

$$c = c'.$$

The problem is that the d'Alembertian differential operator is not invariant under Galilean transformations:

$$\Box^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \Box'^2 f + \frac{v^2}{c^2}\frac{\partial^2 f}{\partial t'^2} - \frac{2v}{c^2}\frac{\partial^2 f}{\partial t' \partial z'}.$$

However:

Theorem 45. *(a) The d'Alembertian differential operator is invariant under the **Lorentz transformation** given by*

$$(x', y', z', t') = \left(x, y, \gamma(z - vt), \gamma\left(t - vc^{-2}z\right)\right).$$

The inverse transform is given by

$$(x, y, z, t) = \left(x', y', \gamma(z' + vt'), \gamma\left(t' + vc^{-2}z'\right)\right).$$

(b) The Lorentz transformation leaves the quadratic form $x^2+y^2+z^2-c^2z^2$ invariant, namely

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2.$$

Proof. (a) Let us find constant A, B, C, D such that the transformation

$$x' = x, \quad y' = y, \quad z' = Az + Bct, \quad ct' = Cz + Dct,$$

leaves the d'Alembertian invariant. By the chain rule,

$$f_{xx} = f_{x'x'}, \quad f_{yy} = f_{y'y'},$$

$$f_z = Af_{z'} + \frac{C}{c} f_{t'}, \quad f_t = Bcf_{z'} + Df_{t'},$$

$$f_{zz} = A\left(Af_{z'z'} + \frac{C}{c} f_{z't'}\right) + \frac{C}{c}\left(Af_{t'z'} + \frac{C}{c} f_{t't'}\right) = A^2 f_{z'z'} + \frac{2AC}{c} f_{z't'} + \frac{C^2}{c^2} f_{t't'},$$

$$f_{tt} = B^2c^2 f_{z'z'} + 2BDcf_{z't'} + D^2 f_{t't'}.$$

Therefore,

$$\begin{aligned}\Box^2 f &= f_{xx} + f_{yy} + f_{zz} - \frac{1}{c^2} f_{tt} \\ &= f_{x'x'} + f_{y'y'} + \left(A^2 - B^2\right) f_{zz} + \frac{2(AC - BD)}{c} f_{z't'} + \frac{C^2 - D^2}{c^2} f_{t't'}.\end{aligned}$$

This shows that $\Box^2 f = \Box'^2 f$ exactly when

$$A^2 - B^2 = 1, \quad AC - BD = 0, \quad D^2 - C^2 = 1.$$

One can easily check that these relations are satisfied for the Lorentz transformation.

(b) Similar arguments work. ∎

Einstein suggested that: *The space and time measurements of the observers O, O' are, in reality, related by the Lorentz transformation.* This was a revolutionary idea in physics. Assuming this mentality, it is reasonable to agree that:

- *Time dilation.* For an event occurring in the origin of O', we have

  $$0 = z' = \gamma(z - vt) \Longrightarrow z = vt \Longrightarrow t' = \gamma\left(t - vz/c^2\right) = \gamma\left(t - tv^2/c^2\right) = t/\gamma.$$

  Doing two measurements,

  $$|t'_2 - t'_1| = |t_2 - t_1|/\gamma < |t_2 - t_1|.$$

  In words: *An observer will measure the moving clock as ticking slower than a clock that is at rest in the observer's own reference frame.*

- *Length contraction.* The length of a rod which is at rest with respect to O', and it situated on the z'-axis from z'_1 to z'_2, becomes smaller when measured with respect to O:

  $$l' = |z'_2 - z'_1| = |\gamma(z_2 - vt) - \gamma(z_1 - vt)| = \gamma|z_2 - z_1| = \gamma l > l.$$

- What could be a mathematical framework to formalize Einstein's idea? In Chapter 2, we defined vectors as triples of real numbers which under the rotations of coordinate systems behave the same way as the components of the position vector (x, y, z) behave. The fundamental invariant quantity here is the square length $x^2 + y^2 + z^2$. If physical laws are expressed in the language of vector calculus, then we can make sure that they give the same results under the rotation of coordinates. To answer the question above, let us denote the spatial coordinates x, y, z by x_1, x_2, x_3, and the time variable t by $x_4 = \sqrt{-1}ct$. We do so, because then, the d'Alembertian operator is given by $\Box^2 = \sum_{\mu=1}^{4} \partial^2/\partial x_\mu^2$, and the quadratic form which is invariant under the Lorentz transform becomes $\sum_{\mu=1}^{4} x_\mu^2$. Then, the **Lorentz point** $x_\mu = (x_1, x_2, x_3, x_4)$ changes as

$$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu, \quad a_{\mu\nu} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{-1}\beta\gamma \\ 0 & 0 & -\sqrt{-1}\beta\gamma & \gamma \end{bmatrix}, \tag{5.15}$$

under the Lorentz transform. Note that the matrix $a_{\mu\nu}$ is orthogonal in the sense that

$$\sum_{\mu=1}^{4} a_{\mu\nu} a_{\mu\lambda} = \begin{cases} 1, & \nu = \lambda, \\ 0, & \nu \neq \lambda. \end{cases}$$

