Notes for *Analysis and Geometry of Several Complex Variables* (CIMAT, Spring 2020)

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Contents

Notations and conventions

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are respectively the set of nonnegative integers, integers, reals and complex numbers.
- For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we set $|\alpha| = \sum \alpha_j$, $\alpha! = \prod \alpha_j$, $a^\alpha = \prod a_j^{\alpha_j}$ and $\partial^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, where $(x_1, \ldots, x_n)$ is the standard coordinates on $\mathbb{R}^n$.
- Neighborhoods are always open.
- Suppose functions f, g defined on some open $D \subseteq \mathbb{C}^m$ and let $p \in D$. The expression "$f = O(g)$ around p" means $|f| \leq C|g|$ for some constant $C > 0$ on some neighborhood of p. The expression "$f = o(g)$ around p" means $f(z)/g(z) \to 0$ as $z \to p$.
- For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ we set $|z|_2 := \sqrt{\sum |z_j|^2}$ and $|z|_\infty := \max |z_j|$.
- $\operatorname{dist}(A, B)$, for $A, B \subseteq \mathbb{C}^m$, denotes the Euclidean distance between A and B, namely $\inf\{|a - b|_2 : a \in A, b \in B\}$.
- The domain, kernel (null space) and range of a linear map A are denoted by Dom_A, Ker_A and Ran_A.
- $\|f\|_X = \sup_{x \in X} |f(x)|$ is the uniform norm of function $f : X \to \mathbb{C}$ continuous on compact topological space X.
- $\mathcal{D}(U) = C_c^\infty(U)$, $U \subseteq \mathbb{R}^n$ open, is the space of smooth compactly supported functions on U.
- $C^k(U)$, $k \in \{0, 1, \ldots, \infty\}$, $U \subseteq \mathbb{R}^n$ open, is the set of functions $f : U \to \mathbb{C}$ which has continuous derivative up to total order k.
- $C^k(\overline{U})$, $k \in \{0, 1, \ldots, \infty\}$, $U \subseteq \mathbb{R}^n$ open, is the set of functions $f : \overline{U} \to \mathbb{C}$ which admit an extension to a C^k function on a neighborhood of $\overline{U}$. Alternatively, by a classical theorem of Borel [Lee, page 27], it is exactly the set of $C^k(U)$ functions f such that each partial derivative $f^{(\alpha)}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sum \alpha_j \leq k$, admits a continuous extension to $\overline{U}$.
- $W^{2,s}(U)$, $s \in \{0, 1, \ldots, \infty\}$, $U \subseteq \mathbb{R}^n$ open, is the Sobolev space of Lebesgue square-integrable functions f on U such that all of their distributional derivatives of total order $\leq s$ are represented by L^2 functions.

Chapter 1

What is this course about?

The main theme of this course on Several Complex Variables (SCV) is the study of holomorphic and harmonic functions on open subsets of $\mathbb{C}^m$, $m > 1$. A recurring situation is the close interaction between the function theory and the geometry of open sets specially their boundary. Examples of this interaction from single variable complex analysis are the definability of the logarithm of nowhere-zero holomorphic functions on simply connected domains [Ahl, page 142], or the solvability of the Dirichlet problem on domains whose complement is such that no connected component reduces to a point [Ahl, page 251]. Important topics in the function theory of SCV are: Solvability of certain partial differential equations, approximations of holomorphic functions, extensions of holomorphic functions, boundary values/zero sets/growth rates of holomorphic/harmonic functions, interpolations of points by holomorphic functions satisfying certain growth conditions, etc. More specifically, a large part of SCV is devoted to extend the following high points of single variable complex analysis to higher dimensions:

- The theorems of Runge and Mergelyan on the uniform approximations of holomorphic functions by rational or polynomial functions [Rud-RCA, chapter 13, 20].

- Riemann's theorem about holomorphic extensions on isolated singularities [Ahl, page 124].

- The theorems of Mittag-Leffler [Rud-RCA, 13.10] and Weierstrass [Rud-RCA, 15.11] on the representations of meromorphic functions by infinite sums and products, and some of their immediate consequences: The interpolation problem [Rud-RCA, 15.13], the division problem [Rud-RCA, 15.15] and Poincaré's problem [Rud-RCA, 15.12] that meromorphic functions are ratios of holomorphic functions.

- Riemann mapping theorem on the classification of simply connected opens of the complex plane up to biholomorphism [Ahl, page 230].

- The theorem of Fatou on the almost everywhere existence of radial limits of bounded holomorphic functions on the unit disk [Rud-RCA, 11.32][Cara, volume II, page 43].

- Inner-outer factorization of Hardy functions on the unit disk [Rud-RCA, 17.17][Dur, 2.8].

- Beurling's theorem on the characterization of shift-invariant subspaces of the Hardy space of the unit disc [Rud-RCA, 17.21].

We continue our discussion of the interaction between function theory and the geometry of domains. To elaborate more on the term "geometry": A certain biholomorphic-invariant notion of convexity for open subsets of $\mathbb{C}^m$, called pseudoconvexity, will play an important role throughout the course. This notion is weaker than the usual notion of convexity, and any open in $\mathbb{C}$ is pseudoconvex; that is why it does not show up in undergraduate complex analysis. The core of the course is around analysis not geometry; however as much as time permits we try to briefly sketch the connections between SCV and important topics in geometry such as Hermitian symmetric spaces, biholomorphic-invariant metrics, Kähler geometry, CR geometry, etc.

In this introductory chapter, we try to give a taste of SCV through two surprising discoveries of early twentieth century mathematics that gives SCV some of its distinguished flavor compared to the single variable complex analysis. To do that we need some notations and definitions given in the next section.

1.1 Holomorphic functions

The complex space $\mathbb{C}^m$ consists of all m-tuples of complex numbers. It is coordinated by complex variables $z_j = x_j + \sqrt{-1}y_j$, $j = 1, \ldots, m$, and their conjugates $\overline{z}_j = x_j - \sqrt{-1}y_j$, where x_j, y_j are usual real coordinates of $\mathbb{R}^{2m}$. Infinitesimal variations of real and complex variables are related by:

$$dz_j = dx_j + \sqrt{-1}dy_j, \quad d\overline{z}_j = dx_j - \sqrt{-1}dy_j. \tag{1.1}$$

Dually we have differential operators:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right). \tag{1.2}$$

These all mean that the infinitesimal variation of a smooth function f defined on some open subset of $\mathbb{C}^n$ is given by:

$$df = \sum_j \frac{\partial f}{\partial x_j}dx_j + \sum_j \frac{\partial f}{\partial y_j}dy_j = \sum_j \frac{\partial f}{\partial z_j}dz_j + \sum_j \frac{\partial f}{\partial \overline{z}_j}d\overline{z}_j. \tag{1.3}$$

(If you are not comfortable with the terms "infinitesimal variations" and their dual "differential operators", accept (1.1), (1.2) and the first equality in (1.3) as definitions, and then verify the second equality in (1.3). Using complex variables instead of real ones saves life and makes formulas much more illuminating.) The first and second summands in the last expression in (1.3) are respectively denoted by ∂f and $\overline{\partial} f$. So d as an operator acting on functions splits as $d = \partial + \overline{\partial}$. The operator $\overline{\partial}$ is called the **d-bar** (or **del bar**) **operator**. It is absolutely the most important differential operator in SCV. A large part of this course is devoted to study it, specially Chapter 4.

Exercise: Show that $d^c \log|z|^2 = d\theta/2\pi$ where $d^c := \frac{\sqrt{-1}}{4\pi}\left(\overline{\partial} - \partial\right)$ and $z = r\exp(\sqrt{-1}\theta)$. (*Hint.* Locally, $\log|z|^2 = \log z + \log\overline{z}$.)

A **domain** in $\mathbb{C}^m$ is a connected open subset. Domains are usually denoted by D or Ω. A complex-valued function $f : D \to \mathbb{C}$ defined on open $D \subseteq \mathbb{C}^m$ is called **holomorphic** (or **complex analytic**) (notations: $f \in \mathcal{O}(D)$ or $f \in \mathrm{Holo}(D)$) if any of the following equivalent conditions holds:

1. f is holomorphic with respect to each of its arguments separately while others are kept fixed, namely for each $z \in D$ and each $j = 1, \ldots, m$, the complex limit $\lim_{\lambda \to 0}(f(z + \lambda e_j) - f(z))/\lambda$ exists, where e_j is the point of $\mathbb{C}^m$ with 1 on the j-th entry and zero elsewhere.

2. f is locally representable by convergent power series, namely for each point p in D there exists a power series $\sum_{\alpha \in \mathbb{N}^m} a_\alpha (z-p)^\alpha$ centered around p which is absolutely convergent on some nonempty (open) neighborhood of ζ and that the value of the series on that neighborhood coincides with $f(z)$. Here we are using multi-index notation: $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$, $(z-p)^\alpha = \prod_{1 \le j \le m} (z_j - p_j)^{\alpha_j}$. Note that the order of summation does not matter because we have required absolute convergence.[1]

3. f is continuously first-order differentiable (notation: $f \in C^1(D)$) and satisfies the **Cauchy-Riemann equation** $\overline{\partial} f = 0$, namely $\partial f / \partial \overline{z}_j = 0$ for all j. If $f = u + \sqrt{-1}v$, where u, v are real-valued, then the Cauchy-Riemann equation is $\partial u / \partial x_j = \partial v / \partial y_j$ and $\partial u / \partial y_j = -\partial v / \partial x_j$. Equivalently, f is Lebesgue integrable on every compact subset of D (notation: $f \in L^1_{\mathrm{loc}}(D)$) and satisfies $\partial f / \partial \overline{z}_j = 0$ in the distributional sense namely $\int_D f \partial \varphi / \partial \overline{z}_j = 0$ for every C^∞ function φ compactly supported in D.

The equivalence of these definitions is proved in Chapter 3, Remarks 40 and 57.

A complex-valued function f defined on an arbitrary subset A of $\mathbb{C}^m$ is called **holomorphic** if there is a holomorphic function on some open containing A which extends f. A $\mathbb{C}^n$-valued function defined on an open of $\mathbb{C}^m$ is **holomorphic** if all of its components are so.

Exercise: Show that the compositions of holomorphic maps are holomorphic, more precisely, if $f(z)$ is a holomorphic map defined on open $D \subseteq \mathbb{C}^m$ with values in open $D' \subseteq \mathbb{C}^n$ and $g(w)$ a holomorphic map from D' to $\mathbb{C}^k$ then $g(w) = g(f(z))$ is holomorphic on D. (*Hint.* Use the third definition. The chain rule in complex coordinates for C^1 functions is

$$g_z = g_w f_z + g_{\overline{w}} \overline{f}_z, \quad g_{\overline{z}} = g_w f_{\overline{z}} + g_{\overline{w}} \overline{f}_{\overline{z}}.$$

Insisting on using the second definition leads to a mess [KP, 2.2.8,1.4.2].)

[1] There is a result sometimes called Abel's Lemma (Lemma 39) which guarantees that if a power series is convergent in some nonempty neighborhood of its center with some summation order (namely a linear order on $\mathbb{N}^m$) then there is a smaller nonempty neighborhood where the series is absolutely convergent. Therefore in the definition above the absolute convergence can be replaced by convergence with respect to some (hence all) linear ordering of $\mathbb{N}^m$.

1.2 Some distinctions between the analysis of one and several complex variables

1.2.1 Hartogs extension phenomenon

For every domain D in the complex plane one can find a holomorphic function which can not be extended holomorphically to any larger domain. To see this first find a sequence of points in D which accumulates at every point of the boundary of D and nowhere else[2], and then by Weierstrass' infinite product theorem from single variable complex analysis (Theorem 20 or [Rud-RCA, 15.11]) construct a holomorphic function on D that vanishes at these points and nowhere else. This is the function we are looking for. Q.E.D. (See also the last exercise of this section.) In great contrast to this:

Theorem (Hartogs extension theorem; $G \setminus K$ version)**.** *Assume open $G \subseteq \mathbb{C}^m$, $m > 1$, and compact $K \subseteq G$ such that $G \setminus K$ is connected. (For example K can be a single point of G.) Then every holomorphic function on $G \setminus K$ can be extended holomorphically to G.*

We will be able to prove this in Chapter 3 (Theorem 24), but at the moment let us show it for the special case $m = 2$, $G = \{|z_1| < 1, |z_2| < 1\}$, $K = \{|z_1| \leq 1/2, |z_2| \leq 1/2\}$. The situation can be visualized in a coordinate system with axis $|z_1|$ and $|z_2|$, the so-called **absolute space**. Let f be holomorphic on $G \setminus K$. *First proof using Cauchy integral formula.* The expression

$$\frac{1}{2\pi\sqrt{-1}} \int_{|\zeta_2|=2/3} \frac{f(z_1, \zeta_2) d\zeta_2}{\zeta_2 - z_2}$$

defines a smooth function holomorphic on $G' := \{|z_1| < 1, |z_2| < 2/3\}$, which agrees with f on $\{1/2 < |z_1| < 1, |z_2| < 2/3\}$, so by the rigidity of holomorphic functions[3] agrees with f also on $G' \cap (G \setminus K)$. Therefore the function defined by f on $G \setminus K$ and by the integral formula above on G' holomorphically extends f to G. *Second proof using Laurent series.* For each fixed z_1 with $|z_1| < 1$, $f(z_1, z_2)$ is holomorphic with respect to z_2 at least on the annulus $1/2 < |z_2| < 1$, so has the Laurent series representation:

$$f(z_1, z_2) = \sum_{n=-\infty}^{\infty} a_n(z_1) z_2^n, \quad a_n(z_1) = \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta_2|=2/3} \frac{f(z_1, \zeta_2) d\zeta_2}{\zeta_2^{n+1}}.$$

When $1/2 < |z_1| < 1$ we have the extra information that $f(z_1, z_2)$ is holomorphic with respect to z_2 on the disk $|z_2| < 1$, so for such z_1 negative powers do not show up in the Laurent series, namely $a_n(z_1) = 0$ for $n < 0$ and $1/2 < |z_1| < 1$. On the other hand, the integral formula shows that $a_n(z_1)$ is holomorphic on whole $|z_1| < 1$, so by the rigidity of holomorphic functions it should be identically zero for $n < 0$ and $|z_1| < 1$. Therefore our Laurent series is in fact a Taylor series. Q.E.D.

[2] For example, enumerate all points in D with rational coordinates by a sequence q_j, and for each j choose a point p_j of distance less than $1/j$ to the boundary which belongs to the largest open ball around q_j which is contained in D.

[3] If two holomorphic functions agree on a nonempty open then they agree on the whole connected component of their common domain of definition containing that open. This can be easily proved by the power series representation of holomorphic functions.

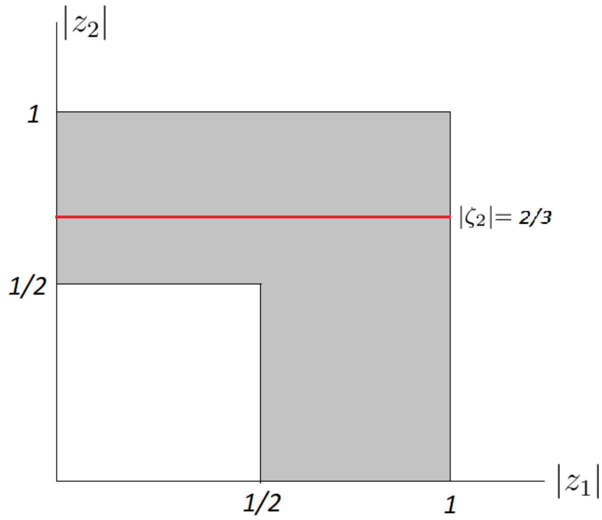


Figure 1: Proof of the Hartogs extension theorem.

Argument above also works when K is a single point. Applying this version to holomorphic functions and their inverses shows that in great contrast to single variable complex analysis: *The zeros and removable singularities of holomorphic functions of more than one complex variables are never isolated.*

With the same methods one can prove:

Theorem 1 (Hartogs extension theorem; H version)**.** *Every holomorphic function on*

$$H_m = \{z \in \mathbb{C}^m : (1/2 < |z_1| < 1, |z_j| < 1, \forall j > 1) \text{ or } (|z_1| < 1, |z_j| < 1/2, \forall j > 1)\}$$

extends holomorphically to $\{z \in \mathbb{C}^m : |z_j| < 1, \forall j\}$.

Exercise: (1) Prove Theorem 1 by both a Cauchy integral formula and a Laurent series. For a proof just using Taylor series refer [Ohs, page 36]. (2) Does the theorem remain valid if H_m is replaced by the following?

$$\{z \in \mathbb{C}^m : (1/2 < |z_1| < 1, |z_j| < 1, \forall j > 1) \text{ or } (|z_1| < 1, 1/2 < |z_j| < 2/3, \forall j > 1)\}$$

Exercise: Let $D \subseteq \mathbb{C}$ be open. This exercise construct in several steps a holomorphic function on D which can not be extended holomorphically across any boundary point. (1) Choose a sequence of points p_j which is dense in ∂D, and mutually disjoint line segments γ_j normal to the boundary at p_j; (2) Setting $f_j := 1/(z - p_j)$ choose nonzero constants c_j small enough such that $|c_j f_j(z)| < 2^{-j}$ for any $z \in \mathbb{C}$ which belongs to the compact set $\{\text{dist}(z, \partial D) \geq 2^{-j}\} \cup \bigcup_{k<j} \gamma_k$; (3) Show that $f := \sum c_j f_j$ converges uniformly on compact subsets of D, so represents a holomorphic function on D. (4) Show that $|\sum_{k \neq j} c_k f_k(z)|$ is bounded on γ_j, so $|f(z)|$ blows up as z approaches p_j along γ_j; (5) Deduce that f can not be extended holomorphically across any boundary point of D.

1.2.2 Balls and polydiscs are not biholomorphically equivalent

Here is the second fundamental distinction between the analysis of single and several complex variables. Recall Riemann's mapping theorem in single variable complex anal-

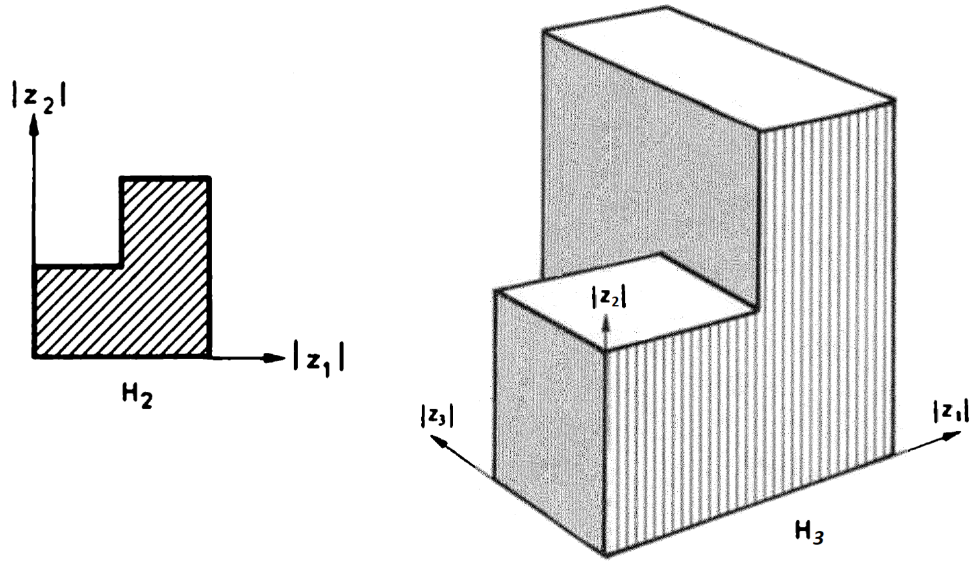


Figure 2: Hartogs domain H [Kaup, page 35].

ysis [Ahl, page 230]: *Every simply connected domain in $\mathbb{C}$ which is not the whole $\mathbb{C}$ is biholomorphic to the unit disk.* In great contrast to this theorem:

Theorem 2. *In $\mathbb{C}^m$, $m > 1$, the unit ball $\{|z_1|^2 + \cdots + |z_m|^2 < 1\}$ and polydisk $\{|z_1| < 1, \ldots, |z_m| < 1\}$ are not biholomorphic*[4].

Proof. By contradiction, assume a biholomorphism $F : B \to P$ from the unit ball to the unit polydisk. By post-composition with Mobius transformations of one complex variable we can assume that F preserves the origin, $F(0) = 0$. Fix a unimodular complex number λ (unimodular means $|\lambda| = 1$), and consider the conjugation map $G : B \to B$ given by $G(z) = F^{-1}(\overline{\lambda} F(\lambda z))$. G is a holomorphic map, $G(0) = 0$, and by the chain rule $G'(0) = \text{id}$. Here prime means the complex Jacobian, namely for $G = (G_j)$, G' is the $m \times m$ matrix with entries $\partial G_j / \partial z_k$. We assert that $G = \text{id}$. (This is called H. Cartan's uniqueness theorem.) If not, the power series representation of G takes the form $G(z) = z + Az^k +$ [higher order terms], where $k > 1$, z^k denotes the column vector with entries all monomials $z_1^{n_1} \cdots z_m^{n_m}$ of degree $n_1 + \cdots + n_m = k$, and A is a nonzero matrix of complex numbers. Then the composition $G^n : B \to B$ of G with itself n-times has Taylor expansion $G^n(z) = z + nAz^k +$ [higher order terms]. By integration over unimodular complex variable μ we get $\int \mu^{-k} G^n(\mu z) = nAz^k$. Making n large gives a contradiction because the left hand side is bounded for every z. This contradiction shows that $G = \text{id}$, or equivalently $F(\lambda z) = \lambda F(z)$. Both sides of this latter identity are holomorphic with respect to λ, so it holds for all $|\lambda| \leq 1$. Differentiating with respect to λ and setting $\lambda = 0$ shows that F is linear. Therefore F induces an invertible linear map on $\mathbb{C}^m$ which maps B bijectively onto P, hence sends the boundary of B bijectively onto the boundary of P. However the boundary of P contains a real line segment $\{(1, t, 0, \cdots, 0) : t \in [-1, 1]\}$ but there is no such thing in the boundary of B. This is a contradiction. Here is another contradiction. For any collection of unimodular

[4] A bijective map which all the components of it and its inverse are holomorphic.

complex numbers λ_j, $j = 1, \dots, m$, since all e_j and $\sum_j \lambda_j e_j$ are on the boundary of P, so all the points $w_j := F^{-1}(e_j)$ and $\sum \lambda_j w_j$ are on the boundary of B, namely of norm 1. Integrating $1 = |\sum \lambda_j w_j|_2^2 = \sum \lambda_j \bar{\lambda}_k \langle w_j, w_k \rangle$ with respect to (λ_j) gives $1 = m$. ■

Other proofs are given in [Ran, page 24][Kaup, page 10][Nar, page 70]. Finding enough geometric invariants to classify domains up to biholomorphism is a deep unsolved problem in SCV that we discuss in Section 10.3.

1.2.3 Other distinctions

Here we list some other distinctive character of SCV compared to analysis of one complex or several real variables:

1. When $m > 1$ there exist biholomorphic mappings from $\mathbb{C}^m$ onto a proper subset of it. Such mappings are known as Fatou-Bieberbach mappings [Boas, section 3.1].

2. An injective holomorphic map $D \to \mathbb{C}^m$ defined on an open $D \subseteq \mathbb{C}^m$, $m \geq 1$, is biholomorphism onto its image, namely has holomorphic inverse [Ran, page 29]. This is not true for real variables: $x \mapsto x^3$ is a smooth map from $\mathbb{R}$ to $\mathbb{R}$ whose inverse is not smooth.

3. Any function which is holomorphic in each variable separately is automatically continuous. This is called Hartogs separate analyticity theorem [Hör, 2.2.8]. However

$$\begin{cases} x^2/(x^2+y^2), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

   is real analytic (defined in page 59) in each variable separately but discontinuous at the origin.

4. A holomorphic function on a polydisc has power series representation around the center point valid on the whole polydisc. However $1/(1+x^2)$ is real analytic on whole $\mathbb{R}$ but its power series around the origin has radius of convergence 1.

We discuss some of these phenomena later.

1.3 A remark about our method

We end this chapter with a remark about the method we are going to use in order to get hands on deep "global" results of SCV, most notably Hartogs extension theorem, holomorphic extensions, Cousin problems, Levi's problem, Cartan theorems A and B, etc. The main issue is to develop a technique for gluing local data in order to construct global objects. Through time several different techniques has been developed, for example:

1. Sheaf theory combined with Weierstrass' division theorem, abstract commutative algebra, Cauchy-Weil integrals, extensive plurisubharmonic function theory, complicated

inductive arguments, and numerous clever ideas of Oka (most notably the embedding into higher dimensions in order to simplify the geometry) [Oka][Nog, Kaup, GR]. These are developed by Oka, Cartan, etc. Here a holomorphic function is a function locally representable by convergent power series (the second definition in page 7).

2. Partial differential equations (PDE) [Hör, Ohs, CS]. This is suggested by Riemann, developed by Kohn, Hörmander, Catlin, etc. Here a holomorphic function is a smooth function satisfying Cauchy-Riemann equations (the third definition in page 7).

3. Integral representations [Ran, HL, LM]. These are developed by Henkin, Grauert, Range, etc. Here a holomorphic function f is a function reproduced by a convolution integral $f(z) = \int f(\zeta)K(z,\zeta)$ where $K(z,\zeta)$ is a fairly explicit kernel holomorphic with respect to z. There are several different constructions of these kernels.

We will choose the PDE approach. If time permits we briefly sketch the other two.

Chapter 2

Holomorphic functions of one complex variable

References: [Hör, chapter 1].

We start by reviewing the first properties of holomorphic functions of one complex variable that can be deduced from Cauchy-Pompeiu integral formula. Later in this chapter we solve the d-bar problem and Cousin problems in one complex variable. The key to their solution is an approximation theorem for holomorphic functions discovered by Runge. These topics are developed in such a way that guides the generalization to higher dimensions.

Before we start it is helpful to remind from undergraduate analysis the conditions under which differentiation commutes with integration and taking limit:

Theorem. *(1) If (X, μ) is a measurable space and $f : X \times (a, b) \to \mathbb{C}$ is a function such that $f(x, t)$ is integrable for every t and $|\partial f/\partial t(x, t)| \leq g(x)$ for some integrable function g and every x, t then $\int_X f(x, t)d\mu(x)$ is differentiable and $\frac{d}{dt}\int_X f(x, t)d\mu(x) = \int_X \frac{\partial f}{\partial t}(x, t)d\mu(x)$. (2) If $f_j : (a, b) \to \mathbb{C}$ is a sequence of differentiable functions such that df_j/dt converges uniformly and f_j converges for at least one point $t_0 \in (a, b)$ then f_j converges uniformly to a differentiable function and $\frac{d}{dt}\lim f_j = \lim \frac{df_j}{dt}$.*

Proof. Both are immediate from the mean value theorem for differentiation and Lebesgue dominated convergence theorem; refer [Fol, 2.27][Apo, 9.13, 10.39]. ■

2.1 First properties of holomorphic functions

To develop the basic properties of holomorphic functions our starting point is the following integral formula.

Theorem 3 (Cauchy-Pompeiu and Cauchy integral formulas)**.** *Let $D \subseteq \mathbb{C}$ be a bounded open with C^1 boundary. (1) For every $f \in C^1(\overline{D})$ and $z \in D$ we have*[1]

$$f(z) = \frac{1}{2\pi\sqrt{-1}}\left(\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta + \int_D \frac{\partial f/\partial\overline{\zeta}(\zeta)}{\zeta - z}d\zeta \wedge d\overline{\zeta}\right).$$

[1] As usual the line integral $\int_{\partial D}$ is taken in the counterclockwise direction, namely D is always on the

(2) For every $f \in C(\overline{D}) \cap \mathcal{O}(D)$ and $z \in D$ we have

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. (1) Fix $z \in D$. Applying Stokes' theorem $\int_{\partial M} \omega = \int_M d\omega$ to the complex-valued differential 1-form $\omega(\zeta) = f(\zeta)(\zeta - z)^{-1} d\zeta$ and the (oriented) surface $M = \{\zeta \in D : |\zeta - z| > \epsilon\}$ for $\epsilon > 0$ smaller than the distance of z to the boundary of D, we have:

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{|\zeta - z| = \epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_M \frac{\partial f / \partial \overline{\zeta}(\zeta)}{\zeta - z} d\overline{\zeta} \wedge d\zeta, \tag{2.1}$$

because $d\omega = \partial f / \partial \overline{\zeta}(\zeta)(\zeta - z)^{-1} d\overline{\zeta} \wedge d\zeta$. (If you are not familiar with Stokes' theorem [Rud-PMA, 10.33][Lee, 16.11], applying Green's theorem $\oint_C M dx + N dy = \int (\partial N / \partial x - \partial M / \partial y) dx dy$ to $M = f(\zeta)(\zeta - z)^{-1}$ and $N = \sqrt{-1} f(\zeta)(\zeta - z)^{-1}$ also gives (2.1).) Now send ϵ to 0. Since $\zeta \mapsto (\zeta - z)^{-1}$ is integrable on any bounded domain around its singular point $\zeta = z$, by the Lebesgue dominated convergence theorem the limit of the area integral as $\epsilon \to 0$ exists. This limit value is the meaning of the area integral in the statement of the theorem. The second line integral after using parametrization $\zeta = z + \epsilon \exp(\sqrt{-1}\theta)$, $\theta \in [0, 2\pi]$, equals $\int \sqrt{-1} f(\zeta) d\theta$, hence approaches $2\pi\sqrt{-1} f(z)$ as $\epsilon \to 0$.

(2) If $f \in C^1(\overline{D}) \cap \mathcal{O}(D)$ the result is immediate from (1). The general case follows after a limiting process. ∎

Theorem 4. *Homomorphic functions are smooth. Complex derivatives of holomorphic functions are holomorphic.*

Proof. Apply the Cauchy integral formula. The integrand $f(\zeta)(\zeta - z)^{-1}$ and all its partial derivatives with respect to $\mathrm{Re} z$, $\mathrm{Im} z$ and $\overline{z}$ are integrable, because the region of integration is compact with z being uniformly away from it, $f(\zeta)$ is continuous, and all partial derivatives of $(\zeta - z)^{-1}$ are integrable. Therefore one can exchange integration with the differential operators $\partial / \partial \mathrm{Re} z$, $\partial / \partial \mathrm{Im} z$ and $\partial / \partial \overline{z}$. ∎

Theorem 5 (Bergman's estimate)**.** *Let f be a holomorphic function on open $D \subseteq \mathbb{C}$. For any compact $K \subseteq D$, open $K \subseteq D' \subseteq D$, and nonnegative integer k there exists constant $C = C(K, D', k) > 0$ such that*

$$\left\| \partial^k f / \partial z^k \right\|_K \leq C \|f\|_{L^1(D')}.$$

Proof. Apply Cauchy-Pompeiu formula to $f\psi$ on D' where ψ is a smooth bump function [Fol, 8.18][Nes, 2.5][Lee, 2.25] compactly supported on D' and equals 1 on some neighborhood of K. ∎

Theorem 6 (Weierstrass)**.** *If a sequence of holomorphic functions on open $D \subseteq \mathbb{C}$ converges uniformly on compact sets of D then the limit function is itself holomorphic. Furthermore, for any positive integer k the sequence of k-th order complex derivatives converges uniformly on compacts to the k-th order complex derivative of the limit function.*

left side of the movement of $d\zeta$. The second term $\int_D$ is an area integral by setting $d\zeta \wedge d\overline{\zeta} := -2\sqrt{-1} d\mu(\zeta)$ where μ is the Lebesgue measure on the plane $\mathbb{C} \cong \mathbb{R}^2$; however this integral is improper because of the singularity at $\zeta = z$, and is understood in the Cauchy's principal value sense namely the limit of the same integrand over $D \setminus \{|\zeta - z| < \epsilon\}$ as $\epsilon \to +0$.

Proof. Let f_j be that sequence, with limit function f. Bergman's estimate applied to $f_j - f_k$ shows that $\partial f_j/\partial z$ converges uniformly on compacts. Since $\partial f_j/\partial \overline{z} = 0$ it follows that $\partial f_j/\partial x$ as well as $\partial f_j/\partial y$ converge uniformly on compacts. Therefore $f \in C^1$ and $\partial f/\partial \overline{z} = \lim \partial f_j/\partial \overline{z} = 0$. ∎

Theorem 7 (Montel's compactness theorem). *A sequence of holomorphic functions on open $D \subseteq \mathbb{C}$ which is locally equibounded contains a subsequence that converges uniformly on compact subsets D.*

Proof. Let f_j be that sequence. According to the Arzela-Ascoli theorem [Fol, 4.44] we need to check equicontinuity: For every $z \in D$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $|z - w| < \delta$ then $|f_j(z) - f_j(w)| < \epsilon$ for all j. Take δ small enough such that the closed disk L of radius δ centered at z is contained in D. For every $w \in L$ we have $f_j(z) - f_j(w) = \int_w^z \partial f_j/\partial \zeta d\zeta$, where we are taking the compact straight line segment from w to z. By the Bergman's estimate and locally equiboundedness hypothesis we have $\|\partial f_j/\partial \zeta\|_L < C$ for some finite number C. So $|f_j(z) - f_k(w)| < C|z - w|$, and we are done by shrinking δ to ϵ/C if needed. ∎

Theorem 8 (Power series representations, Cauchy's estimate). *Let f be a holomorphic function on the disk $\{|z| < R\}$. Then:*

$$f(z) = \sum_{n \geq 0} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta,$$

for any $|z| < R$ and $0 < r < R$. The convergence is absolute and uniform on compacts. We also have the estimate

$$|a_n| \leq r^{-n} \sup_{|z|=r} |f(z)|,$$

for any $0 < r < R$.

Proof. Fix the point z with $|z| < R$, and choose $|z| < r < R$. In the Cauchy integral formula for $f(z)$ on $\{|\zeta| < r\}$, use geometric series representation $(\zeta - z)^{-1} = \sum_n z^n \zeta^{-n-1}$ valid for $|z| < |\zeta|$. Since the convergence in geometric series is absolute and uniform on compacts one can interchange summation with integration and the resulting series also converges absolutely and uniformly on compacts [Apo, 9.9][Rud-PMA, 7.16]. ∎

Theorem 9 (Identity theorem). *Let f and g be two holomorphic functions on connected open $D \subseteq \mathbb{C}$. Then f equals g everywhere on D if any of the following conditions holds: (1) f along with all its higher complex derivatives match the corresponding values of g at some point of D; (2) f and g agree on some nonempty open subset of D. (3) f and g agree on a set of points which accumulates at some point of D.*

Proof. (1) Consider $h := f - g$. The set S of points in D where all $\partial^k h/\partial z^k$, $k \geq 0$, vanish is nonempty (by hypothesis), closed (by continuity) and open (by power series representation). Therefore $S = D$ because D is connected.

(2) Immediate from (1).

(3) By contradiction assume that $h := f - g$ have zeros which accumulate at $p \in D$, but is not identically zero. The first Taylor coefficient a_0 of $h = \sum a_j (z - p)^j$ is zero

but since h it is not identically zero there is a smallest one c_k which is nonzero. Then $h(z) = z^k H(z)$ where H is a holomorphic function with $H(0) = c_k \neq 0$. By continuity H is nonzero on some neighborhood of a. This is a contradiction. ■

Case (3) above fails in several complex variables as the simple example $f(z_1, z_2) = z_1 z_2$ shows. In fact, a simple application of the Hartogs extension theorem mentioned in Chapter 1 shows that zeros of holomorphic functions of more than one complex variable are never isolated.