Equivalently, the inverse transformation is given by

$$x_\mu = \sum_{\nu=1}^{4} x'_\nu a_{\nu\mu}.$$

To simplify notations, we accept the **Einstein summation convention**: Always sum over repeated indices. Then, (5.15) is written as

$$x'_\mu = a_{\mu\nu} x_\nu.$$

Those physical quantities which does not change under the Lorentz transform are called **Lorentz scalars**. **Lorentz vectors** (or **four-vectors**) are those quantities $A_\mu = (A_1, A_2, A_3, A_4)$ which under the Lorentz transform change exactly the same way as the Lorentz position vector (x_1, x_2, x_3, x_4) changes:

$$A'_\mu = a_{\mu\nu} A_\nu.$$

For example, if Φ is a Lorentz scalar field, then

$$\Box\Phi := \left(\frac{\partial\Phi}{\partial x_\mu}\right)_{\mu=1,2,3,4},$$

called the **Lorentz gradient** of Φ, is a Lorentz vector field, because

$$\frac{\partial\Phi}{\partial x'_\mu} = \frac{\partial\Phi}{\partial x_\nu}\frac{\partial x_\nu}{\partial x'_\mu} = a_{\mu\nu}\frac{\partial\Phi}{\partial x_\nu}.$$

On the other hand, if A_μ is a Lorentz vector field, then

$$\Box \cdot A_\mu := \sum_{\mu=1}^{4} \frac{\partial A_\mu}{\partial x_\mu},$$

called the **Lorentz divergence** of A_μ, is a Lorentz scalar field, because

$$\sum_\mu \frac{\partial A'_\mu}{\partial x'_\mu} = \sum_{\mu,\nu} \frac{\partial A'_\mu}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\mu} = \sum_{\mu,\nu} \frac{\partial A'_\mu}{\partial x_\nu} a_{\mu\nu} = \sum_\nu \frac{\partial}{\partial x_\nu} \left(\sum_\nu A'_\mu a_{\mu\nu} \right) = \sum_\nu \frac{\partial A_\nu}{\partial x_\nu}.$$

Therefore, if Φ is a Lorentz scalar field, then

$$\Box^2 \Phi := \Box \cdot (\Box \Phi) = \sum_{\mu=1}^{4} \frac{\partial^2 \Phi}{\partial x_\mu^2} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2},$$

called the **Lorentz Laplacian** (or **d'Alembertian**) of Φ, is a Lorentz scalar field. **Lorentz tensors of rank two** are those quantities $T_{\mu\nu}$, consisting of 16 components, which under the Lorentz transform change as

$$T'_{\mu\nu} = a_{\mu\lambda} a_{\nu\sigma} T_{\lambda\sigma}.$$

For example, if A_μ is a Lorentz vector field, then

$$T_{\mu\nu} := \left(\frac{\partial A_\mu}{\partial x_\nu} \right)_{\mu,\nu=1,2,3,4}$$

is a Lorentz tensor of rank two, because

$$\frac{\partial A'_\mu}{\partial x'_\nu} = \frac{\partial}{\partial x'_\nu} (a_{\mu\lambda} A_\lambda) = a_{\mu\lambda} \frac{\partial A_\lambda}{\partial x'_\nu} = a_{\mu\lambda} \frac{\partial A_\lambda}{\partial x_\sigma} \frac{\partial x_\sigma}{\partial x'_\nu} = a_{\mu\lambda} a_{\nu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma}. \tag{5.16}$$

Lorentz tensors of higher rank can be defined similarly. The upshot of all these is that: *For physical laws to give the same result in inertial frames, they show be expressed in terms of Lorentz tensors.*

Next, we are going express Maxwell's equations (Section 5.1.3) in terms of Lorentz tensors. The continuity equation

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

can be expressed as

$$\Box \cdot J_\mu = 0,$$

if we introduce the **source Lorentz vector field**

$$J_\mu = \left(J_1, J_2, J_3, \sqrt{-1} c\rho \right).$$

Accordingly, the Lorentz gauge (5.6) can be expressed as

$$\Box \cdot A_\mu = 0,$$

if we introduce the **potential Lorentz vector field**

$$A_\mu = \left(A_1, A_2, A_3, \sqrt{-1}\Phi/c\right).$$

Then, the equations (5.7) are combined into

$$\Box^2 A_\mu = -\mu_0 J_\mu \quad \text{for each } \mu = 1,2,3,4.$$

The definitions

$$B = \nabla \times A, \quad E = -\nabla\Phi - \frac{\partial A}{\partial t}$$

combine into

$$F_{\mu\nu} := \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} = \begin{bmatrix} 0 & B_3 & -B_2 & -\sqrt{-1}E_1/c \\ -B_3 & 0 & B_1 & -\sqrt{-1}E_2/c \\ B_2 & -B_1 & 0 & -\sqrt{-1}E_3/c \\ \sqrt{-1}E_1/c & \sqrt{-1}E_2/c & \sqrt{-1}E_3/c & 0 \end{bmatrix} \tag{5.17}$$

The main observation here is that: *$F_{\mu\nu}$ is a Lorentz-tensor field of rank two, called the **field-intensity Lorentz vector field**.* The reason is that, based on computations (5.16), we have

$$F'_{\mu\nu} = \frac{\partial A'_\mu}{\partial x'_\nu} - \frac{\partial A'_\nu}{\partial x'_\mu} = a_{\mu\lambda}a_{\nu\sigma}\frac{\partial A_\lambda}{\partial x_\sigma} - a_{\nu\sigma}a_{\mu\lambda}\frac{\partial A_\sigma}{\partial x_\lambda} = a_{\mu\lambda}a_{\nu\sigma}F_{\lambda\sigma}.$$

The axioms

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times B = \mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t}$$

combine into

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \quad \text{for each } \mu = 1,2,3,4,$$

and the axioms

$$\nabla \cdot B = 0, \quad \nabla \times E = -\frac{\partial B}{\partial t}$$

combine into

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0 \quad \text{for each pairwise distinct } \mu,\nu,\lambda = 1,2,3,4.$$