Theorem 10 (Open mapping theorem)**.** *Nonconstant holomorphic functions on connected opens of the complex plane map open subsets to open ones.*

Proof. [Ahl, page 132] gives an elegant proof based on the argument principle. Here is a more elementary proof taken from [Nar, page 6]. Assuming a nonconstant function f holomorphic on some open disk U around the origin and that $f(0) = 0$, it suffices to show that some neighborhood of 0 is taken by f. By the identity theorem there exists $\epsilon > 0$ such that f never vanishes on $0 < |z| \leq \epsilon$. Let $\delta > 0$ denote the distance of $\{f(z) : |z| = \epsilon\}$ from the origin. For any complex number w not contained in $f(U)$, since $g(z) := (f(z) - w)^{-1}$ is holomorphic on U, it follows that

$$\frac{1}{|w|} = |g(0)| \leq \sup_{|z|=\epsilon} |g(z)| \leq \frac{1}{||w| - \delta|},$$

hence $|w| \geq \delta/2$. Contrapositively $\{|w| < \delta/2\} \subseteq f(U)$. *Another proof.* By the identity theorem the zeros of nonconstant holomorphic functions of one complex variable are discrete, so $f(z) = z^m F(z)$ for some nonnegative integer m and holomorphic functions F with $F(0) \neq 0$. By continuity F is nowhere zero on some sufficiently small disk around origin, so finding a holomorphic m-th root of $F(z)$ we get $f(z) = G(z)^m$ where $G(z)$ is holomorphic, $G(0) = 0$ and $G'(0) \neq 0$. The inverse function theorem [Apo, 13.16][Rud-PMA, 9.24] implies that $G(z)$ is a local diffeomorphism, so some neighborhood D' of the origin is taken by G. $F(D')$ is then a neighborhood of the origin. ■

Theorem 11 (Maximum modulus principle)**.** *(1) The modulus of a nonconstant holomorphic function on a connected open of the complex plane has no local maximum. (2) If f is a holomorphic on bounded open $D \subseteq \mathbb{C}$ and continuous up to boundary then the global maximum of the modulus of f in $\overline{D}$ is attained at the boundary.*

Proof. Open mapping theorem immediately gives (1), and (2) is immediate from (1). Here is an independent proof for (1). Let the modulus of a nonconstant holomorphic function f on domain $D \subseteq \mathbb{C}$ attain a local maximum at $p \in D$, so there exists $\epsilon > 0$ such that $|f(z)| \leq |f(p)|$ for $|z - p| \leq \epsilon$. Putting the trivial case $f(p) = 0$ aside, by the Cauchy integral formula we have

$$\int_0^{2\pi} \left(1 - \frac{f(p + \epsilon \exp(\sqrt{-1}\theta))}{f(p)}\right) d\theta = 0.$$

The real part of the integrand is ≥ 0, and $= 0$ happens exactly when the integrand is zero. This forces the continuous integrand to be identically zero. *Another argument.*

There should be a point q with $r := |q - p| < \epsilon$ such that $|f(q)| < |f(p)|$, because otherwise $|f|$ becomes constant on $|z - p| < \epsilon$ and then the Cauchy-Riemann equation forces f to be constant on $|z - p| < \epsilon$ and hence constant on whole D by the identity theorem. By continuity there is a nonempty open subset of $0 \le \theta \le 2\pi$ such that $|f(p + r\exp(\sqrt{-1}\theta))| < |f(p)|$, so

$$\int_0^{2\pi} \left|f(p + r\exp(\sqrt{-1}\theta))\right| d\theta/2\pi < |f(p)|.$$

This contradicts the estimation $|f(p)| \le \int_0^{2\pi} |f(p + r\exp(\sqrt{-1}\theta))| d\theta/2\pi$ coming from the Cauchy integral formula. ∎

Exercise: (1) Show that if the modulus of a nonconstant holomorphic function on a connected open of the complex plane has a local minimum at a point then the function vanishes at that point. (2) Show that if a function is holomorphic on a bounded open subset of the plane, continuous up to boundary and constant on the boundary then the function is identically constant.

2.2 Runge's approximation theorem with applications

Theorem 12 (Runge's approximation theorem). *For every open $D \subseteq \mathbb{C}$ and compact $K \subseteq D$ the followings are equivalent:*

(1; topological condition) K adds no **hole** *to D, in the sense that $D \setminus K$ has no component compactly supported in D.*

(2; functional analysis condition) $\mathcal{O}(D)$ is **dense** *in $\mathcal{O}(K)$, in the sense that every holomorphic function on K can be uniformly approximated on K by holomorphic functions on D.*

(3; function theory condition) K is **holomorphically convex** *in D, in the sense that for any $z \in D \setminus K$ there exists some holomorphic function f on D such that $|f(z)| > \sup_K |f|$.*

In case any of these equivalent conditions hold, (D, K) is called a **Runge pair**.

Example: (1) $\{|z| \le 1\}$ has no hole itself (it is simply connected), so adds no hole to whatever open which contains it. (2) $\{|z| = 1\}$ adds a hole to $\{|z| < 2\}$, but not to $\{0 < |z| < 2\}$.

Remark 13. (1) This is the version of Runge's approximation theorem that we will need for the rest of this chapter. The more famous version says: *For any compact $K \subseteq \mathbb{C}$ and any $P \subseteq \mathbb{C}$ which contains at least one point in each bounded component of $\mathbb{C} \setminus K$, every holomorphic function on K can be uniformly approximated on K by rational functions with poles in P.* This latter version can be proved with exactly the same techniques [Rud-RCA, 13.6]. An elementary proof is given in [Sar, page 115]. (2) By Theorem 8 holomorphic functions on $\mathbb{C}$ can be uniformly approximated on compacts by polynomials. Therefore for the special case $D = \mathbb{C}$ one deduces from Theorem 12 that: *For $K \subseteq \mathbb{C}$ compact, $\mathbb{C} \setminus K$ is connected if and only if every holomorphic function on K can be uniformly approximated on K by polynomials.* This is another useful version of Runge's theorem [Rud-RCA, 13.7, 13.8]. ∎

Proof of Theorem 12. (2 or 3⇒1) Assume (1) fails. Then $D \setminus K$ has a component O which is compactly supported in D. Note that $\partial O \subseteq K$. By the maximum principle

$$\|f\|_{\overline{O}} \le \|f\|_K, \quad \forall f \in \mathcal{O}(D), \tag{2.2}$$

which contradicts (3) for any $z \in O$. Now let (2) hold. Fix $\zeta \in O$. Applying (2) to $f(z) := (z-\zeta)^{-1} \in \mathcal{O}(K)$ gives a sequence f_n of holomorphic functions on D which converge uniformly on K to f. Applying (2.2) to $f_n - f_m$ shows that f_n converges uniformly on $\overline{O}$ to some limit function F. Note that F is holomorphic on O, continuous on $\overline{O}$, and equals f on ∂O namely $(z-\zeta)F(z) = 1$ on ∂O. This latter identity persists on $\overline{O}$ by the maximum principle applied to $z \mapsto (z-\zeta)F(z) - 1$. This gives a contradiction when $z = \zeta$.

(1⇒2) Fix an arbitrary $f \in \mathcal{O}(K)$. Consider f as an element of the space $C(K)$ of continuous functions on K equipped with uniform norm. Since the dual of $C(K)$ is given by (regular Borel) measures, according to Hahn-Banach theorem [Rud-RCA, 5.19][Bre, 1.8] we need to check that any measure μ on K which is orthogonal to $\mathcal{O}(D)$ (namely $\int g d\mu = 0$ for all $g \in \mathcal{O}(D)$) is also orthogonal to f. Let ψ be a smooth bump function compactly supported on some neighborhood of K where f is holomorphic on, and ψ equals 1 on some neighborhood of K. By Cauchy-Pompeiu $f(z) = (2\pi\sqrt{-1})^{-1} \int f(\zeta)\psi_{\overline{\zeta}}(\zeta)(\zeta - z)^{-1} d\zeta \wedge d\overline{\zeta}$ for every $z \in K$, so applying Fubini's theorem:

$$\int f(z) d\mu(z) = \frac{1}{2\pi\sqrt{-1}} \int f(\zeta)\psi_{\overline{\zeta}}(\zeta)\varphi(\zeta) d\zeta \wedge d\overline{\zeta},$$

where $\varphi(\zeta) = \int (\zeta - z)^{-1} d\mu(z)$. It suffices to show that the function φ defined on $\mathbb{C} \setminus K$ is identically zero. Fix an arbitrary point $z \in \mathbb{C} \setminus K$. Clearly φ is holomorphic. It also vanishes on the unbounded component of $\mathbb{C} \setminus K$ because $(\zeta - z)^{-1}$ is a uniform sum of monomials $z^n \in \mathcal{O}(D)$ on $|\zeta| \ge 2\sup_{w \in K} |w|$. Let O be an arbitrary bounded component of $\mathbb{C} \setminus K$. Because of our topological assumption O intersects $\mathbb{C} \setminus D$, so let ζ_0 be a point in the intersection. Then $\partial^k \varphi / \partial \zeta^k (\zeta_0) = (-1)^k k! \int (\zeta_0 - z)^{-k-1} d\mu(z)$ vanishes because $(\zeta_0 - z)^{-k-1}$ is holomorphic on D. By the identity theorem φ vanishes on whole O.

(1 and 2⇒3) Fix $z \in D \setminus K$. Choose a closed disc L centered at z with $L \subseteq D \setminus K$. The components of $D \setminus (K \cup L)$ are the same as those of $D \setminus K$ apart from the fact that L has been removed from exactly one of them. Therefore $K \cup L$ adds no hole to D. Applying (2) to the function which is 0 in a neighborhood of K and is 1 in a neighborhood of L gives $f \in \mathcal{O}(D)$ such that $\|f\|_K < 2^{-1}$ and $\|f - 1\|_L < 2^{-1}$. This f satisfies (3). ■

Here is a justification for the appellation "holomorphically convex". Recall that the convex hull of a compact $K \subseteq \mathbb{R}^n$ is defined as the smallest closed convex set containing it, or equivalently the intersection of all half-spaces containing K [Rud-FA, 3.4][Bre, 1.7][Hör-Conv, 2.1.11]. Therefore in function theory terms the convex hull of K equals $\hat{K}_{L(\mathbb{R}^n)} := \{x \in \mathbb{R}^n : f(x) \le \sup_K f, \forall f \in L(\mathbb{R}^n)\}$, where $L(\mathbb{R}^n)$ denotes the space of all $\mathbb{R}$-multilinear functions $\mathbb{R}^n \to \mathbb{R}$. Now note that the third condition in Theorem 12 exactly says $K = \hat{K}_{\mathcal{O}(D)}$ where $\hat{K}_{\mathcal{O}(D)} = \{z \in D : |f(z)| \le \sup_K |f|, \forall f \in \mathcal{O}(D)\}$ is called the **holomorphically convex hull** of K in D.

Exercise: Show that an open $U \subseteq \mathbb{R}^n$ is convex if and only if for every compact $K \subseteq U$ the set $\hat{K}_{L(U)} := \{x \in U : f(x) \le \sup_K f, \forall f \in L(U)\}$ is compactly supported in U.

The next theorem gathers the basic properties of holomorphically convex hulls in one complex variable.

Theorem 14. *For any open $D \subseteq \mathbb{C}$ and compact $K \subseteq D$ we have:*

(1) $\hat{K}_{\mathcal{O}(D)}$ is bounded, closed in D, contains K and contained in the convex hull of K.

(2) $\operatorname{dist}(K, \mathbb{C} \setminus D) = \operatorname{dist}(\hat{K}_{\mathcal{O}(D)}, \mathbb{C} \setminus D)$. (Here dist *stands for the Euclidean distance.) Furthermore, $\hat{K}_{\mathcal{O}(D)}$ is compact.*

(3) $(D, \hat{K}_{\mathcal{O}(D)})$ is a Runge pair.

(4) There exists an exhaustion of D by compacts K_j, each (D, K_j) a Runge pair. (Exhaustion means $D = \bigcup K_j$, K_j contained in the interior of K_{j+1}.)

(5) $\hat{K}_{\mathcal{O}(D)}$ is obtained from K by filling the holes K add to D, more precisely, $\hat{K}_{\mathcal{O}(D)}$ is the union of K and the components of $D \setminus K$ which are compactly supported in D.

Proof. (1) Testing the definition of $\hat{K}$ for the coordinate function $z \in \mathcal{O}(D)$ shows that $\hat{K}$ is contained in the smallest closed disk around origin which contains K, so is bounded. $\hat{K}$ is closed in D because if a sequence p_j of points of $\hat{K}$ converges to $p \in D$, by the continuity of holomorphic functions $p \in \hat{K}$. $K \subseteq \hat{K}$ is trivial. Testing the definition of $\hat{K}$ for $\exp(az) \in \mathcal{O}(D)$, $a \in \mathbb{C}$, shows that $\hat{K}$ is contained in the intersection of all half-spaces containing K, namely the usual convex hull of K.

(2) $K \subseteq \hat{K}$ gives $\geq$. For any $\zeta \in \mathbb{C} \setminus D$ the function $z \mapsto (z - \zeta)^{-1}$ is holomorphic on D, so by the very definition of $\hat{K}$ we have $|z - \zeta|^{-1} \leq \operatorname{dist}(\zeta, K)^{-1}$ for any $z \in \hat{K}$. This gives the other direction $\leq$. $\hat{K}$ is closed in $\mathbb{C}^m$ because if a sequence p_j of points of $\hat{K}$ converges to $p \in \mathbb{C}^m$, by the equality we have just proved $p \in D$, and then by the continuity of holomorphic functions $p \in \hat{K}$.

(3) $\hat{\hat{K}} = \hat{K}$ is immediate from the definition of holomorphically convex hulls.

(4) Choose an exhaustion of D by compacts L_j [Fol, 4.39][Lee, A.60]. Set $K_1 := \hat{L}_1$. Assuming that $K_1, \ldots, K_{j-1}$ has been defined, choose some $n_j > j$ such that K_{j-1} is contained in the interior of L_{n_j} and set $K_j := \hat{L}_{n_j}$.

(5) For any component O of $D \setminus K$ compactly supported in D the maximum principle gives $\|f\|_{\overline{O}} \leq \|f\|_K$ for all $f \in \mathcal{O}(D)$, so $O \subseteq \hat{K}$. Therefore the union L of K with all such components is contained in K. On the other hand, L is a compact adding no hole to D, so by the Runge's approximation theorem $L = \hat{L}$, and $\hat{L}$ contains $\hat{K}$ just because L contains K. ■

2.2.1 Application I: d-bar problem in one complex variable

A fundamental problem in complex analysis is the solvability of the d-bar problem $\overline{\partial} u = f$. Here is our first attack on this problem.

Theorem 15 (d-bar problem in one complex variable)**.** *For every C^k, $k \in \{1, \ldots, \infty\}$, function f on open $D \subseteq \mathbb{C}$ there exists a C^k function u on D such that $\partial u / \partial \overline{z} = f$.*

We start by proving a special case:

Lemma 16 (d-bar problem in one complex variable; compactly supported data)**.** *For every C^k, $k \geq 1$, function f compactly supported in open $D \subseteq \mathbb{C}$, the function u defined by*

$$u(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta} \tag{2.3}$$

is a C^k function on D (not necessarily compactly supported) which solves $\partial u / \partial \overline{z} = f$.

Proof. We first treat the special case $D = \mathbb{C}$. The integral defining u is improper and understood in Cauchy's principal value sense, so we should be extra careful applying the d-bar operator to it. Specially, the integration does not commute with the d-bar operator unless z is outside the support of f. In this exceptional case the integral is not improper anymore and $\partial u / \partial \overline{z} = 0$. The trick is to do a change of variables first:

$$u(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f(\zeta + z) d\zeta \wedge d\overline{\zeta}}{\zeta}.$$

Now our integrand and all its derivatives are integrable (because the region of integration is bounded, $\zeta \mapsto f(\zeta + z)$ is smooth, and ζ^{-1} is integrable), so differentiation commutes with integration. Therefore u is smooth and we have:

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial f / \partial \overline{z}(\zeta + z) d\zeta \wedge d\overline{\zeta}}{\zeta} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial f / \partial \overline{\zeta}(\zeta) d\zeta \wedge d\overline{\zeta}}{\zeta - z}.$$

This last expression equals $f(z)$ by Cauchy-Pompeiu integral formula because f is compactly supported. Now we treat the general case of D. Fix $z_0 \in D$, and find a smooth bump function ψ compactly supported on D which equals 1 on some neighborhood of z_0. The partition of unity $1 = \psi + (1 - \psi)$ splits u into two terms:

$$u = u_1 + u_2, \quad u_1(z) = \int_{\mathbb{C}} \frac{\psi(\zeta) f(\zeta) d\zeta \wedge d\overline{\zeta}}{2\pi\sqrt{-1}(\zeta - z)}, \quad u_2(z) = \int_{\mathbb{C}} \frac{(1 - \psi(\zeta)) f(\zeta) d\zeta \wedge d\overline{\zeta}}{2\pi\sqrt{-1}(\zeta - z)}.$$

The integrand of the first integral is compactly supported, so the special case $D = \mathbb{C}$ we have just proved implies $\partial u_1 / \partial \overline{z}(z_0) = \psi(z_0) f(z_0) = f(z_0)$. On the other hand, z_0 is outside the support of the integrand of the second integral, so the explanations in the first paragraph of this proof implies $\partial u_2 / \partial \overline{z}(z_0) = 0$. Therefore $\partial u / \partial \overline{z}(z_0) = f(z_0)$. ∎

If μ is not compactly supported then the integral in formula (2.3) might not exist. In the following we use Runge's approximation to solve the d-bar problem in general.