All these show that: *From the viewpoint of the special theory of relativity, not E, B but their combination (5.17) is the essential notion.* In different inertial frames, electric and magnetic field transform into each other. The transformation rule for E, B comes from the transformation rule for the field-intensity Lorentz tensor field $F_{\mu\nu}$:

$$F'_{\mu\nu} = a_{\mu\lambda}a_{\nu\sigma}F_{\lambda\sigma}.$$

Writing it up, we have

$$\begin{cases}(E_1', E_2', E_3') = (\gamma(E_1 - c\beta B_2), \gamma(E_2 + c\beta B_1), E_3)\,, \\ (B_1', B_2', B_3') = (\gamma(B_1 + \beta E_2/c), \gamma(B_2 - \beta E_1/c), B_3)\,,\end{cases}$$

or, equivalently,

$$E_\parallel' = E_\parallel, \quad B_\parallel' = B_\parallel, \quad E_\perp' = \gamma\left(E_\perp + \overrightarrow{v} \times B_\perp\right), \quad B_\perp' = \gamma\left(B_\perp - c^{-2}\overrightarrow{v} \times E_\perp\right),$$

where $\overrightarrow{v} = v\widehat{z}$, and $\parallel, \perp$ denote the components of vectors parallel and perpendicular to v, respectively. Therefore,

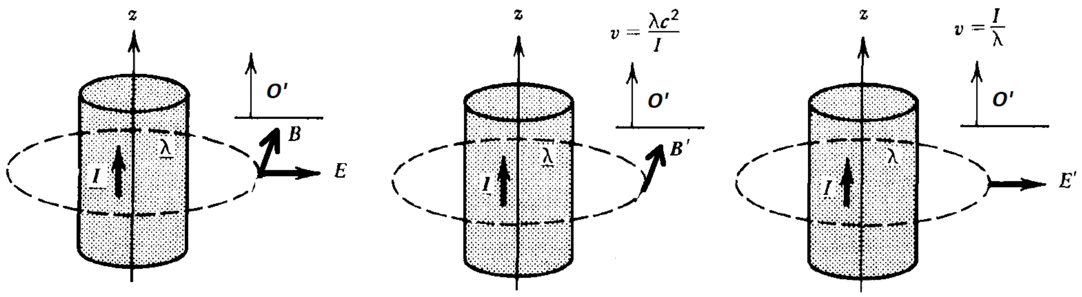
$$E' = E_\parallel + \gamma\left(E_\perp + \overrightarrow{v} \times B_\perp\right) = \frac{E\cdot\overrightarrow{v}}{v^2}\overrightarrow{v} + \gamma\left(E - \frac{E\cdot\overrightarrow{v}}{v^2}\overrightarrow{v} + \overrightarrow{v} \times \left(B - \frac{B\cdot\overrightarrow{v}}{v^2}\overrightarrow{v}\right)\right)$$
$$= \gamma\left(E + \overrightarrow{v} \times B\right) - (\gamma - 1)\overrightarrow{v}\frac{E\cdot\overrightarrow{v}}{v^2},$$

and similarly for B':

$$\begin{cases}E' = \gamma\left(E + \overrightarrow{v} \times B\right) - (\gamma-1)\frac{\overrightarrow{v}\cdot E}{v^2}\overrightarrow{v}, \\ B' = \gamma\left(B - \frac{\overrightarrow{v}\times E}{c^2}\right) - (\gamma-1)\frac{\overrightarrow{v}\cdot B}{v^2}\overrightarrow{v}.\end{cases} \tag{5.18}$$

Exercise: Derive the transformation rule for the charge and current densities ρ, J from the transformation rule for the Lorentz vector field J_μ.

Example 46. Consider a thin, infinitely-long cylinder along the z-axis, with a uniform linear charge density λ, and a steady, uniformly distributed current I. With respect to



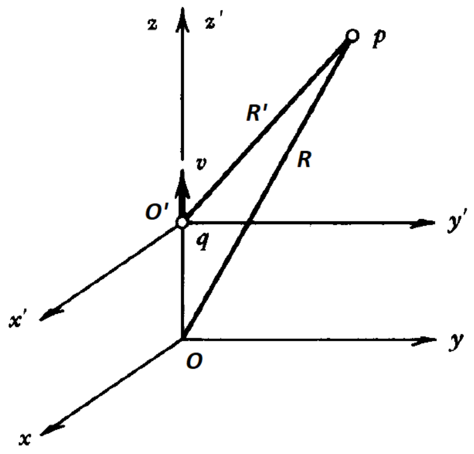
the observer O at rest with respect to the cylinder, based on the formulas derived in Sections (3.1) and (4.1), we have the electric and magnetic fields given by

$$E = \frac{\lambda}{2\pi\epsilon_0 r}\widehat{r}, \quad B = \frac{\mu_0 I}{2\pi r}\widehat{\varphi}.$$

With respect to the observer O' moving with velocity v along the z-axis, according to the transformation rules (5.18), the electric and magnetic fields are given by

$$E' = \frac{\gamma}{2\pi r}\left(\frac{\lambda}{\epsilon_0} - \mu_0 I v\right)\widehat{r}, \quad B' = \frac{\gamma}{2\pi r}\left(\mu_0 I - \frac{v\lambda}{c^2\epsilon_0}\right)\widehat{\varphi}.$$

This shows that if $v = \lambda c^2/I$ (respectively, $v = I/\lambda$), then the electric (respectively, magnetic) effects disappear. ■



Example 47. Consider a point charge q moving with constant velocity v along the z-axis. With respect to the observer O' installed on the charge, based on the formulas derived in Sections (3.1) and (4.1), we have the electric and magnetic potentials and fields are given by

$$\Phi' = \frac{q}{4\pi\epsilon_0 R'}, \quad A' = 0, \quad E' = \frac{q}{4\pi\epsilon_0 R'^3}\overrightarrow{R}', \quad B' = 0.$$

Using the transformation rule of the Lorentz scalar field A_μ and the Lorentz vector field $F_{\mu\nu}$, the electric and magnetic potentials and fields with respect to the observer O are given by

$$\Phi = \frac{\gamma q}{4\pi\epsilon_0 R'}, \quad A = \frac{\gamma\mu_0 v q}{4\pi R'}\widehat{z}, \quad E = \frac{q\gamma^3(1-\beta^2)}{4\pi\epsilon_0 R'^3}\left(r\widehat{r} + (z-vt)\widehat{z}\right), \quad B = \frac{\mu_0 v q\gamma r}{4\pi R'^3}\widehat{\varphi}.$$

■

Exercise: Read the example in [Fey, Volume II, Section 13.6], and explain it in your own words.