Proof of Theorem 15. Find an exhaustion of D by compact subsets $K_1, K_2, \ldots$ such that each (D, K_j) is a Runge pair (Theorem 14). Choose smooth bump function ψ_j compactly supported in D which equals 1 in some neighborhood of K_j, and form the partition of unity $\varphi_1 := \psi_1$, $\varphi_j := \psi_j - \psi_{j-1}$ for $j > 1$. The partition of unity $1 = \sum \varphi_j$ splits our initial arbitrary data f into compactly supported pieces $f = \sum \varphi_j f$, so by Lemma 16 we have solution $u_j \in C^\infty(\mathbb{R}^2)$ for each piece of data: $\partial u_j / \partial \overline{z} = \varphi_j f$. Formally, $u := \sum u_j$ solves the d-bar problem: $\partial u / \partial \overline{z} = \sum \varphi_j f = f$, however the summation might not converge. The idea is to use Runge's approximation to modify summands u_j by adding holomorphic functions in order to make the sum converge. Note that u_j is holomorphic on K_{j-1} because φ_j vanishes in some neighborhood of K_{j-1}. By Runge's approximation find v_j holomorphic on D with $\|u_j - v_j\|_{K_{j-1}} < 2^{-j}$. We assert that $u := \sum(u_j - v_j)$ solves the d-bar problem. To prove this fix arbitrary compact $K \subseteq D$, and find positive integer l such that K is contained in K_j for all $j \geq l$. The summation from $j = l$ to $j = \infty$ consists of terms which are holomorphic on K_{l-1}, and the sum converges uniformly on K_{l-1} (because of the uniform estimate $\|u_j - v_j\|_{K_{j-1}} < 2^{-j}$), so by Theorem 6 it converges to a function which is holomorphic in the interior of K_{l-1}. Therefore $u \in C^\infty(D)$, and we have $\partial u/\partial \overline{z} = \sum \varphi_j f = f$. ■

Remark 17. Ehrenpreis and Malgrange proved a vast generalization of Lemma 16: *For every polynomial $P \in \mathbb{C}[x_1, \ldots, x_n] \setminus \{0\}$ the equation $P(\partial/\partial x_1, \ldots, \partial/\partial x_n)u = f$ has a smooth solution u on $\mathbb{R}^n$ if f is smooth and compactly supported in $\mathbb{R}^n$.* There are many proofs of this result in the literature [Fol-PDE, 1.54][Rud-FA, page 210–215][Wag]. ■

Remark 18. For a finite and compactly supported complex Borel measure μ in the complex plane, the expression $\mathcal{C}[\mu](z) := \int_{\mathbb{C}}(z-\zeta)^{-1}d\mu(\zeta)$ is called the **Cauchy transform** of μ. [Bell] proves the fundamental results of single variable complex analysis (including the Riemann mapping theorem) using Cauchy transform. Deep research-level investigations about the Cauchy transform can be found in [Tol, Paj]. ■

2.2.2 Application II: Cousin problems in one complex variable

Cousin problems are about meromorphic functions. We start by giving two definitions for such functions. The first one which is more abstract has the advantage of that it also works for several complex variables.

For each point z in the complex plane let $\mathcal{O}_z$ be the set of holomorphic functions at z (namely holomorphic on $\{z\}$), modulo the equivalence relation $\sim$ given by $f \sim g$ if and only if $f = g$ on some neighborhood of z. The equivalence class associated to function f which is holomorphic at z is denoted by f_z, and is called the **germ** of f at z. $\mathcal{O}_z$ is the pointwise (better term: "stalkwise" or "fiberwise") model for holomorphic functions. $\mathcal{O}_z$ is an integral domain by the identity theorem, and its field of fraction is denoted by $\mathcal{M}_z$. $\mathcal{M}_z$ is the stalkwise model for meromorphic functions which we now define. A **meromorphic function** on open $D \subseteq \mathbb{C}$ consists of any of the following equivalent set of data:

1. A function $\varphi : D \to \bigcup_{z \in \mathbb{C}} \mathcal{M}_z$ with $\varphi(z) \in \mathcal{M}_z$ for all $z \in D$ which is locally the ratio of holomorphic functions, in the sense that for every $z \in D$ there exist holomorphic functions f and g on some neighborhood of z such that $\varphi(\zeta) = f_\zeta / g_\zeta$ on that neighborhood.

2. A function F holomorphic on D minus a discrete set of points P called **poles** such that $1/F$ after declaring to be zero at the poles is holomorphic on P.

The correspondence between φ and F is as follows. Having F set $\varphi(z) = F_z$ for $z \in D \setminus P$, and $\varphi(z) = 1/(1/F)_z$ for $z \in P$. Now suppose φ is given. For $z \in D$ let $\varphi(z)$ be given by the ratio f_z/g_z of germs of holomorphic functions. Note that g is not identically zero, because division by zero is not allowed in the definition of the field of fractions $\mathcal{M}_z$. If f is identically zero set $F(z) := 0$, otherwise by the identity theorem write $f(\zeta) = (\zeta - z)^k f_1(\zeta)$ and $g(\zeta) = (\zeta - z)^l g_1(\zeta)$ where f_1 and g_1 are holomorphic functions at z with $f_1(z) \neq 0 \neq g_1(z)$, and set:

$$F(z) := \begin{cases} 0, & k > l \\ \infty, & k < l \\ f_1(z)/g_1(z), & k = l \end{cases}.$$

The set of meromorphic functions on D is denoted by $\mathcal{M}(D)$.

Theorem 19 (Mittag-Leffler)**.** *Let D be an open subset of the complex plane.*

(1; classical formulation) Let z_j be a sequence of distinct points in D which does not accumulate at any point of D. For each j let f_j be a function meromorphic in a neighborhood of z_j with only pole at z_j. Then there exists a meromorphic function f on D with only poles at z_j such that $f - f_j$ is holomorphic in a neighborhood of z_j.

(2; another classical formulation) Let $D = \bigcup_{j\geq 1} D_j$ be an open covering. For each j let f_j a function meromorphic on D_j such that each $f_j - f_k$ is holomorphic on $D_j \cap D_k$. Then there exists a meromorphic function f on D such that each $f - f_j$ is holomorphic on D_j.

(3; cohomological formulation) The first Čech cohomology $H^1(\{D_j\}, \mathcal{O})$ vanishes, in the sense that for any open covering $D = \bigcup_{j\geq 1} D_j$ and any data $(g_{jk} \in \mathcal{O}(D_j \cap D_k))_{j,k\geq 1}$ satisfying $g_{jk} + g_{kl} + g_{lk} = 0$ on $D_j \cap D_k \cap D_l$ there exists $(h_j \in \mathcal{O}(D_j))_{j\geq 1}$ such that $g_{jk} = h_k - h_j$ on $D_j \cap D_k$. In words: Every Čech 1-cocyle is a 1-coboundary.

Proof. It is straightforward to show that (1) and (2) are equivalent. (3) readily implies (2), because setting $g_{jk} = f_j - f_k$, we can find $h_j \in \mathcal{O}(D_j)$ such that $f_j - f_k = h_k - h_j$. Then the function f defined on D by setting $f := f_j + h_j$ on each D_j is well-defined and has the required property asked in (2). We give two proofs for (1) and one for (3) based on Runge's approximation theorem and the d-bar problem.

(1; first proof) Each f_j has the unique representation $\sum_{k\geq m_j} a_{jk}(z - z_j)^k$ with integer m_j and complex numbers a_{jk}. The idea is to use the Runge's approximation theorem to find holomorphic functions φ_j on D such that $f := \sum_{j\geq 1}(f_j - \varphi_j)$ satisfies the desired properties. By Theorem 14.(4) find an exhaustion of D by compacts K_j such that each (D, K_j) is a Runge pair. Since z_j do not accumulate in D, after passing to a subsequence of K_j, one can assume that K_j does not contain any of $z_j, z_{j+1}, \ldots$. Therefore f_j is holomorphic on K_j, so Runge's approximation gives $\varphi_j \in \mathcal{O}(D)$ such that $\|f_j - \varphi_j\|_{K_j} < 2^{-j}$. Then each tail $\sum_{j\geq k}(f_j - \varphi_j)$ of f converges uniformly on K_k to a function which is holomorphic on the interior of K_k. Therefore f has the desired properties.

(1; second proof) Choose neighborhood U_j of z_j contained in the domain of definition of f_j such that $U_j \cap U_k = \emptyset$ for all j, k. Let ψ_j be a smooth bump function compactly supported in U_j which equals 1 on some neighborhood of z_j. The idea is to find a modification of $\sum \psi_j f_j$ which works as our desired function f. The expression $g := \sum \partial(\psi_j f_j)/\partial\bar{z}$ initially defined on $\bigcup U_j \setminus \{z_j\}$, after extension by zero to whole D, is a smooth function. Note that $g = 0$ on some neighborhood of each z_j. By Theorem 15 find u satisfying $\partial u/\partial\bar{z} = g$. Then $f := -u + \sum \psi_j f_j$ is our desired function.

(3) Choose a smooth partition of unity ψ_j subordinate to the covering $D = \bigcup D_j$ [Lee, 2.23], namely ψ_j is a smooth function on D supported in D_j, each point of D has a neighborhood which all but finitely many of ψ_j vanish, $0 \leq \psi_j \leq 1$, and $\sum \psi_j = 1$ on D. If the conclusion of the theorem holds then multiplying $g_{jk} = g_k - g_j$ by ψ_j and adding over j we get $g_k = h_k + u$ where $h_k = \sum \psi_j g_{jk}$ and $u = \sum \psi_j g_j$. Set $h_k := \sum_j \psi_j g_{jk}$. Then h_k is smooth on D_k. Using $g_{jk} + g_{kl} + g_{lj} = 0$ we have $h_k - h_l = g_{lk}$, so $\partial h_k/\partial \overline{z} = \partial h_l/\partial \overline{z}$ on $D_j \cap D_l$, hence setting $H := \partial h_k/\partial \overline{z}$ on D_k we get a well-defined smooth function H on D. Choosing any solution of $\partial u/\partial \overline{z} = -H$ by Theorem 15 we are done by setting $g_k := h_k + u$. ∎

This proof is nonconstructive because Hahn-Banach extension theorem (so the Zorn's lemma) is used in the proof of the underlying theorems 12, 15. An easy constructive proof in the special case $D = \mathbb{C}$ is given in [Ahl, page 188].

Theorem 20 (Weierstrass' infinite product theorem)**.** *Let D be an open subset of the complex plane.*

(1; Classical formulation) Let $\{z_j\}_{j\geq 1}$ be a sequence of distinct points in the open set $D \subseteq \mathbb{C}$ which does not accumulate at any point of D. For each j let n_j be an arbitrary integer. (n_j is to be thought of as the order zeros if it is positive and the order of poles if it is negative.) Then there is a meromorphic function f on D such that f is holomorphic and nonzero except at the points z_j, and $f(z)/(z - z_j)^{n_j}$ is holomorphic and nonzero in a neighborhood of z_j.

(2; Another classical formulation) Let $D = \bigcup_{j\geq 1} D_j$ be an open covering. For each j let f_j be a function meromorphic on D_j such that each f_j/f_k is holomorphic on $D_j \cap D_k$. Then there exists a nonzero meromorphic function f on D such that each f/f_j is holomorphic on D_j.

(3; Cohomological formulation) The first Čech cohomology $H^1(\{D_j\}, \mathcal{O})$ vanishes, in the sense that for any open covering $D = \bigcup_{j\geq 1} D_j$ and any data[2] $(g_{jk} \in \mathcal{O}^*(D_j \cap D_k))_{j,k\geq 1}$ satisfying $g_{jk}g_{kl}g_{lk} = 1$ on $D_j \cap D_k \cap D_l$ there exists $(h_j \in \mathcal{O}^*(D_j))_{j\geq 1}$ such that $g_{jk} = h_k/h_j$ on $D_j \cap D_k$.

Proof. (1) Exhaust D by compacts K_j such that each (D, K_j) is a Runge pair. We are done by constructing rational function f_j having the desired poles and zeros (counting multiplicities) in K_j and holomorphic functions g_j in D such that the following estimation holds:

$$\left\| f_{j+1} f_j^{-1} e^{g_j} - 1 \right\|_{K_j} < 2^{-j}, \tag{2.4}$$

because then

$$f := f_1 \prod_{j\geq 1} f_{j+1} f_j^{-1} e^{g_j},$$

works as our desired function. Start by choosing any rational function f_1 having the desired poles and zeros in K_1. Assume that $f_1, \ldots, f_j, g_1, \ldots, g_{j-1}$ have been constructed. Choose a rational function F_{j+1} having the desired poles and zeros in K_{j+1}. Note that $F_{j+1} f_j^{-1}$ is a nowhere zero holomorphic function on a neighborhood U_j on K_j. If U_j is

[2] $\mathcal{O}^*(U)$ for an open $U \subseteq \mathbb{C}$ stands for the set of nowhere-zero holomorphic functions on U.

simply connected then we are done by applying Runge's approximation theorem to a well-defined branch of $\log\left(F_{j+1}f_j^{-1}\right)$. Without simply connectedness assumption we proceed as follows. Assume $F_{j+1}f_j^{-1} = A\prod(z-p_k)^{n_k}$ where $A \in \mathbb{C}$, $p_k \in D \setminus K_j$, $p_k \in \mathbb{Z}$. Since K_j adds no hole to D we can find for each k a point $p'_k \in D \setminus K_{j+1}$ such p'_k and p_k belong to the same component of $D \setminus K$. This makes $\log((z-p_k)/(z-p'_k))$ well-defined [Ahl, page 116]. Therefore correcting F_{j+1} by $f_{j+1} := F_{j+1}\prod(z-p_k)^{-n_k}$ gives a well-defined branch

$$\log(f_{j+1}f_j^{-1}) = \log A + \sum m_k \log((z-p_k)/(z-p'_k)),$$

and we are done by repeating the procedure for simply connectedness case. ■

An immediate consequence of this theorem is: *Meromorphic functions on an open $D \subseteq \mathbb{C}$ are ratios of holomorphic functions on D.* This statement is also true for the so-called pseudoconvex opens $D \subseteq \mathbb{C}^m$ (or more generally Stein spaces) [Hör, 7.4.6], but never holds for compact Riemann surfaces (because every compact Riemann surface supports nonconstant meromorphic functions [Don-RS, page 114] but not nonconstant holomorphic functions according to the maximum modulus principle).

Theorem 21 (Interpolation problem)**.** *For every open $D \subseteq \mathbb{C}$, every sequence p_j of distinct points in D which does not accumulate at any point of D and every sequence c_j of complex numbers there exists a holomorphic function f on D such that $f(p_j) = c_j$ for all j.*

Proof. By Weierstrass theorem find a holomorphic function g on D which has p_j as zeros of multiplicity one. By Mittag-Leffler theorem find a meromorphic function h on D whose principle part at p_j is $c_j/((z-p_j)g'(p_j))$. Then $f := gh$ works as our desired function. ■

Exercise: Prove the following generalization of the interpolation theorem: *For every open $D \subseteq \mathbb{C}$, sequence p_j of distinct points in D which does not accumulate at any point of D, nonnegative integers m_j and complex numbers c_{jk}, $k = 0, \ldots, m_j$, there exists a holomorphic function f on D such that $f^{(k)}(p_j) = c_{jk}$ for all $j = 1, 2, \ldots$ and $k = 0, \ldots, m_j$.* (*Hint.* Imitate the proof of Theorem 21. g has p_j as zero of multiplicity $m_j + 1$. This time use principal parts $\sum_{1 \le l \le m_j+1} a_{jl}(z-p_j)^{-l}$, where a_{jl} are determined such that gh has the desired Taylor expansion $\sum_{0 \le k \le m_j} c_{jk}/k!\ (z-p_j)^k + o((z-p_j)^{m_j})$ around p_j.)

Exercise: Let D be an open subset of the complex plane. Prove that every finitely generated ideal of the algebra $\mathcal{O}(D)$ of holomorphic functions on D is principal in the sense that for every positive integer n and $g_j \in \mathcal{O}(D)$, $j = 1, \ldots, n$, there exists $g, f_j, G_j \in \mathcal{O}(D)$ such that $g_j = gG_j$ and $g = \sum f_j g_j$. (*Hint.* Without loss of generality assume that D is connected, and no g_j is identically zero. By Weierstrass theorem find $g \in \mathcal{O}(D)$ such that for every $z \in D$ the order of z as a zero of g equals the minimum over j of the order of z as a zero of g_j. Then $G_j := g_j/g$ can be seen as holomorphic functions on D which have no common zero. By induction on n and using Weierstrass theorem and the generalization of the interpolation theorem proved in the previous exercise show that there exist $f_j \in \mathcal{O}(D)$ such that $1 = \sum f_j G_j$.) More in this direction can be found in [BG, section 3.5] .

Chapter 3

Holomorphic functions of several complex variables

References: [Hör, chapter 2][Ran, chapter 2].

We start by proving the first properties of holomorphic functions of several complex variables that can be deduced from iterated use of single variable Cauchy integral formula. Then we give two proofs for Hartogs $G \setminus K$ extension theorem (page 8), first via the d-bar problem for compactly supported data, and second by a certain higher dimensional generalization of Cauchy integral formula discovered by Bochner and Martinelli. The rest of the chapter is devoted to a thorough study of the first two most fundamental notions of the function theory of SCV: domains of holomorphy and pseudoconvexity.

Our initial definition for holomorphic functions is the third one in page 7: C^1 functions satisfying Cauchy-Riemann equations. The equivalence between this definition and others is discussed in Remark 40. For a development of the subject starting from the power series definition refer [Ohs, Nar][Die, volume I, chapter 9].

3.1 First properties of holomorphic functions

Theorem 22 (First properties of holomorphic functions). *(1; Cauchy integral formula for polydiscs) Let $P \subseteq \mathbb{C}^m$ be the open polydisc $\{|\zeta_1 - a_1| < r_1, \ldots, |\zeta_m - a_m| < r_m\}$, $r_j > 0$, $a_j \in \mathbb{C}$. For any $f \in C(\overline{P}) \cap \mathcal{O}(P)$ and $z = (z_1, \ldots, z_m) \in P$ we have*

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^m} \int_{|\zeta_1 - a_1| = r_1, \ldots, |\zeta_m - a_m| = r_m} \frac{f(\zeta_1, \ldots, \zeta_m)}{(\zeta_1 - z_1) \cdots (\zeta_m - z_m)} d\zeta_1 \wedge \cdots \wedge d\zeta_m.$$

In other words we are iteratively taking counterclockwise complex line integrals over circles $\{|\zeta_j - a_j| = r_j\}$.

(2) Homomorphic functions are smooth. Complex derivatives of holomorphic functions are holomorphic.

(3; Bergman's estimate) Let f be a holomorphic function on open $D \subseteq \mathbb{C}^m$. For any compact $K \subseteq D$, open $K \subseteq D' \subseteq D$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ there exists

constant $C_\alpha = C_\alpha(K, D', \alpha)$ *such that*

$$\|\partial^\alpha f/\partial z^\alpha\|_K \leq C_\alpha \|f\|_{L^1(D')},$$

where $\partial^\alpha f/\partial z^\alpha = \partial^{\alpha_1}/\partial z_1^{\alpha_1} \cdots \partial^{\alpha_m}/\partial z_m^{\alpha_m} f$.

(4; Weierstrass) If a sequence of holomorphic functions on open $D \subseteq \mathbb{C}^m$ *converges uniformly on compact sets of* D *then the limit function is itself holomorphic. Furthermore, for any multi-index* $\alpha \in \mathbb{N}^m$ *the sequence of* α*-th order complex derivatives converges uniformly on compacts to the* α*-th order complex derivative of the limit function.*

(5; Montel's compactness theorem) A sequence of holomorphic functions on open $D \subseteq \mathbb{C}^m$ *which is locally equibounded contains a subsequence that converges uniformly on compact subsets* D.