Exercise: Show that $E \cdot E - c^2 B \cdot B$ is a Lorentz scalar field.

Chapter 6

Topics

6.1 Antennas

Reference: [Che, Chapter 11]

6.2 Monopole principal bundle

Reference: [Nab, Chapter 0], [Jam, Chapter 12]

6.3 The linking number of knots

Suppose two (smooth) closed oriented curves C, C' in the three dimensional space, which do not intersect. Using electromagnetic theory, Gauss discovered a topological invariant which describes the number of times that each curve winds around the other [Jam, Chapter 12]. To derive this invariant, we assume a uniform and steady current I flowing along C', and compute the circulation of the produced magnetic field along C. The magnetic field is given by (4.3), so by the axiom (4.2), we have

$$n\mu_0 I = \oint_C B \cdot \overrightarrow{dl} = \frac{\mu_0}{4\pi} \oint_C \oint_{C'} \frac{I\,\overrightarrow{dl}' \times \overrightarrow{R}_0}{R_0^2} \cdot \overrightarrow{dl},$$

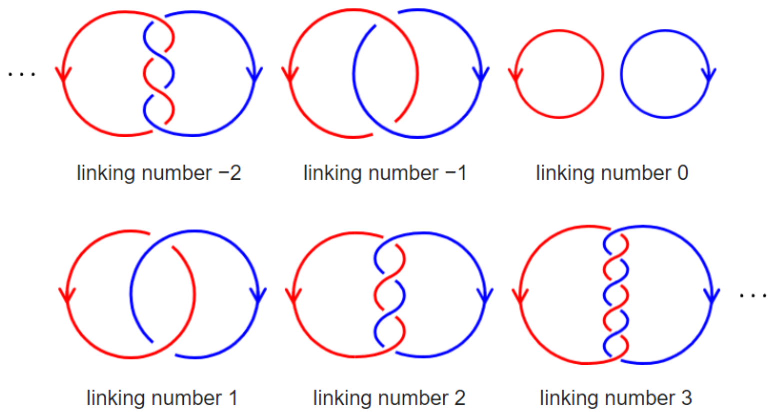
where n denotes the number of times that the current I crosses a surface S whose boundary is C. The integer-valued quantity

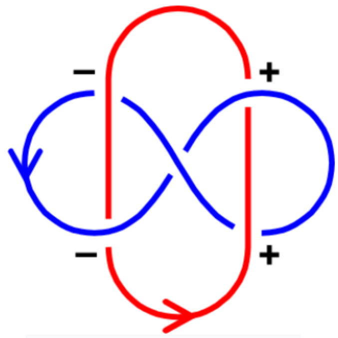
$$n = \frac{1}{4\pi} \oint_C \oint_{C'} \frac{\overrightarrow{R}_0 \times \overrightarrow{dl} \cdot \overrightarrow{dl}'}{R_0^2}$$

is called the **linking number of knots** C, C'. In cartesian coordinates, the linking number is given by the following expression

$$\frac{1}{4\pi} \oint_C \oint_{C'} \frac{(x-x')\,(dydz'-dzdy') + (y-y')\,(dzdx'-dxdz') + (z-z')\,(dxdy'-dydx')}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{\frac{3}{2}}}.$$

The linking number is an integer-valued topological invariant. Reversing the orientation of either of the curves negates the linking number, while reversing the orientation of both curves leaves it unchanged.





The two curves of the Whitehead link have linking number zero.

Exercise: Prove that the winding number of an oriented closed curve in the xy-plane equals its linking number with the z-axis.

Appendix A

Table of conformal mappings [Chu, Appendix 2]

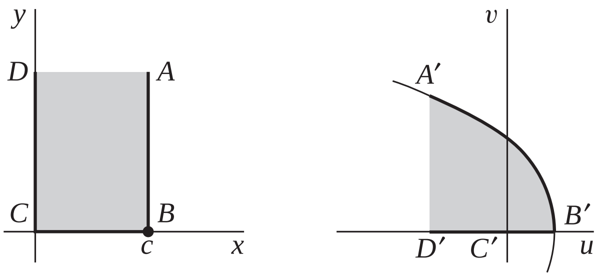


FIGURE 3
$w = z^2$;
$A'B'$ on parabola $v^2 = -4c^2(u - c^2)$.

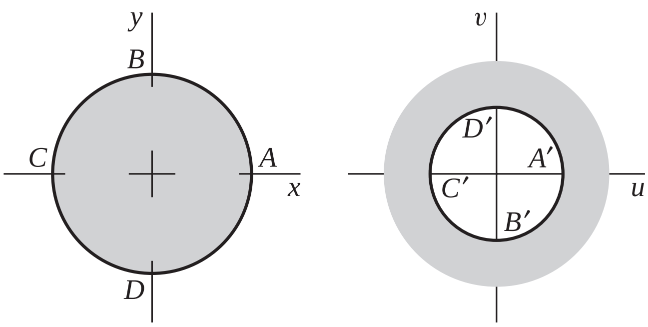


FIGURE 4
$w = 1/z$.

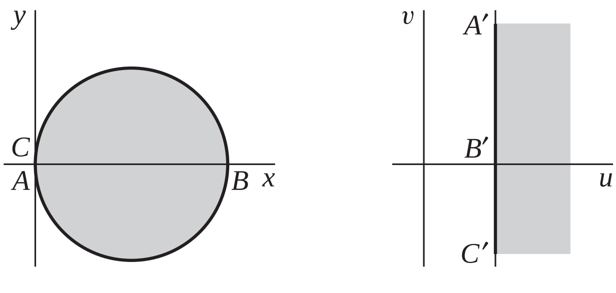


FIGURE 5
$w = 1/z$.

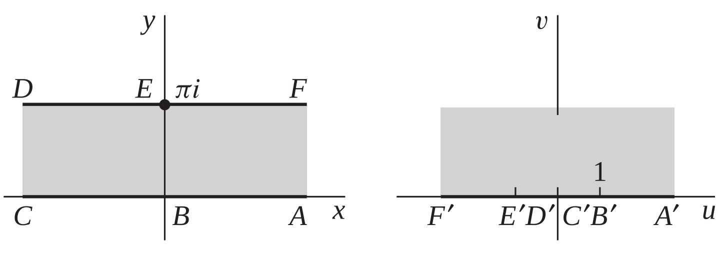


FIGURE 6
$w = \exp z$.

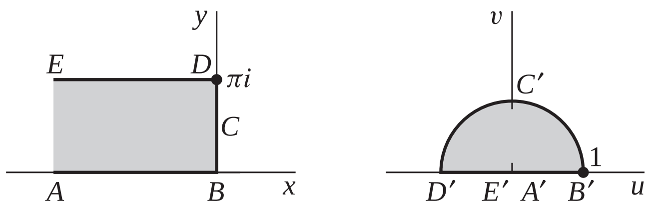


FIGURE 7
$w = \exp z$.

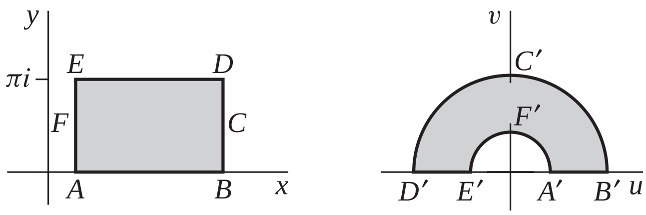


FIGURE 8
$w = \exp z$.

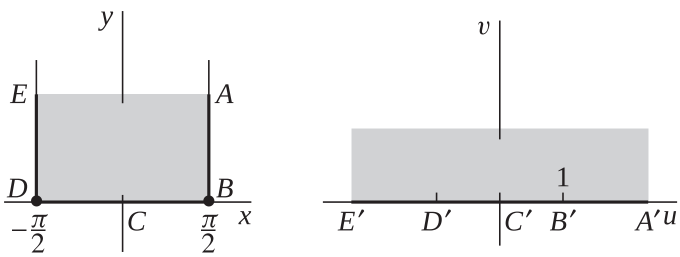


FIGURE 9
$w = \sin z$.

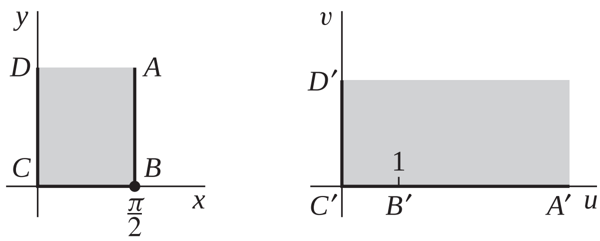


FIGURE 10
$w = \sin z$.

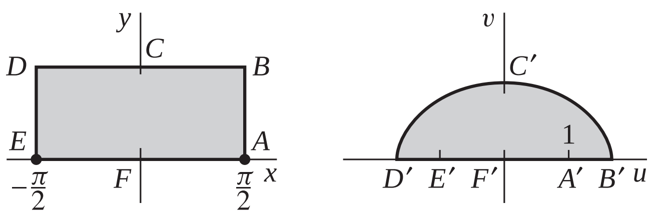


FIGURE 11
$w = \sin z$; BCD on line $y = b$ $(b > 0)$,

$B'C'D'$ on ellipse $\dfrac{u^2}{\cosh^2 b} + \dfrac{v^2}{\sinh^2 b} = 1$.