(6; Power series representation and Cauchy's estimate) Every function f *holomorphic on the open polydisk* $D = \{|\zeta_1| < R_1, \ldots, |\zeta_m| < R_m\}$ *has a unique power series representation* $f(z) = \sum_{\alpha\in\mathbb{N}^m} a_\alpha z^\alpha$ *which* **converges normally** *in the sense that that* $\sum \|a_\alpha z^\alpha\|_K$ *converges for any compact* $K \subseteq D$. *Furthermore, the Taylor coefficients* a_α *satisfy*

$$a_\alpha = \frac{\partial^\alpha f/\partial z^\alpha(0)}{\alpha!} = \frac{1}{(2\pi\sqrt{-1})^m}\int_{|\zeta_1|=r_1,\ldots,|\zeta_m|=r_m} \frac{f(\zeta_1,\ldots,\zeta_m)}{\zeta_1^{\alpha_1+1}\cdots\zeta_m^{\alpha_m+1}} d\zeta_1\wedge\cdots\wedge d\zeta_m$$

$$= \frac{\prod r_j^{-\alpha_j}}{(2\pi)^m}\int_{[0,2\pi]^m} f\left(r_1 e^{\sqrt{-1}\theta_1}, \ldots, r_m e^{\sqrt{-1}\theta_m}\right) e^{-\sqrt{-1}\sum\alpha_j\theta_j} d\theta_1\cdots d\theta_m,$$

$$|a_\alpha| \leq r_1^{-\alpha_1}\cdots r_m^{-\alpha_m} \sup_{|z_1|=r_1,\ldots,|z_m|=r_m} |f(z)|,$$

for every $0 < r_1 < R_1, \ldots, 0 < r_m < R_m$.

(7; Identity theorem) Let f *and* g *be two holomorphic functions on domain* $D \subseteq \mathbb{C}^m$. *Then* f *equals* g *everywhere on* D *if any of the following conditions holds: (i)* f *along with all its higher complex derivatives match the corresponding values of* g *at some point of* D*; (ii)* f *and* g *agree on some nonempty open subset of* D.

(8; Maximum modulus principle) The modulus of a nonconstant holomorphic function on a domain of $\mathbb{C}^m$ *has no local maximum. If a function* f *is holomorphic on a domain of* $\mathbb{C}^m$ *and is continuous up to boundary then the global maximum of the modulus of* f *is attained at the boundary.*

(9; Open mapping theorem) Nonconstant holomorphic functions on domains of $\mathbb{C}^m$ *map open subsets to open ones.*

Proof. (1) Apply the single variable Cauchy integral formula m times to a slightly smaller polydisc, and then take limit.

(2,4,5,7,8) The proof of the single variable case works.

(3) Cover K by finitely many open polydiscs contained in D', and then apply the corresponding single variable result m times. *Another argument.* Fix $z \in K$. For any open polydisc P_R of multi-radii R centered at z and contained in D', writing Cauchy integral formula for $f(z)$ on P, applying $\partial^\alpha/\partial z^\alpha$, and doing the trivial estimation gives

$$|f^{(\alpha)}(z)| \leq (2\pi)^{-m}\alpha! R^{-\alpha}\int_{\partial_0 P_R} |f| d\theta_1\cdots d\theta_m,$$

where $\partial_0 P_R$ denotes the Cartesian product of the boundary circles of product factors of P_R. On the other hand

$$\begin{aligned}\|f\|_{L^1(D')} \geq \|f\|_{L^1(P_R)} &= \int_{r\in\prod[0,R_j]}\left(\int_{\partial_0 P_r}|f|d\theta_1\cdots d\theta_m\right) r_1dr_1\cdots r_mdr_m \\ &\geq C_{\alpha,m}\int_{r\in\prod[0,R_j]}|f^{(\alpha)}(z)|r^\alpha r_1dr_1\cdots r_mdr_m = C'_{\alpha,m,R}|f^{(\alpha)}(z)|.\end{aligned}$$

(6) Similar to the proof of the single variable case. This time use geometric series

$$\prod_{1\leq j\leq m}(\zeta - z_j)^{-1} = \sum_{\alpha\in\mathbb{N}^m} z_1^{\alpha_1}\cdots z_m^{\alpha_m}\zeta_1^{-\alpha_1-1}\cdots\zeta_m^{-\alpha_m-1},$$

for $|z_j| < |\zeta_j| = r_j < R_j$.

(9) Assuming a nonconstant function f holomorphic on some open ball U around the origin and that $f(0) = 0$, it suffices to show that some neighborhood of 0 is taken by f. By the identity theorem there exists $p \in U$ such that $f(p) \neq 0$. Consider the single variable holomorphic function $g(\lambda) := f(\lambda p)$ defined on the closed unit disc $|\lambda| \leq 1$. By the single variable version of the open mapping theorem some neighborhood of $g(0) = 0$ is contained in $g(|\lambda| < 1) \subseteq f(U)$. ■

Exercise: If f is a holomorphic function on $\mathbb{C}^m$ such that $|f(z)| \leq C|z|_2^k$ for some positive real C, some positive integer k and every $z \in \mathbb{C}^m$ then f is a polynomial of total degree $\leq k$.

Exercise: Every function f holomorphic on the closed polyannulus $D = \{r_1 \leq |\zeta_1| \leq R_1, \ldots, r_m \leq |\zeta_m| \leq R_m\}$, $0 \leq r_j < R_j$, has a unique Laurent series representation $f(z) = \sum_{\alpha\in\mathbb{Z}^m} a_\alpha z^\alpha$ which converges normally on D. Furthermore

$$a_\alpha = \frac{1}{(2\pi\sqrt{-1})^m}\int_{|\zeta_1|=\rho_1,\ldots,|\zeta_m|=\rho_m}\frac{f(\zeta_1,\ldots,\zeta_m)}{\zeta_1^{\alpha_1+1}\cdots\zeta_m^{\alpha_m+1}}d\zeta_1\wedge\cdots\wedge d\zeta_m,$$

for any $r_j < \rho_j \leq R_j$. Moreover, if $r_j = 0$ for some j then $a_\alpha = 0$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_j < 0$. (*Hint.* Iterate the single variable Laurent expansion.)

Exercise: Show that the normal limit of a sequence of nowhere-zero holomorphic functions on a connected open subset $\mathbb{C}^m$ is either nowhere-zero or identically zero. (*Hint.* First prove the one variable case by the argument principle.)

Exercise: (1) If f is a holomorphic function on the open unit polydisc (respectively ball) of $\mathbb{C}^m$ such that $f(0) = 0$ and $|f| \leq 1$, then $|f(z)| \leq |z|_\infty$ (respectively $|f(z)| \leq |z|_2$). (*Hint.* For a fixed z in the open unit polydisc (respectively ball) apply the maximum principle to the single variable function $\lambda \mapsto f(\lambda z/|z|_\infty)$ defined on the open unit disc of the complex plane.)

3.2 Two proofs for Hartogs extension theorem

In this section we give two proofs for Hartogs $G\setminus K$ extension theorem (page 8). The first is by a d-bar technique, the second by Bochner-Martinelli integral formula. We devote two subsections to develop the fundamental formalism of complex differential geometry in $\mathbb{C}^m$ needed in order to make sense of the d-bar problem as well as to work effectively with Bochner-Martinelli kernels.

3.2.1 Preliminaries: Differential forms in $\mathbb{C}^m$, d-bar problem

Differential forms are generalization of functions, designed by differential geometers to be used for integration on (oriented smooth) manifolds. They have become important in SCV (even for the analysis of opens of $\mathbb{C}^m$, setting aside complex manifolds) because a specific problem formulated in terms of them (the so-called d-bar problem; it is defined at the end of this section.) plays a unifying role in dealing with numerous important problems of SCV, for example holomorphic extensions, interpolations, division, Cousin problems, characterizing domains of holomorphy, etc. Chapter 4 is devoted to the d-bar problem and its applications.

Here we give a quick introduction to differential forms. (Refer [BT][Lee][Rud-PMA, chapter 10] for elaboration.) The familiar calculus expression $\int_a^b f(x)dx$ is the integration of the differential 1-form $f(x)dx$ on the directed open line segment from a to b. The double integral $\int_c^d \int_a^b f(x,y)dxdy$ is the integration of the differential 2-form $f(x,y)dx \wedge dy$ on the oriented rectangle made by the Cartesian product of line segments one from a to b and the other from c to d. Next comes the generalization to higher dimensions.

Let U be an open subset of $\mathbb{R}^n$. The **Grassmann** (or **exterior**) **algebra** G_n is the free $\mathbb{C}$-algebra generated by symbols dx_i, $i = 1, \dots, n$, subject to relations $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for every $i, j \in \{1, \dots, n\}$, where the wedge symbol $\wedge$ denotes the product operation in the algebra. Specially, $dx_i \wedge dx_i = 0$ and $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$ if $k > n$. As a vector space over complex numbers G_n is generated by symbols

$$1,\ dx_1, \dots,\ dx_n,\ dx_1 \wedge dx_2, \dots,\ dx_1 \wedge dx_n, \dots,\ dx_{n-1} \wedge dx_n, \dots,\ dx_1 \wedge \cdots \wedge dx_n,$$

with no relations among them, so is of dimension 2^n.[1] The element $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is denoted by dx_I, $I = (i_1, \dots, i_k)$, and is called the **Grassmann** (or **exterior**) **monomial** associated to the tuple $I = (i_1, \dots, i_k)$. The length of I is $|I| = k$. If $i_1 < \cdots < i_k$ then I is called a **shuffle** of $\{1, \dots, n\}$. We also set $dx_\emptyset = 1$ for the empty shuffle $I = \emptyset$ of length zero. Example: Elements of G_2 have unique representations of the form $\alpha + \beta dx_1 + \gamma dx_2 + \delta dx_1 \wedge dx_2$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and can be added and multiplied together and with complex numbers naturally, for example $\beta dx_1 + \gamma dx_2 + \delta dx_1 \wedge dx_2$ wedged with $\beta' dx_1 + \gamma' dx_2 + \delta' dx_1 \wedge dx_2$ gives $(\beta\gamma' - \gamma\beta')dx_1 \wedge dx_2$. A complex-valued smooth **differential form** ω on U is an element of the tensor product algebra $C^\infty_\bullet(U) := C^\infty(U) \otimes_{\mathbb{C}} G_n$ namely a formal finite sum

$$\omega = \sum \omega_I dx_I,$$

[1] If this seems hard to digest, first convince yourself that the familiar algebra $\mathbb{C}[x_1, \dots, x_n]$ of polynomials in n variables with complex coefficients is the free $\mathbb{C}$-algebra generated by symbols x_i, $i = 1, \dots, n$, subject to relations $x_i x_j = x_j x_i$. As a vector space over $\mathbb{C}$ this algebra of polynomials is generated by monomials (or words) $x_1^{i_1} \cdots x_{i_n}^{i_n}$ with $i_1, \dots, i_n$ ranging on nonnegative integers. Specially, $x_1^0 \cdots x_n^0$ is the empty word and can be denoted by $\emptyset$ or 1.

where the coefficients ω_I are complex-valued smooth function on U. If some ω_I vanishes identically then the corresponding term $\omega_I dx_I$ can be dropped from the sum. The expression $1dx_I$, where the coefficient is the constant function 1, is identified with dx_I. The expression $fdx_\emptyset$, where f is a smooth function, is identified with f. Equality, addition and scalar multiplies of forms are defined naturally. Note that if all ω_I vanish everywhere then ω is the zero element of the algebra. After gathering similar terms and dropping zero ones, the summation $\sum \omega_I dx_I$ can be assumed to be taken over shuffles only; to emphasize that this is the case we put a prime after summation: $\omega = \sum' \omega_I dx^I$. The differential form $\sum' \omega_I dx^I$ is said to be **of degree** k (or a k**-form**) if all I with nonzero ω_I are of length k. The vector space of k-forms is denoted by $C_k^\infty(U)$. Note that forms of degree 0 are nothing but smooth functions. The product operation in the algebra of differential forms is called the **wedge product**; it is given by:

$$\sum \omega_I dx_I \wedge \sum \eta_J dx_J = \sum \omega_I \eta_J dx_I \wedge dx_J.$$

Specially, the wedge of a smooth function f with a form $\omega = \sum \omega_I dx_I$ is $\sum f\omega_I dx_I$, and is denoted by $f\omega$ instead of $f \wedge \omega$. Example: On $\mathbb{C}^4$ the form $fdx_1 \wedge dx_2 + gdx_3 \wedge dx_4$ wedged with itself equals $2fgdx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. The **exterior derivative** of the form $\omega = \sum \omega_I dx_I$ is defined by

$$d\omega = \sum d\omega_I \wedge dx_I \quad \text{where} \quad d\omega_I = \sum_{j=1}^{n} \frac{\partial \omega_I}{\partial x_j} dx_j.$$

Example: On $\mathbb{C}^2$ we have $d(x_1 dx_2) = dx_1 \wedge dx_2$. The **integral** of a top form $\omega = fdx_1 \wedge \cdots \wedge dx_n$ is defined by

$$\int_U \omega = \int_U f dx_1 \cdots dx_n,$$

where $dx_1 \cdots dx_n$ is the Lebesgue measure. Assuming a smooth map $F : V \to \mathbb{R}^n$, $y \mapsto x$, defined on an open $V \subseteq \mathbb{R}^N$ and with values in U, the **pullback** of the differential form $\omega = \sum \omega_I(x) dx_I$ on U along F is the differential form $F^*\omega$ on V obtained from ω by replacing each $\omega_I(x)$ by $\omega_I(x(y))$, and each dx_i (in dx_I) by $\sum_{1\le j\le n} \partial x_i/\partial y_j dy_j$. Specially, the pullback of a smooth function f on U is the composite function $f \circ F$ on V. Example: The pullback of the top form $dx_1 \wedge \cdots \wedge dx_n$ on $\mathbb{R}^n$ along a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto x$, is $dy_1 \wedge \cdots \wedge dy_n$ multiplied by the Jacobian determinant $\partial x/\partial y$. Here are some basic facts:

1. The wedge product is associative and distributive over addition from both sides.
2. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ for any k-form ω and l-form η.
3. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for any k-form ω and any form η.
4. $dd\omega = 0$ for any form ω. In notations: $d^2 = 0$.
5. $F^* d\omega = dF^*\omega$ for any form ω and smooth map F.

6. $G^*F^*\omega = (F \circ G)^*\omega$ for any form ω and smooth maps F, G.

7. $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$ for any forms ω, η and smooth map F.

8. For every bijective smooth map $F : V \to U$ with smooth inverse between U and an open $V \subseteq \mathbb{R}^n$ we have

$$\int_U \omega = \pm \int_V F^*\omega, \tag{3.1}$$

where $\pm$ is chosen depending on whether F is orientation preserving/reversing namely the Jacobian determinant of F is everywhere positive/negative.

Now let $U = D$ be an open subset $\mathbb{R}^{2m} \cong \mathbb{C}^m$. It can be coordinated by real functions $x_1, \ldots, x_{2m}$, or holomorphic and anti-holomorphic ones $z_j = x_j + \sqrt{-1}x_{j+m}$, $\overline{z}_j = x_j - \sqrt{-1}x_{j+m}$. Consider the Grassmann algebra G_{2m}, the free $\mathbb{C}$-algebra generated by symbols dx_i, $i = 1, \ldots, 2m$, subject to relations $dx_i \wedge dx_j = -dx_j \wedge dx_i$. It has basis dx_I with I ranging over shuffles of $\{1, \ldots, 2m\}$. Setting $dz_j := dx_j + \sqrt{-1}dx_{j+m}$, $d\overline{z}_j := dx_j - \sqrt{-1}dx_{j+m}$, one gets another basis

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

abbreviated as $dz_I \wedge d\overline{z}_J$, with I and J ranging over shuffles of $\{1, \ldots, m\}$. (Exercise: Verify this.) A complex-valued smooth **differential form** ω on D is an element of the tensor algebra $C^\infty_\bullet(D) := C^\infty(D) \otimes_{\mathbb{C}} G_{2m}$, so can be uniquely represented by

$$\sum{}' \omega_{I,J} dz_I \wedge d\overline{z}_J,$$

where $\omega_{I,J}$ are smooth functions on D. It is said to be **of type** (p, q) (or a (p, q)**-form**) if the summation, after dropping zero terms, is over $|I| = p$ and $|J| = q$. The vector space of (p, q)-forms is denoted by $C^\infty_{p,q}(U)$. Note that, putting trivialities aside, every form is a unique sum of k-forms, $k = 0, \ldots, 2m$, and every k-form is a unique sum of (p, q)-forms, $p, q = 0, \ldots, m$ such that $p + q = k$.

In Section 1.1 we introduced a natural splitting $d = \partial + \overline{\partial}$ of the exterior derivative acting on functions. Here is the generalization to forms:

$$\begin{aligned} d\underbrace{\sum_{I,J} \omega_{I,J} dz_I \wedge d\overline{z}_J}_{\omega} &= \sum_{I,J} d\omega_{I,J} \wedge dz_I \wedge d\overline{z}_J = \sum_{I,J,k} \left(\frac{\partial \omega_{I,J}}{\partial z_k} dz_k + \frac{\partial \omega_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \right) \wedge dz_I \wedge d\overline{z}_J \\ &= \underbrace{\sum_{I,J,k} \frac{\partial \omega_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J}_{\partial\omega} + \underbrace{\sum_{I,J,k} \frac{\partial \omega_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J}_{\overline{\partial}\omega}. \end{aligned}$$

Here are some useful facts:

1. For a smooth function f on $D \subseteq \mathbb{C}^m$ consider df, ∂f and $\overline{\partial} f$ as functions of $t \in \mathbb{C}^m$ in the following natural way:

$$(\partial f)(t) = \sum \frac{\partial f}{\partial z_j} t_j, \quad (\overline{\partial} f)(t) = \sum \frac{\partial f}{\partial \overline{z}_j} \overline{t_j}, \quad (df)(t) = (\partial f)(t) - (\overline{\partial} f)(t).$$

   Then ∂f is $\mathbb{C}$-linear and $\overline{\partial} f$ is $\mathbb{C}$-antilinear in the sense that:

$$(\partial f)(\sqrt{-1}t) = \sqrt{-1}(\partial f)(t), \quad (\overline{\partial} f)(\sqrt{-1}t) = -\sqrt{-1}(\overline{\partial} f)(t).$$

   Specially, f is holomorphic exactly when df is $\mathbb{C}$-linear.

2. $\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q}\omega \wedge \overline{\partial}\eta$ for any (p,q)-form ω and any form η. The same is true for ∂ instead of $\overline{\partial}$.

3. $\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0$. (*Proof.* Since $d = \partial + \overline{\partial}$ acting on (p,q)-forms splits them respectively into $(p+1,q)$ and $(p,q+1)$ forms it follows that $0 = d^2 = \partial^2 + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}^2$ acting on (p,q)-forms splits them respectively into $(p+2,q)$, $(p+1,q+1)$ and $(p,q+2)$-forms, and we know that forms of different type can not be equal.)

4. A necessary condition on a form ω to be expressed as $\omega = \overline{\partial}\eta$ is that η is **$\overline{\partial}$-closed**, namely $\overline{\partial}\eta = 0$.

5. ∂ and $\overline{\partial}$ commute with the pullback along holomorphic maps. (This is because the pullback of forms along holomorphic maps does not change the type.)

A (p,q)-form ω is called **holomorphic** if $q = 0$ and $\overline{\partial}\omega = 0$. Equivalently, $\omega = \sum \omega_I dz_I$ with all functions ω_I holomorphic.

For any vector space $\mathcal{F}$ of functions (or distributions) we use $\mathcal{F}_{p,q}$ to denote the space of forms of type (p,q) with coefficients in $\mathcal{F}$. Usual choices for $\mathcal{F}$ are: $C^\infty_{p,q}$ (smooth), $L^2_{p,q}$ (Lebesgue square integrable), $L^2_{p,q,\mathrm{loc}}$ (locally square integrable), $W^{2,s}_{p,q}$ (Sobolev with weak differentiability index s), etc.

The **d-bar problem** is about the solvability of the equations $\overline{\partial}u = f$ on an open $D \subseteq \mathbb{C}^m$ (more generally on complex manifolds) with respect to (p,q)-forms u when the data f is a $(p,q+1)$-form satisfying the necessary condition $\overline{\partial}f = 0$. Here are some important questions:

1. Do we have smooth solutions $u \in C^\infty_{p,q}(D)$ for smooth data $f \in C^\infty_{p,q+1}(D)$?[2]

2. Do we have solutions smooth to the boundary $u \in C^\infty_{p,q}(\overline{D})$ for data smooth to the boundary $f \in C^\infty_{p,q+1}(\overline{D})$?[3]

3. Do we have norm-controlled solutions $\|u\| \leq C\|f\|$, more precisely, does the exists a solution operator $S : f \mapsto u$ which is bounded $L^p \to L^p$?[4]

   Exercise: (1) In $\mathbb{R}^n$ the form $dx_1 + \cdots + dx_n$ wedged with itself n times equals 0 if $n \geq 2$. (2) In $\mathbb{C}^m$ the form $dz_1 \wedge d\overline{z}_1 + \cdots + dz_m \wedge d\overline{z}_m$ wedged with itself m times equals $m! dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_m \wedge d\overline{z}_m$.

[2] *Answer:* Yes if D is "pseudoconvex" (Chapter 4).

[3] *Answer:* Yes if D is bounded "strongly pseudoconvex" with smooth boundary (Section 5).

[4] *Answer:* Yes if D is bounded "pseudoconvex" and $p = 2$ (Chapter 4), or if D is bounded "strongly pseudoconvex" with smooth boundary and $1 \leq p \leq \infty$ (Section 5).

3.2.2 d-bar problem with compactly supported data, The first proof

Lemma 23 (d-bar problem; compactly supported data; functions)**.** *Let f be a smooth $(0,1)$-form f compactly supported in $\mathbb{C}^m$ satisfying $\overline{\partial} f = 0$. Then there exists a smooth function u on $\mathbb{C}^m$ such that $\overline{\partial} u = f$. If $m > 1$ then u can be chosen to be of compact support.*

Proof. Let $f = \sum f_j d\overline{z}_j$. We assert that the following is a solution:

$$\begin{aligned} u(z) &:= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}^m} \frac{f_1(\zeta, z_2, \ldots, z_m)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}^m} \frac{f_1(\zeta + z_1, z_2, \ldots, z_m)}{\zeta} d\zeta \wedge d\overline{\zeta}. \end{aligned}$$

The second formula shows that u is smooth. $\partial u / \partial \overline{z}_1 = f_1$ by Lemma 16. For $j \neq 1$ we have

$$\begin{aligned} \frac{\partial u}{\partial \overline{z}_j}(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}^m} \frac{\partial f_1 / \partial \overline{z}_j(\zeta, z_2, \ldots, z_m)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}^m} \frac{\partial f_j / \partial \overline{z}_1(\zeta, z_2, \ldots, z_m)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta}, \end{aligned}$$

which equals f_j again by Lemma 16. Now assume $m > 1$. Note that $u(z)$ is holomorphic for $|z_1| + \cdots + |z_m|$ large because f is compactly supported and $\overline{\partial} u = f$. By its very definition $u(z)$ vanishes for large $|z_2| + \cdots + |z_m|$, so the identity theorem forces it to be compactly supported. ■

Even this easy version of the d-bar problem enables us to prove the Hartogs extension theorem:

Theorem 24 (Hartogs extension theorem)**.** *(1) For every open $G \subseteq \mathbb{C}^m$, $m > 1$, and every compact $K \subseteq G$ such that $G \setminus K$ is connected, every holomorphic function on $G \setminus K$ can be extended to a holomorphic function on G. (2) If $G \subseteq \mathbb{C}^m$, $m > 1$, is a bounded open with connected boundary then every function holomorphic on the boundary can be extended holomorphically to $\overline{G}$.*

Proof. (1) Again, in accordance with the general idea of d-bar techniques, we first solve the problem smoothly and then do the required modifications by the d-bar problem. Assuming $f \in \mathcal{O}(G \setminus K)$, we choose a smooth bump function ψ on $\mathbb{C}^m$ compactly supported in D which equals 1 on K, consider $f_1 := (1 - \psi) f \in C^\infty(G)$, and try to find a correction function $\varphi \in C^\infty(G)$ such that $f_2 := f_1 - \varphi$ extends f holomorphically to G. To make f_2 holomorphic φ need to satisfy

$$\overline{\partial} \varphi = \overline{\partial} f_1 = -f \overline{\partial} \psi. \tag{3.2}$$

Pretending $-f\overline{\partial}\psi$ to live in $C_c^\infty(\mathbb{C}^m)$, Lemma 23 gives $\varphi \in C_c^\infty(\mathbb{C}^m)$ satisfying (3.2). The function φ vanishes outside a bounded region and is holomorphic wherever ψ is constant, so on some nonempty open of $G \setminus K$ we have $\varphi = 0 = \psi$ and so $f_2 = f$. Since $G \setminus K$ is connected, by the identity theorem $f_2 = f$ on whole $G \setminus K$.

(2) The same argument works. In fact it easy to see that (1) and (2) imply each other. ∎

The concise d-bar proof above first appeared in [Hör, 2.3.2], idea due to [Ehr]. Another proof using Bochner-Martinelli integral formula is given in page 42. A proof based on the classical "analytic discs method" is recently found [MP]. A generalization by Bochner will be proved in Theorem 107.

Exercise: Let f be a function holomorphic on the bounded open $D \subseteq \mathbb{C}^m$, $m > 1$, which is continuous up to the boundary. If f vanishes at some point of D show that it must vanish at some point on the boundary. If $|f| = 1$ on the boundary show that f is constant. (*Hint.* Apply Hartogs theorem to $1/f$.)

Here is a generalization of Lemma 23, the so-called Dolbeault-Grothendieck lemma:

Lemma 25 (d-bar problem; compactly supported data on polydiscs)**.** *If P and P' are open polydiscs in $\mathbb{C}^m$ with P compactly supported in P', then for any smooth $(p, q+1)$-form f on P' satisfying $\overline{\partial} f = 0$ there exists a smooth (p, q)-form u on P such that $\overline{\partial} u = f$.*

Proof. We argue inductively on $\overline{\partial}$-closed forms f having no $d\overline{z}_l$, $l > k$. If $k = 0$ then $f = 0$ (because $q + 1 \geq 1$), and $u = 0$ solves the d-bar problem. Write $f = d\overline{z}_k \wedge g + h$ where $g = \sum'_{|I|=p,|J|=q} g_{I,J} dz_I \wedge d\overline{z}_J$ and h are smooth forms on P' having no $d\overline{z}_l$, $l \geq k$. In the expansion of $\overline{\partial} f = 0$ the only summand with both $d\overline{z}_k$ and some $d\overline{z}_l$, $l > k$, is $\partial g_{I,J}/\partial \overline{z}_l d\overline{z}_l \wedge d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$, so $\partial g_{I,J}/\partial \overline{z}_l = 0$ for $l > k$. Lemma 16 implies that

$$G_{I,J}(z_1, \ldots, z_m) := \frac{1}{2\pi\sqrt{-1}} \int_D \frac{g_{I,J}(z_1, \ldots, z_{k-1}, \zeta, z_{k+1}, \ldots, z_m)}{\zeta - z_k} d\zeta \wedge d\overline{\zeta},$$

where D is a closed disc in the complex plane sandwiched between the k-th product factors of P and P', defines a smooth function on P satisfying $\partial G_{I,J}/\partial \overline{z}_k = g_{I,J}$. Also $\partial G_{I,J}/\partial \overline{z}_l = 0$ for $l > k$ because $\partial g_{I,J}/\partial \overline{z}_l = 0$ for such l. Set $G := \sum' G_{I,J} dz_I \wedge d\overline{z}_J$. Then $\overline{\partial} G = d\overline{z}_k \wedge g + h_1$ where h_1 has no $d\overline{z}_l$, $l \geq k$. Then $f - \overline{\partial} G = h - h_1$ is a $\overline{\partial}$-closed form having no $d\overline{z}_l$, $l \geq k$, so by the induction hypothesis $f - \overline{\partial} G = \overline{\partial} u$ for some smooth form u on P. Therefore, $f = \overline{\partial}(G + u)$. ∎

The ideas that we used to solve the d-bar problem on the plane from the special case of compactly supported data, can be used to prove the following result. Details can be found in [Gun, page 47][Ohs, page 29].

Proposition 26 (d-bar problem on polydiscs)**.** *If P is an open polydiscs in $\mathbb{C}^m$ then for any smooth $(p, q + 1)$-form f on P satisfying $\overline{\partial} f = 0$ there exists a smooth (p, q)-form u on P such that $\overline{\partial} u = f$.*

Example 27. This example [Gun, volume I, page 51] gives an explicit smooth $(0, 1)$-form on $D = \mathbb{C}^2 \setminus \{(0, 0)\}$ which is $\overline{\partial}$-closed but not a $\overline{\partial}$-boundary. Set $r := \sqrt{|z_1|^2 + |z_2|^2}$. Then the identity $1/z_1 z_2 = \overline{z}_1/z_2 r^2 + \overline{z}_2/z_1 r^2$ valid for $z_1 z_2 \neq 0$ shows that

$$f := \begin{cases} -\overline{\partial}\left(\overline{z}_1/(z_2 r^2)\right), & z_2 \neq 0 \\ \overline{\partial}\left(\overline{z}_2/(z_1 r^2)\right), & z_1 \neq 0 \end{cases},$$

is a well-defined $\overline{\partial}$-closed form on D. Suppose by contradiction that $f = \overline{\partial} u$ for some smooth function u on D. Then $v := z_1 u - \overline{z}_2 / r^2$ is holomorphic on $\mathbb{C}^2 \setminus \{z_1 = 0\}$. As in the proof of the simple case of Hartogs extension theorem given in Chapter 1, setting $v(z_1, z_2)$ equal to

$$\frac{1}{2\pi\sqrt{-1}} \int_{|\zeta_1|=1} \frac{v(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1$$

on $|z_1| < 1$ extends v holomorphically to $\mathbb{C}^2$. However, $v(z)$ blows up as z approaches the origin along $z_1 = 0$. ■

3.2.3 Preliminaries: Hodge star operator in $\mathbb{C}^m$, Integration by parts, Complex Laplacian

For every two complex-valued differential forms $\omega = \sum' \omega_I dx_I$ and $\eta = \sum' \eta_I dx_I$ on open $U \subseteq \mathbb{R}^n$, the function defined by

$$\langle \omega, \eta \rangle := \sum\nolimits' \omega_I \overline{\eta}_I,$$

is called their **pointwise inner product**, and its integral

$$(\omega, \eta) := \int_U \sum\nolimits' \omega_I \overline{\eta}_I d\mu,$$

with respect to the Lebesgue measure is called their **(global) inner product**. Note that the global inner product has all the properties of an inner product on complex vector spaces unless it might be infinite. We define the pointwise/global norms:

$$|\omega| := \langle \omega, \omega \rangle^{1/2}, \quad \|\omega\| := (\omega, \omega)^{1/2}.$$

If $U = D$ is an open of $\mathbb{C}^m \cong \mathbb{R}^{2m}$ then a straightforward calculation shows that

$$\langle \omega, \eta \rangle = \sum_{|I|=p, |J|=q}{}' 2^{p+q} \omega_{I,J} \overline{\eta}_{I,J}, \quad (\omega, \eta) = \int_D \langle \omega, \eta \rangle \, d\mu,$$

where we have used (p,q)-type representations $\omega = \sum' \omega_{I,J} dz_I \wedge d\overline{z}_J$ and $\eta = \sum' \eta_{I,J} dz_I \wedge d\overline{z}_J$. The next theorem introduces a useful duality operator which expresses the inner product of forms in terms of their wedge, and enables us to find compact illuminating formulas for otherwise messy expressions.

Theorem 28 (Hodge star operator in $\mathbb{R}^n$ and $\mathbb{C}^m$)**.** *For every open $U \subseteq \mathbb{R}^n$ and integer $k \in \{0, \ldots, n\}$ there exists a unique map $* : C_k^\infty(U) \to C_{n-k}^\infty(U)$, called the **Hodge star** operator, from the vector space of complex-valued smooth differential forms of degree k on U to the space of forms of degree $n-k$ such that*

$$(\omega, \eta) = \int_U \omega \wedge *\overline{\eta}, \quad \forall \omega, \eta \in C_k^\infty(U),$$

or equivalently

$$\langle \omega, \eta \rangle = \omega \wedge *\overline{\eta}\ dV, \quad \forall \omega, \eta \in C_k^\infty(U),$$

where the conjugate of any form $\eta = \sum \eta_I dx_I \in C_k^\infty(U)$ *is defined by* $\overline{\eta} = \sum \overline{\eta_I} dx_I$ *and* $dV = dx_1 \wedge \cdots \wedge dx_n$*. This operator has the following extra properties:*

(1) $*$ *is a* $C^\infty(U)$*-module homomorphism, and is given by* $*dx_I = \epsilon_{II'}^N dx_{I'}$ *for any two shuffles* I *and* I' *which partition the shuffle* $N := (1, \ldots, n)$*. Here* ϵ *is the sign of the permutation.*

(2; reality) $\overline{*\omega} = *\overline{\omega}$ *for every* $\omega \in C_k^\infty(U)$*.*

(3; duality) $** = (-1)^{k(n-k)}$ *acting on forms of degree* k*.*

(4) $*1 = dV$ *and* $*dV = 1$*.*[5]

(5) For open $D \subseteq \mathbb{C}^m$ *with the standard identification* $\mathbb{C}^m \cong \mathbb{R}^{2m}$ *via* $z_j = x_{2j-1} + \sqrt{-1}x_{2j}$ *we have:*

$$* : C_{p,q}^\infty(D) \to C_{m-p,m-q}^\infty(D),$$

$$** = (-1)^{p+q} \text{ acting on } C_{p,q}^\infty(D),$$

$$* \, dz_I = \frac{(-1)^{|I|(|I|-1)/2}}{2^{m-|I|}\sqrt{-1}^m} dz_I \wedge \left(\bigwedge_{j \in I'} d\overline{z}_j \wedge dz_j \right) = \overline{*d\overline{z}_I}, \tag{3.3}$$

for any two shuffles I *and* I' *which partition the shuffle* $(1, \ldots, m)$*.*

Proof. In order for a map $* : C_k^\infty(U) \to C_{n-k}^\infty(U)$ to satisfy $(\omega, \overline{\eta}) = \int \omega \wedge *\eta$ for every forms ω, η of degree k we must have

$$\int {\sum_I}' \omega_I \eta_I d\mu = \int {\sum_{I,J}}' \omega_I dx_I \wedge \gamma_J dx_J = \int {\sum_I}' \omega_I \gamma_{I'} \epsilon_{II'}^N d\mu,$$

where $\gamma = \sum'_{|J|=n-k} \gamma_J dx_J = *\eta$ and we have used the notations introduced in (1). Since this holds for every ω it follows that $\gamma_{I'} = \epsilon_{II'}^N \eta_I$. The same conclusion holds if $*$ wants to satisfy $\langle \omega, \overline{\eta} \rangle = \omega \wedge *\eta\ dV$ for every ω, η. Having this explicit formula at hand verifying (2,3,4,5) is straightforward. Specially, (3.3) is deduced by comparing two sides of the identity

$$dz_I \wedge *d\overline{z}_I = \langle dz_I, dz_I \rangle\, dV = 2^{|I|} \bigwedge_{j=1}^{2m} dx_j = 2^{|I|} \frac{\sqrt{-1}^m}{2^m} \bigwedge_{j=1}^{m} dz_j \wedge d\overline{z}_j =$$

$$\frac{\sqrt{-1}^m}{2^{m-|I|}} (-1)^{|I|(|I|-1)/2} dz_I \wedge d\overline{z}_I \wedge \left(\bigwedge_{j \in I'} dz_j \wedge d\overline{z}_j \right).$$

■

In the following we assume familiarity with the notion of the **Riemannian volume form** $dV = \sqrt{\det(g)} dx_1 \wedge \cdots \wedge dx_n$ of oriented Riemannian manifolds with boundary (M^n, g) (for us $M = D \cup \partial D$ where $D \subseteq \mathbb{C}^m$ is an open with smooth boundary), and the volume form dS it induces on ∂M [Lee, chapter 15].

[5] Remember our definition of the integral of a top form in page 29; if we have defined $\int f dx_1 \wedge \cdots \wedge dx_n = -\int f dx_1 \cdots dx_n$ there, then we would have gotten $*1 = -dV$ and $*dV = -1$ here. This shows that Hodge star depends on the orientation chosen.