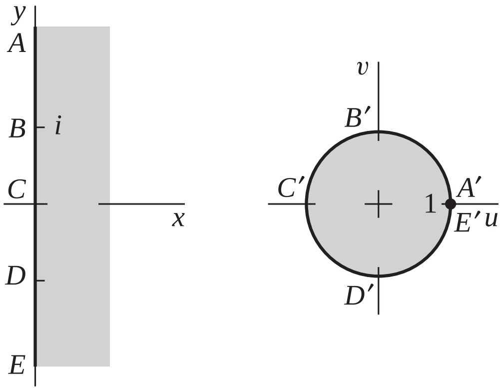


FIGURE 12

$$w = \frac{z-1}{z+1}.$$

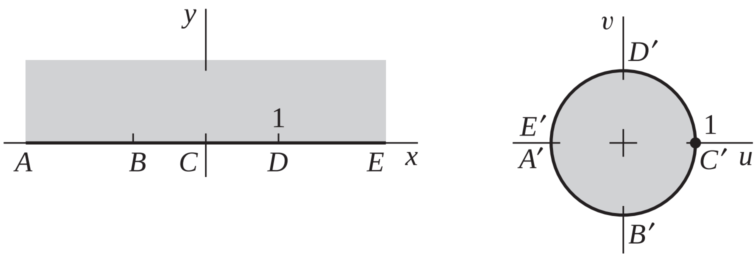


FIGURE 13

$$w = \frac{i-z}{i+z}.$$

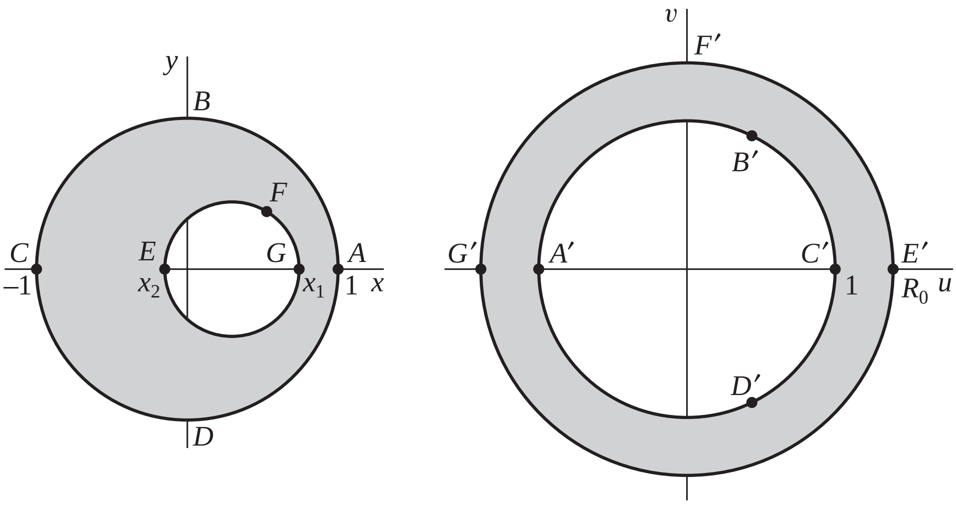


FIGURE 14

$$w = \frac{z-a}{az-1};\ a = \frac{1 + x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2},$$

$$R_0 = \frac{1 - x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 - x_2} \quad (a > 1 \text{ and } R_0 > 1 \text{ when } -1 < x_2 < x_1 < 1).$$

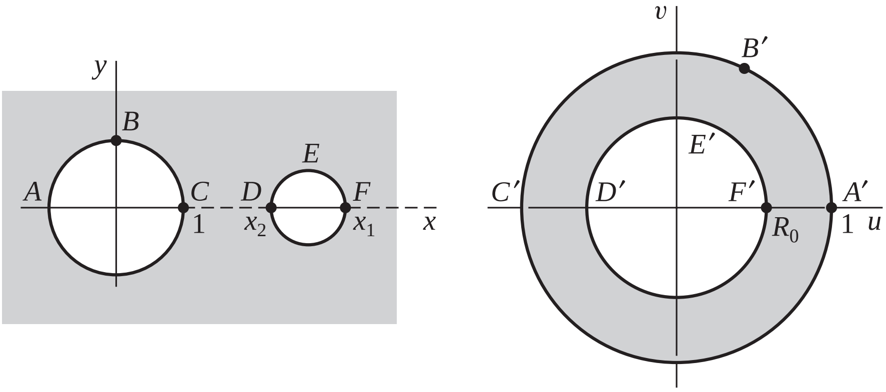


FIGURE 15

$$w = \frac{z-a}{az-1};\ a = \frac{1 + x_1x_2 + \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1+x_2},$$

$$R_0 = \frac{x_1x_2 - 1 - \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1 - x_2}$$ $(x_2 < a < x_1$ and $0 < R_0 < 1$ when $1 < x_2 < x_1)$.

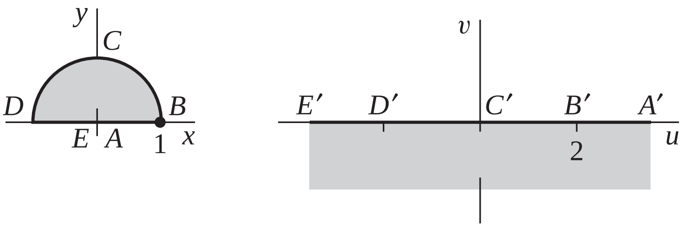


FIGURE 16

$$w = z + \frac{1}{z}.$$

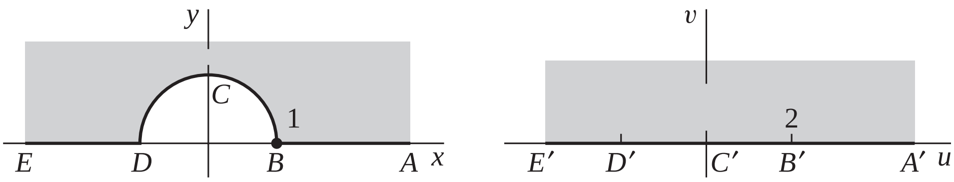


FIGURE 17

$$w = z + \frac{1}{z}.$$

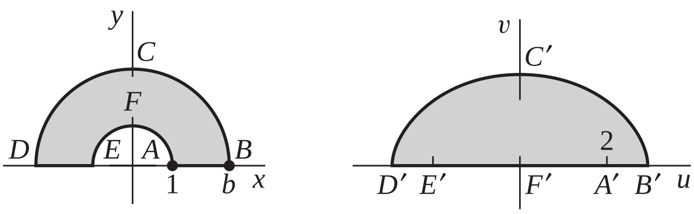


FIGURE 18

$$w = z + \frac{1}{z};\ B'C'D' \text{ on ellipse } \frac{u^2}{(b+1/b)^2} + \frac{v^2}{(b-1/b)^2} = 1.$$

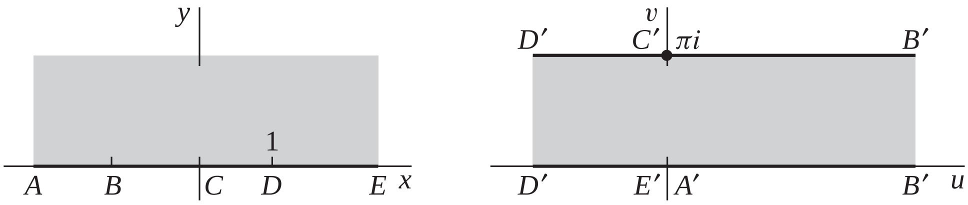


FIGURE 19

$w = \operatorname{Log} \dfrac{z-1}{z+1}$; $z = -\coth \dfrac{w}{2}$.

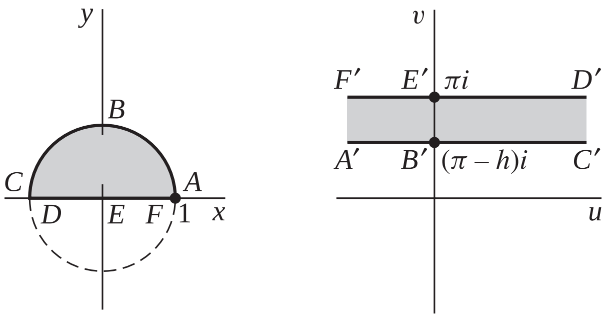


FIGURE 20

$w = \operatorname{Log} \dfrac{z-1}{z+1}$;

ABC on circle $x^2 + (y + \cot h)^2 = \csc^2 h \quad (0 < h < \pi)$.

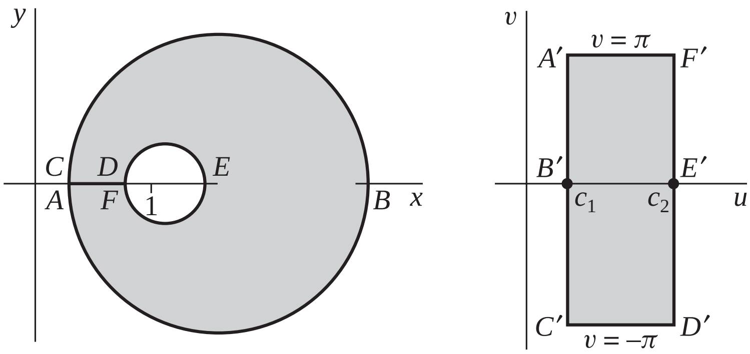


FIGURE 21

$w = \operatorname{Log} \dfrac{z+1}{z-1}$; centers of circles at $z = \coth c_n$, radii: $\operatorname{csch} c_n \quad (n = 1, 2)$.

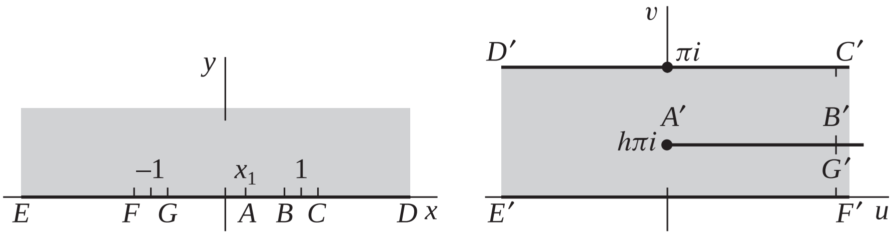


FIGURE 22

$$w = h \ln \frac{h}{1-h} + \ln 2(1-h) + i\pi - h \operatorname{Log}(z+1) - (1-h) \operatorname{Log}(z-1);\ x_1 = 2h - 1 .$$