Theorem 29 (The Riemannian volume form on the boundaries of opens of $\mathbb{R}^m$ and $\mathbb{C}^m$)**.** *Let $U \subseteq \mathbb{R}^n$ be an open with C^1 defining function r. The canonical Riemannian volume form induced on ∂U is given by*

$$\begin{aligned} dS &= i^* \left(\sum_{j=1}^{n} (-1)^{j-1} \nu_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \right) \\ &= i^* \left(\frac{*dr}{|dr|} \right), \end{aligned}$$

where $i : \partial U \hookrightarrow U$ is the inclusion map, i^* is the pullback map of forms and $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal vector field on ∂U. Furthermore, if $U = D$ is an open of $\mathbb{R}^{2m} \cong \mathbb{C}^m$ then

$$dS = 2i^* \left(\frac{*\partial r}{|dr|} \right) = \sqrt{2} i^* \left(\frac{*\partial r}{|\partial r|} \right).$$

Proof. (1) The first formula is exactly "dS equals ν interior product the volume form of U" proved in [Lee, 15.34]. The normalized gradient $dr/|dr|$ is the Riemannian (musical) dual of ν namely $dr/|dr| = \sum \nu_j dx_j$. Since

$$(-1)^{j-1} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n = *dx_j,$$

two formulas coincide.

(2) We use Theorem 28. Since $\partial r = \sum \partial r/\partial z_j dz_j$ and $\langle dz_j, dz_k \rangle = 2\delta_{jk}$ it follows that

$$|\partial r|^2 = \sum 2|\partial r/\partial z_j|^2 = \frac{1}{2} \sum |\partial r/\partial x_{2j-1}|^2 + |\partial r/\partial x_{2j}|^2 = \frac{1}{2}|dr|^2.$$

On the other hand

$$*\partial r = \frac{1}{2^{m-1}\sqrt{-1}^m} \sum_j \frac{\partial r}{\partial z_j} dz_j \wedge \left(\bigwedge_{k \neq j} d\bar{z}_k \wedge dz_k \right) = \frac{1}{(m-1)!\sqrt{-1}} \partial r \wedge \omega^{m-1},$$

where $\omega := \frac{1}{2\sqrt{-1}} \sum_j d\bar{z}_j \wedge dz_j$. Therefore

$$*dr = *\partial r + \overline{*\partial r} = \frac{1}{(m-1)!\sqrt{-1}} \left(\partial r - \bar{\partial} r \right) \wedge \omega^{m-1}.$$

Combined with the trivial equation $0 = d0 = di^*r = i^*dr = i^*\partial r + i^*\bar{\partial} r$ we have $i^*(*dr) = 2i^*(*\partial r)$, and we are done by (1). ∎

For the next theorem we need to know how to integrate a form in $C^1_{n-1}(U)$ over the C^1 boundary of an open $U \subseteq \mathbb{C}^m$. The main issue here, the so-called **orientation**, is to find a way to make a consistent (namely continuous) choice of the $\pm$ ambiguity in transformation law (3.1) for the integration of forms on manifolds. This is resolved if we can choose an open covering of ∂D by coordinates charts such that all the change of coordinates maps $(u_1, \ldots, u_{2m-1}) \mapsto (v_1, \ldots, v_{2m-1})$ have positive Jacobian determinant. This is possible: Fix a global defining function r for D, as defined in Section 3.5.1. By the inverse function theorem $(r, u_1, \dots, u_{2m-1})$ is a local coordinate system for $\mathbb{R}^n$, so $dr \wedge du_1 \wedge \cdots \wedge du_{2m-1}$ is a nowhere-zero C^1 function f_u multiplied by the standard top form $dx_1 \wedge \cdots \wedge dx_{2m}$ of $\mathbb{R}^{2m}$. If f_u is positive choose $(u_1, \dots, u_{2m-1})$, otherwise choose $(u_2, u_1, u_3, u_4, \dots, u_{2m-1})$. More details can be found in [Lee, chapter 15][Ran, chapter 3].

Theorem 30 (Integration by parts)**.** *Assume open $D \subseteq \mathbb{C}^m$.*

(1) For every $\eta \in C^1_{p,q+1}(D)$ there exists a unique $\gamma \in C^1_{p,q}(D)$ such that $(\overline{\partial}\omega, \eta) = (\omega, \gamma)$ for every compactly supported form $\omega \in C^1_{p,q}(D)$. This γ is denoted by $\overline{\partial}^*\eta$ and given by

$$\gamma = - * \partial * \eta = 2(-1)^{p+1} \sum_{|I|=p, |K|=q}' \sum_{j=1}^{m} \sum_{|J|=q+1}' \epsilon^J_{jK} \frac{\partial \eta_{I,J}}{\partial z_j} dz_I \wedge d\overline{z}_K. \tag{3.4}$$

*The differential operator $\overline{\partial}^* : C^\infty_{p,q+1}(D) \to C^\infty_{p,q}(D)$ is called the **formal adjoint** of $\overline{\partial}$.*

*(2) If D is bounded and with C^1 boundary then for every $\omega \in C^1_{p,q}(\overline{D})$ and $\eta \in C^1_{p,q+1}(\overline{D})$ we have **integration by parts formulas***

$$(\overline{\partial}\omega, \eta) - \left(\omega, \overline{\partial}^*\eta\right) = \int_{\partial D} \omega \wedge *\overline{\eta},$$
$$\left(\overline{\partial}^*\eta, \omega\right) - (\eta, \overline{\partial}\omega) = - \int_{\partial D} \overline{\omega} \wedge *\eta.$$

Proof. (1) Using properties of the Hodge star operator and Stokes' theorem

$$(\overline{\partial}\omega, \eta) = \int_D \overline{\partial}\omega \wedge *\overline{\eta} = \int_D \overline{\partial}(\omega \wedge *\overline{\eta}) - \int_D (-1)^{p+q} \omega \wedge \overline{\partial} * \overline{\eta} =$$
$$\int_D d(\omega \wedge *\overline{\eta}) - \int_D \omega \wedge * * \overline{\partial} * \overline{\eta} = 0 - \int_D \omega \wedge \overline{* * \partial * \eta} = -(\omega, *\partial * \eta).$$

Since

$$\overline{\partial}\omega = \sum_{I,K}' \sum_j \frac{\partial \omega_{I,K}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_K = (-1)^p \sum_{I,K}' \sum_j \frac{\partial \omega_{I,K}}{\partial \overline{z}_j} dz_I \wedge d\overline{z}_j \wedge d\overline{z}_K =$$
$$(-1)^p \sum_{I,K,J}' \sum_j \epsilon^J_{jK} \frac{\partial \omega_{I,K}}{\partial \overline{z}_j} dz_I \wedge d\overline{z}_J,$$

the equation $(\omega, \gamma) = (\overline{\partial}\omega, \eta)$ says

$$2^{p+q} \sum_{I,K}' \int \omega_{I,K} \overline{\gamma_{I,K}} dV = 2^{p+q+1} (-1)^p \sum_{I,K,J}' \sum_j \epsilon^J_{jK} \int \frac{\partial \omega_{I,K}}{\partial \overline{z}_j} \overline{\eta_{I,J}} dV =$$
$$- 2^{p+q+1} (-1)^p \sum_{I,K,J}' \sum_j \epsilon^J_{jK} \int \omega_{I,K} \frac{\partial \overline{\eta_{I,J}}}{\partial \overline{z}_j} dV,$$

where we have done integration by parts in the last step. Since this equation holds for every compactly supported ω we get the messy formula in (3.4).

(2) The first formula is proved exactly like (1) but this time $\int_D d(\omega \wedge *\overline{\eta})$ equals $\int_{\partial D} \omega \wedge *\overline{\eta}$ instead of 0. The second formula is immediate from the first. ∎

Integration by part formulas for arbitrary first-order differential operators is given in [Tay, volume I, page 178][FK, page 13].

Theorem 31 (Complex Laplacian on $\mathbb{C}^m$)**.** *The **complex** (or **Kohn**) **Laplacian** $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is related to the Hodge Laplacian [War, chapter 6][Jos-DG, chapter 3] $\Delta = dd^* + d^*d$ via $\Box = \frac{1}{2}\Delta$. Furthermore, $\Box$ acts diagonally in the sense that*

$$\Box\omega = \sum_{I,J}{}' (\Box\omega_{I,J})\, dz_I \wedge d\overline{z}_J, \quad \Box\omega_{I,J} = -2\sum_{j=1}^{m} \frac{\partial^2\omega_{I,J}}{\partial z_j \partial \overline{z}_j},$$

for every $\omega = \sum' \omega_{I,J} dz_I \wedge d\overline{z}_J$.

Proof. Fix ω of type (p,q). In the course of the proof L, I, A, K are shuffles of $\{1,\dots,m\}$ of lengths $q-1$, q, q, $q+1$ respectively, and j,k are indices ranging on $\{1,\dots,m\}$. Having this in mind without confusion we can suppress summation notations. Using (3.4) we have

$$\frac{-1}{2}\Box\omega = \frac{\partial^2\omega_{I,A}}{\partial\overline{z}_j\partial z_k}\epsilon^A_{kL}\epsilon^J_{jL} dz_I \wedge d\overline{z}_J + \frac{\partial^2\omega_{I,A}}{\partial\overline{z}_j\partial z_k}\epsilon^K_{jA}\epsilon^K_{kJ} dz_I \wedge d\overline{z}_J. \tag{3.5}$$

The second sum equals $\frac{\partial^2\omega_{I,A}}{\partial\overline{z}_j\partial z_k}\epsilon^{jA}_{kJ} dz_I \wedge d\overline{z}_J$. It splits into two sums depending whether $j = k$ or $j \neq k$; the first sum gives the desired expression in the statement of the theorem, so we need to show that the second sum S cancels the first sum in (3.5). This is true because: to have nonzero term in S, the shuffles A and J must be formed by adding k and j, respectively, to some common shuffle L, and we have

$$\epsilon^{jA}_{kJ} = \epsilon^{jA}_{jkL}\epsilon^{jkL}_{kjL}\epsilon^{kjL}_{kJ} = -\epsilon^A_{kL}\epsilon^{jL}_J.$$

So far we have shown that

$$\Box\omega = -2\sum_{I,J}{}' \sum_{j=1}^{m} \frac{\partial^2\omega_{I,J}}{\partial\overline{z}_j\partial z_j} dz_I \wedge d\overline{z}_J.$$

A similar computation [Jos-DG, page 112] shows that

$$\Delta\omega = -4\sum_{I,J}{}' \sum_{j=1}^{m} \frac{\partial^2\omega_{I,J}}{\partial\overline{z}_j\partial z_j} dz_I \wedge d\overline{z}_J.$$

∎

Theorem 32 (The Riemannian volume form of the sphere)**.** *Consider the differential forms*

$$\alpha = \left(dd^c|z|_2^2\right)^m,$$

$$\beta = i^*\left(d^c \log|z|_2^2 \wedge \left(dd^c \log|z|_2^2\right)^{m-1}\right),$$

where $d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$ and $i : \mathbb{S}^{2m-1} \hookrightarrow \mathbb{C}^m$ is the inclusion map of the unit sphere. Then:

(1) α is a constant times the Riemannian volume form of $\mathbb{C}^m$, and $\int_{|z|_2<1}\alpha = 1$.
(2) β is a constant times the Riemannian volume form of $\mathbb{S}^{2m-1}$, and $\int_{\mathbb{S}^{2m-1}}\beta = 1$.
(3)

$$\beta = \frac{1}{(2\pi\sqrt{-1})^m} i^* \left(\partial|z|_2^2 \wedge \left(\partial\overline{\partial}|z|_2^2\right)^{m-1}\right).$$

Proof. Note that $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$. (1) Since $dd^c|z|^2 = \frac{\sqrt{-1}}{2\pi}\sum dz_j \wedge d\overline{z}_j$ and $dz_j \wedge d\overline{z}_j = -2\sqrt{-1}dx_{2j-1} \wedge dx_{2j}$ it follows that

$$\alpha = \frac{\sqrt{-1}^m}{(2\pi)^m} m! dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_m \wedge d\overline{z}_m = \frac{m!}{\pi^m} \bigwedge_{1\le j\le 2m} dx_j.$$

We are done by noticing that $\pi^m/m!$ is the volume of the unit ball of $\mathbb{C}^m$ [Fol, 2.55].

(2,3) To compute the Riemannian volume form dS of $\mathbb{S}^{2m-1}$ we use Theorem 29. A defining function of the unit ball is $r = |z|^2 - 1$, so $dr = 2\sum_{1\le j\le 2m} x_j dx_j$, $|dr| = 2|z|^2$, $i^*|dr| = 2$, hence $dS = i^*(*\partial r)$.

$$* \partial r = \sum_{j=1}^m \overline{z}_j(*dz_j) = \frac{1}{2^{m-1}\sqrt{-1}^m} \sum_j \overline{z}_j dz_j \wedge \left(\bigwedge_{k\ne j} d\overline{z}_k \wedge dz_k\right) =$$
$$\frac{1}{(m-1)!2^{m-1}\sqrt{-1}^m} \sum_j \overline{z}_j dz_j \wedge \left(\sum_j d\overline{z}_j \wedge dz_j\right)^{m-1} =$$
$$\frac{1}{(m-1)!2^{m-1}\sqrt{-1}^m}\partial|z|^2 \wedge (\overline{\partial}\partial|z|^2)^{m-1}.$$

Therefore

$$dS = \frac{1}{(m-1)!2^{m-1}\sqrt{-1}^m}\left(i^*\partial|z|^2 \wedge (i^*\overline{\partial}\partial|z|^2)^{m-1}\right). \tag{3.6}$$

Since $i^*d|z|^2 = di^*|z|^2 = d1 = 0$ it follows that

$$i^*\partial|z|^2 = -i^*\overline{\partial}|z|^2 \quad \text{hence} \quad i^*\partial|z|^2 \wedge i^*\overline{\partial}|z|^2 = 0.$$

With the same arguments

$$i^*\partial \log|z|^2 = -i^*\overline{\partial}\log|z|^2,$$
$$i^*\partial \log|z|^2 = i^*\left(|z|^{-2}\partial|z|^2\right) = i^*\partial|z|^2,$$
$$i^*\partial\overline{\partial}\log|z|^2 = i^*\partial\left(|z|^{-2}\overline{\partial}|z|^2\right) = i^*\left(\partial|z|^{-2} \wedge \overline{\partial}|z|^2 + |z|^{-2} \wedge \partial\overline{\partial}|z|^2\right) = 0 + i^*\partial\overline{\partial}|z|^2.$$

These equation combined with (3.6) gives

$$dS = -\frac{1}{(m-1)!2^m\sqrt{-1}^m}\left(i^*(\overline{\partial}-\partial)\log|z|^2 \wedge (i^*\overline{\partial}\partial\log|z|^2)^{m-1}\right)$$
$$= \frac{2\pi^m}{(m-1)!} i^*\left(d^c \log|z|^2 \wedge (dd^c\log|z|^2)^{m-1}\right).$$

We are done by noticing that $2\pi^m/(m-1)!$ is the volume of the unit sphere of $\mathbb{C}^m$ [Fol, 2.54]. ∎

3.2.4 Bochner-Martinelli formula for functions, The second proof

The Cauchy integral formula (Theorem 22.(1)) gives the values of holomorphic functions on polydisks in terms of their values on (some part of) the boundary. This formula compared to Cauchy-Pompeiu formula of one complex variable has two restrictions: (1) It applies only to polydiscs; (2) It applies only to holomorphic functions. The following is a generalization of Cauchy-Pompeiu formula to SCV which gives the values of smooth functions on arbitrary opens of $\mathbb{C}^m$ in terms of their integrals over the opens and their boundaries:

Theorem 33 (Bochner-Martinelli formula for functions)**.** *Let $D \subseteq \mathbb{C}^m$ be a bounded open with C^1 boundary. (1) For every $f \in C^1(\overline{D})$ we have*

$$\int_{\partial D} f(\zeta)K_0(\zeta,z) - \int_D \overline{\partial} f(\zeta) \wedge K_0(\zeta,z) = \begin{cases} f(z), & z \in D \\ 0, & z \in \mathbb{C}^m \setminus \overline{D} \end{cases},$$

where

$$K_0(\zeta,z) := \frac{(m-1)!}{(2\pi\sqrt{-1})^m} \sum_{j=1}^m \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|_2^{2m}} d\zeta_j \wedge \left(\bigwedge_{1\le k\le m, k\neq j} d\overline{\zeta}_k \wedge d\zeta_k \right) \tag{3.7}$$

a smooth form on $\mathbb{C}^m \times \mathbb{C}^m \setminus \{z=\zeta\}$ of type $(m, m-1)$ with respect to ζ is called the **Bochner-Martinelli kernel for functions***.*

(2) For every $f \in C(\overline{D}) \cap \mathcal{O}(D)$ we have

$$\int_{\partial D} f(\zeta)K_0(\zeta,z) = \begin{cases} f(z), & z \in D \\ 0, & z \in \mathbb{C}^m \setminus \overline{D} \end{cases}.$$

(3) Other representations of K_0 are

$$K_0(\zeta,z) = \frac{1}{(2\pi\sqrt{-1})^m}\beta^{-m}\partial_\zeta\beta \wedge \left(\overline{\partial}_\zeta\partial_\zeta\beta\right)^{m-1}, \qquad \beta := |\zeta - z|_2^2, \tag{3.8}$$

$$= \frac{1}{(2\pi\sqrt{-1})^m}\partial_\zeta B \wedge \left(\overline{\partial}_\zeta\partial_\zeta B\right)^{m-1}, \qquad B := \log|\zeta - z|_2^2, \tag{3.9}$$

$$= - * \partial_\zeta \Gamma(\zeta,z), \tag{3.10}$$

where

$$\Gamma(\zeta,z) = \frac{2}{\sigma_{2m-1}} \times \begin{cases} -\log|\zeta - z|, & m=1 \\ \frac{1}{(2m-2)}|\zeta - z|_2^{2-2m}, & m>1 \end{cases}, \tag{3.11}$$

is twice the **Newtonian potential** *(or the* **Green function of the free space** *$\mathbb{C}^m$), and $\sigma_{2m-1} = 2\pi^m/(m-1)!$ is the volume of the unit sphere of $\mathbb{C}^m$.*