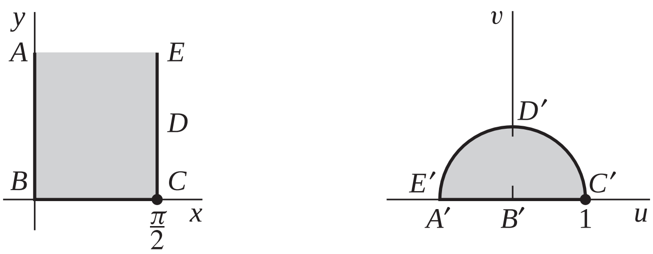


FIGURE 23

$$w = \left(\tan \frac{z}{2}\right)^2 = \frac{1-\cos z}{1+\cos z}.$$

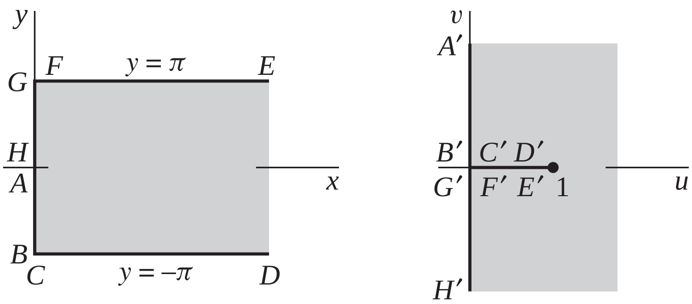


FIGURE 24

$$w = \coth \frac{z}{2} = \frac{e^z+1}{e^z-1}.$$

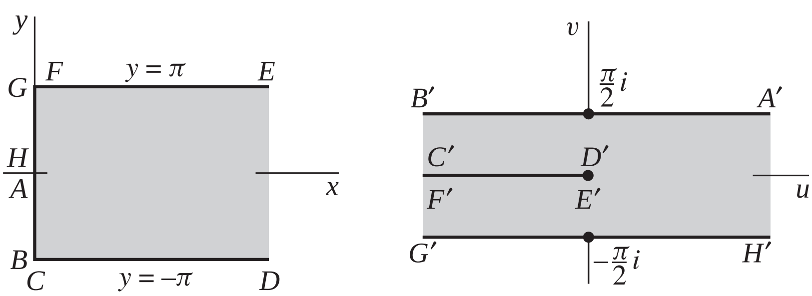


FIGURE 25

$$w = \operatorname{Log}\left(\coth \frac{z}{2}\right).$$

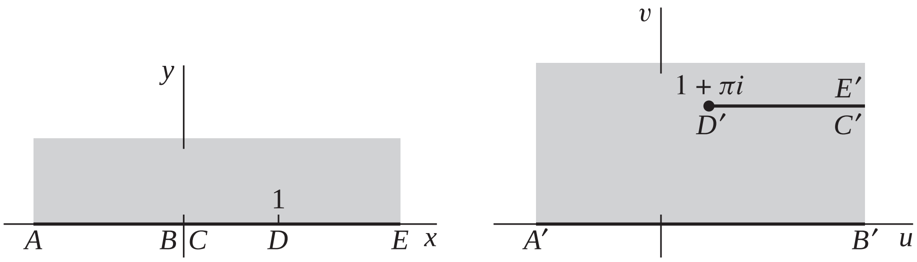


FIGURE 26

$w = \pi i + z - \text{Log } z.$

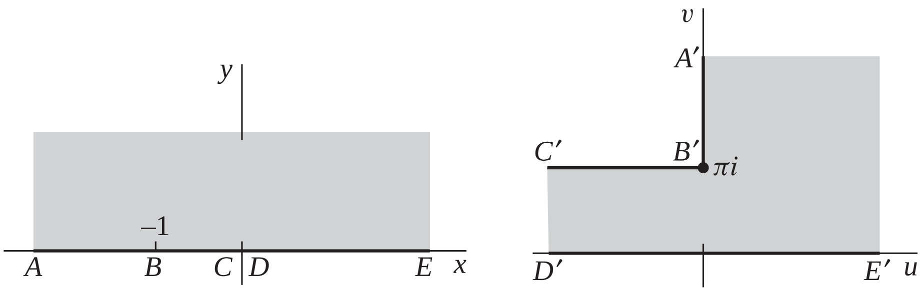


FIGURE 27

$$w = 2(z+1)^{1/2} + \text{Log } \frac{(z+1)^{1/2} - 1}{(z+1)^{1/2} + 1}.$$

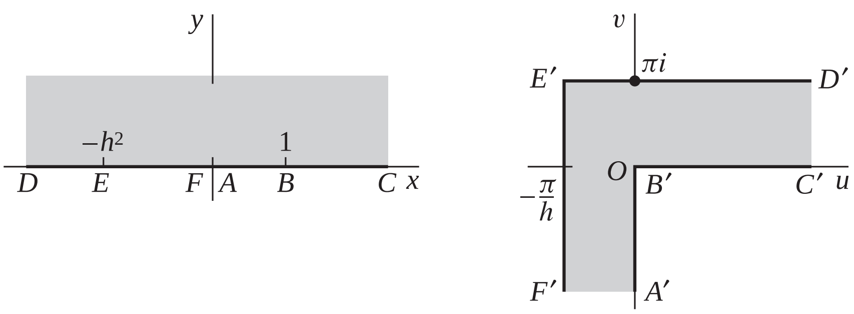


FIGURE 28

$$w = \frac{i}{h}\text{Log } \frac{1 + iht}{1 - iht} + \text{Log } \frac{1+t}{1-t}; \ t = \left(\frac{z-1}{z+h^2}\right)^{1/2}.$$

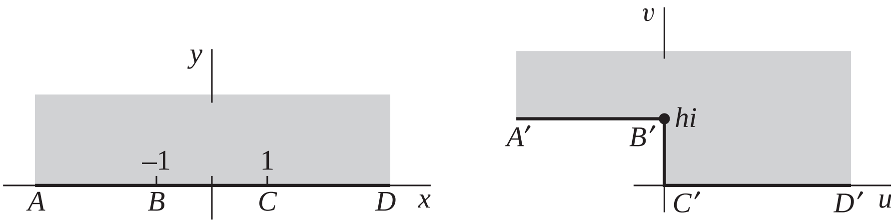


FIGURE 29

$w = \dfrac{h}{\pi}[(z^2 - 1)^{1/2} + \cosh^{-1} z].$*

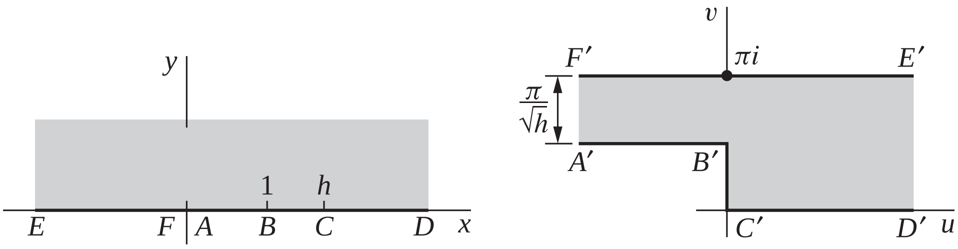


FIGURE 30

$$w = \cosh^{-1}\left(\frac{2z - h - 1}{h - 1}\right) - \frac{1}{\sqrt{h}} \cosh^{-1}\left[\frac{(h + 1)z - 2h}{(h - 1)z}\right].$$

*See Exercise 3, Sec. 122.

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