Proof. (1) We first check the following two facts:

$$\partial_\zeta\left(f(\zeta)K_0(\zeta,z)\right) = 0 \quad \text{and} \quad \overline{\partial}_\zeta K_0(\zeta,z) = 0. \tag{3.12}$$

The left hand side of the first equation is a form of type $(m+1, m-1)$, hence identically zero. For the second one, setting $\alpha_j := d\bar{\zeta}_j \wedge d\zeta_j$, $\xi_j := \zeta_j - z_j$, we have

$$\frac{(2\pi\sqrt{-1})^m}{(m-1)!}\bar{\partial}_\zeta K_0 = \sum_{j,l} \left(\delta_{lj}|\xi|^{-2m} - m\bar{\xi}_j\xi_l|\xi|^{-2m-2}\right) d\bar{\xi}_l \wedge d\xi_j \wedge \bigwedge_{k\neq j} \alpha_k =$$

$$|\xi|^{-2m-2}\left(\sum_j \left(|\xi|^2 - m\bar{\xi}_j\xi_j\right)\right)\alpha_1 \wedge \cdots \wedge \alpha_m = 0.$$

If $z \notin \overline{D}$ then by Stokes' theorem and (3.12) we have

$$\int_{\partial D} f(\zeta)K_0(\zeta, z) = \int_D d_\zeta\left(f(\zeta)K_0(\zeta, z)\right) = \int_D \bar{\partial} f(\zeta) \wedge K_0(\zeta, z).$$

Now fix $z \in D$. Exactly similar to Cauchy-Pompeiu formula (case $m = 1$ here), to prove this generalization we apply Stokes' theorem $\int_{\partial M} \omega = \int_M d\omega$ to $\omega(\zeta) = f(\zeta)K_0(\zeta, z)$ and $M = \{\zeta \in D : |\zeta - z|_2 > \epsilon\}$ where $0 < \epsilon < \operatorname{dist}(z, \partial D)$:

$$\int_{\partial D} f(\zeta)K_0(\zeta, z) - \int_{|\zeta - z| = \epsilon} f(\zeta)K_0(\zeta, z) = \int_M \bar{\partial} f(\zeta) \wedge K_0(\zeta, z).$$

Since $\zeta \mapsto |\zeta - z|^{-2m}$ is integrable on any bounded domain around its singular point $\zeta = z$, by Lebesgue dominated convergence theorem the limit of the integral $\int_M$ as $\epsilon \to 0$ exists. This limit value is the meaning of the volume integral $\int_D$ in the statement of the theorem. It remains to show that the integral $\int_{|\zeta - z| = \epsilon}$ tends $f(z)$ as $\epsilon \to 0$. Another application of the Stokes's theorem shows that:

$$\int_{|\zeta - z| = \epsilon} K_0(\zeta, z) = \frac{(m-1)!\epsilon^{-2m}}{(2\pi\sqrt{-1})^m}\int_{|\zeta - z| < \epsilon} m\bigwedge_{k=1}^m d\bar{\zeta}_k \wedge d\zeta_k = \frac{m!\epsilon^{-2m}}{\pi^m}\int_{|\zeta - z| < \epsilon} \bigwedge_{k=1}^{2m} dx_k = 1,$$

because $m!/\pi^m$ is the volume of the unit ball in $\mathbb{C}^m$. Therefore

$$\int_{|\zeta - z| = \epsilon} f(\zeta)K_0(\zeta, z) = f(z) + \int_{|\zeta - z| = \epsilon} (f(\zeta) - f(z))K_0(\zeta, z).$$

The second integral tends to 0 as $\epsilon \to 0$ because its integrand is dominated by a constant times $\epsilon \times \epsilon^{1-2m}$.

(2) If $f \in C^1(\overline{D}) \cap \mathcal{O}(D)$ the result is immediate from (1). The general case follows after a limiting process.

(3) That (3.8) equals (3.7) is straightforward. (3.9) equals (3.8) because

$$\bar{\partial}\partial B = \bar{\partial}(\beta^{-1}\partial\beta) = -\beta^{-2}\bar{\partial}\beta \wedge \partial\beta + \beta^{-1}\bar{\partial}\partial\beta,$$

$$\partial B \wedge \bar{\partial}\partial B = 0 + \beta^{-2}\partial\beta \wedge \bar{\partial}\partial\beta, \quad \partial B \wedge (\bar{\partial}\partial B)^2 = \beta^{-3}\partial\beta \wedge (\bar{\partial}\partial\beta)^2, \quad \text{etc.}$$

Finally using the properties of Hodge star we can put K_0 in the following concise form:

$$K_0(\zeta, z) = \frac{(m-1)!}{(2\pi\sqrt{-1})^m}\sum_{j=1}^m \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2m}}\left(2^{m-1}\sqrt{-1}^m * d_\zeta(\zeta_j - z_j)\right) =$$

$$= \frac{(m-1)!}{2\pi^m} * \sum_{j=1}^m \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2m}}\partial_\zeta(\zeta_j - z_j) = - * \partial_\zeta\Gamma.$$

■

Lemma 34. *Setting*

$$K_1(\zeta,z) := \frac{m-1}{(2\pi\sqrt{-1})^m}|\zeta-z|_2^{-2m}\times \left(\sum_{j=1}^m(\overline{\zeta}_j-\overline{z}_j)d\zeta_j\right)\wedge\left(\sum_{j=1}^m d\overline{\zeta}_j\wedge d\zeta_j\right)^{m-2}\wedge\left(\sum_{j=1}^m d\zeta_j\wedge d\overline{z}_j\right),$$

we have

$$\overline{\partial}_\zeta K_0(\zeta,z)=0,\quad \overline{\partial}_\zeta K_1(\zeta,z)=-\overline{\partial}_z K_0(\zeta,z),$$

on $\mathbb{C}^m\times\mathbb{C}^m\setminus\{z=\zeta\}$.

Proof. The first one was already proved during the proof of Theorem 33. Here is another proof based on the representation (3.10) and the properties of Hodge star:

$$\overline{\partial}_\zeta K_0 = -\overline{\partial}_\zeta * \overline{\partial}_\zeta\Gamma = - * *\overline{\partial}_\zeta * \overline{\partial}_\zeta\Gamma = - * \partial_\zeta\overline{\partial}_\zeta\Gamma = - * \square\Gamma = -\frac{1}{2} * \Delta\Gamma = 0,$$

because it is famous that Γ is harmonic. The second one is a straightforward calculation [CS, page 39]. For a conceptual proof refer [Ran, pages 148, 155]. ∎

Another proof for Theorem 24. We prove the second formulation. Choose a connected neighborhood $U\subsetneq\mathbb{C}^m$ of the boundary such that f is holomorphic on, and domains $D_1\subset\subset D\subset\subset D_2$ with connected smooth boundaries such that $\overline{D}_2\setminus D_1\subsetneq U$. The Bochner-Martinelli formula applied to $D_2\setminus\overline{D}_1$ gives

$$f(z)=F_2(z)-F_1(z),\quad F_j(z)=\int_{\partial D_j}f(\zeta)K_0(\zeta,z),\quad z\in D_2\setminus\overline{D}_1.$$

Although $K_0(\zeta,z)$ is not holomorphic in z but F_j is holomorphic on $\mathbb{C}^m\setminus\partial D_j$ because:

$$\overline{\partial}_z K_0(\zeta,z)=-\overline{\partial}_\zeta K_1(\zeta,z)=-\overline{\partial}_\zeta K_1(\zeta,z)-\partial_\zeta K_1(\zeta,z)=-d_\zeta K_1(z,\zeta),$$

so by Stokes' theorem $\overline{\partial}F_j(z)=0$ if $z\in\mathbb{C}^m\setminus\partial D_j$. (Note that this is true for every continuous f.) The representation (3.7) shows that $F_1(z)\to 0$ as $|z|_2\to\infty$, so in the splitting $z=(z',z_m)$ if $|z'|_2$ is large enough then $F_1(z)$ is a bounded holomorphic function with respect to z_m on the whole complex plane, so Liouville theorem and the identity theorem together force F_1 to vanish on $\mathbb{C}^m\setminus\overline{D}_1$. Then $F_2(z)$ works as our desired extension. ∎

Remark 35. The strange formula of K_0 in (3.7) comes from the potential theory as we elaborate now. The Newtonian potential $\frac{1}{2}\Gamma$ in (3.11) is the fundamental solution of the Laplacian in the sense that $\Delta\frac{1}{2}\Gamma$ equals the Dirac unit mass distribution [Fol, pages 291, 300][Jos-RS, page 103][Helm, 1.4.2]. Therefore for every compactly supported C^2 function in $\mathbb{C}^m$ we have

$$\begin{aligned}f(z)&=\int_{\mathbb{C}^m}\Delta f(\zeta)\frac{1}{2}\Gamma(\zeta,z)d\mu(\zeta)=\frac{1}{2}(\Delta f,\Gamma)=(\square f,\Gamma)=\left(0+\overline{\partial}_\zeta^*\overline{\partial}_\zeta f,\Gamma\right)=(\overline{\partial}_\zeta f,\overline{\partial}_\zeta\Gamma)\\&=\int\overline{\partial}f\wedge *\partial_\zeta\Gamma dV.\end{aligned}$$

Comparison with Bochner-Martinelli formula gives $K_0 = - * \partial_\zeta \Gamma$. The general case of non-compactly supported data is treated similarly. More generally there are Bochner-Martinelli kernels for reproducing smooth $(0,q)$-forms, $q = 0, \dots, m$:

$$K_q(\zeta, z) := \frac{(-1)^{q(q-1)/2}}{(2\pi\sqrt{-1})^m} \binom{m-1}{q} \partial_\zeta B \wedge \left(\overline{\partial}_\zeta \partial_\zeta B\right)^{m-q-1} \wedge \left(\overline{\partial}_z \partial_\zeta B\right)^q.$$

For details refer [Ran, chapter 4]. ■

3.2.5 Bochner-Martinelli formula for differential forms

Theorem 36 (Bochner-Martinelli formula for forms)**.** *Let $D \subseteq \mathbb{C}^m$ be a bounded open with C^1 boundary. Then for every $f \in C^1_{0,q}(\overline{D})$, $q = 0, \dots, m$, we have*

$$\int_{\partial D} f(\zeta) K_q(\zeta, z) - \int_D \overline{\partial} f(\zeta) \wedge K_q(\zeta, z) - \overline{\partial} \int_D f(\zeta) K_{q-1}(\zeta, z) = \begin{cases} f(z), & z \in D \\ 0, & z \in \mathbb{C}^m \setminus \overline{D} \end{cases}.$$

Proof. ■

The Bochner-Martinelli integral formula for forms gives the following generalization of the first part of Lemma 23:

Theorem 37. *For every bounded open $D \subseteq \mathbb{C}^m$ and every $f \in C^1_{0,q}(\overline{D})$, $q = 1, \dots, m$, satisfying $\overline{\partial} f = 0$ there exists $u \in C^1_{0,q-1}(D)$ such that $\overline{\partial} u = f$ on D.*

Proof. Apply Bochner-Martinelli formula to $f\psi$ where ψ is an appropriate smooth bump function which equals 1 on $\overline{D}$. ■

3.3 Domains of convergence of power series

The set of points which a power series $\sum_{n \geq 0} a_n z^n$ of one complex variable converges might be hard to describe, but the interior of this set is simple: either the empty set or an open disk around the origin [Ahl, page 38]. We want to solve the same problem in SCV. The **domain of convergence of power series** $\sum_{\alpha \in \mathbb{N}^m} a_\alpha z^\alpha$ is defined to be the interior of the set of points $z \in \mathbb{C}^m$ that the series is absolutely convergent. To characterize these sets we need some terminology. An open $D \subseteq \mathbb{C}^m$ is a **Reinhardt** (or **multi-circular**) **domain** (around the origin) if $(z_1, \dots, z_m) \in D$ implies $(w_1, \dots, w_m) \in D$ for every w such that $|w_j| = |z_j|$ for all j. (Contrary to the usual usage of the term "domain", we allow Reinhardt domains to be disconnected.) If the latter implication holds for every w with $|w_j| \leq |z_j|$ for all j, then we have a **complete Reinhardt domain**. An open $D \subseteq \mathbb{C}^m$ is called **logarithmically convex** if $\{x \in \mathbb{R}^m : x = (\log |z_j|), \exists z = (z_j) \in D\}$ is convex. Note that a Reinhardt domain D is logarithmically convex if and only if for any two points z, w in D with all coordinates nonzero and any real number $0 < \lambda < 1$ the point $(|z_j|^\lambda |w_j|^{1-\lambda})$ is in D. We can now express our main theorem:

Theorem 38 (Characterization of domains of convergence)**.** *A subset of $\mathbb{C}^m$ is the domain of convergence of a power series (centered at the origin) if and only if it is a logarithmically convex complete Reinhardt domain.*

The *only if* part is easy and shown in this section. The other direction is deeper and will be proved in the next section (Theorem 48). A long but elementary proof is given in [Boas, page 7–10].

Lemma 39 (Abel's lemma). *Consider a powers series $\sum a_\alpha z^\alpha$. (1) If $|a_\alpha \zeta^\alpha| \leq C$ for some point $\zeta \in \mathbb{C}^m$, some positive real C and all multi-indices α (this happens, for example, if the series is convergent at $z = \zeta$ for some linear order on multi-indices), then the power series converges normally on the polydisk $\{|z_1| < |\zeta_1|, \dots |z_m| < |\zeta_m|\}$, called the open* **silhouette** *of ζ. (2) A point $\zeta \in \mathbb{C}^m$ belongs to the domain of convergence of the power series if and only if $|a_\alpha| \leq C \prod_{1 \leq j \leq m} (|\zeta_j| + \epsilon)^{-\alpha_j}$ for some positive reals $C, \epsilon > 0$ and all multi-indices α.*

Proof. (1) The open silhouette of ζ is exhausted by compacts $\{|z_j| \leq r_j |\zeta_j|, \forall j\}$, $0 < r_j < 1$, and on each such compact we have $|a_\alpha z^\alpha| \leq C r^\alpha$. (2) Immediate from (1) combined with Cauchy's estimate on polydiscs (Theorem 22.(6)). ∎

Remark 40. We are now able to verify the equivalence between the different definitions of holomorphic functions given in Chapter 1, page 7. That the power series definition implies the other two follows from Abel's lemma combined with Theorem 22.(4). That the third definition implies the power series one is Theorem 22.(6). Finally assume a function holomorphic in the first sense. It is a remarkable fact in SCV that any function which is holomorphic in each variable separately is automatically continuous. (This is called Hartogs separate analyticity theorem [Hör, 2.2.8]. We do not prove it here.) Accepting this fact the proof of the Cauchy integral formula for polydiscs (Theorem 22.(1)) remains valid, and we again have its corollary Theorem 22.(6). ∎

Example: The domain of convergence of a power series of more than one complex variable might be empty even if the series converges on points besides its center. For example $\sum_{(n_1,n_2)\in\mathbb{N}^2} n_1! z_1^{n_1} z_2^{n_2}$ absolutely converges only on $\{0\} \times \{|z_2| < 1\}$, which has no interior point.

Exercise: Find the domain of convergence of $\sum_{n\geq 0} z_1^n z_2^{n^2}$.

Exercise: Find power series in two complex variables with the followings as their domain of convergence: (1) $\{|z_1| < 1, |z_2| < 1\}$; (2) $\{|z_1||z_2| < 1\}$; (3) $\{|z_1| < 1\}$. (4) $\{|z_1|^2 + |z_2|^2 < 1\}$. *Hint.* For the last one consider the power series $\sum_{n\geq 0} (z_1^2 + \theta_n z_2^2)^n$, $\theta_n := \exp(\sqrt{-1}n)$, and note that the sequence θ_n is dense in the unit circle [Lee, 4.20].

Proof of the only if part of Theorem 38. Consider the power series $\sum a_\alpha z^\alpha$. Abel's lemma implies that the domain of convergence is complete Reinhardt. For any two points z, w in the domain of convergence and any real $0 < \lambda < 1$, by Abel's lemma we have

$$|a_\alpha| = |a_\alpha|^\lambda |a_\alpha|^{1-\lambda} \leq \left(\frac{C}{\prod(|z_j| + \epsilon)^{\alpha_j}}\right)^\lambda \left(\frac{C}{\prod(|w_j| + \epsilon)^{\alpha_j}}\right)^{1-\lambda} \leq \frac{C}{\prod(|z_j|^\lambda |w_j|^{1-\lambda} + \epsilon')^{\alpha_j}},$$

for some positive reals C, ϵ, ϵ' and all multi-indices α. Therefore Abel's lemma again shows that the point $(|z_j|^\lambda |w_j|^{1-\lambda})$ is in the domain of convergence. *Another argument.* If $\sum a_\alpha z^\alpha$ and $\sum a_\alpha w^\alpha$ are absolutely convergent then applying Holder's inequality with conjugate exponents $1/\lambda$ and $1/(1-\lambda)$ (for any $0 < \lambda < 1$) shows that $\sum a_\alpha \zeta^\alpha$, where

$\zeta = \prod |z_j|^\lambda |w_j|^{1-\lambda}$, is also absolutely convergence. Applying this observation to some small neighborhoods of the points in the domain of convergence gives the result. ■

We end this section with several interesting theorems about holomorphic functions on Reinhardt domains, complete or not.

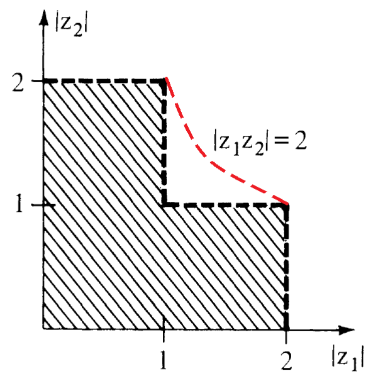
Theorem 41 (Global power series representations)**.** *Every holomorphic function on a complete Reinhardt domain has a unique power series representation valid on the whole domain.*

Proof. Uniqueness is by the identity theorem. Let f be a holomorphic function on the complete Reinhardt domain D. Note that D is a union of polydiscs P centered at the origin. By our statement of the power series representation of functions holomorphic on polydisks (Theorem 22.(6)), f on each P can be represented by a power series, but all these power series have the same coefficients because the coefficients are determined by the complex derivatives of f at the origin. ■

Theorem 42 (Hartogs extension theorem; complete Reinhardt domains version)**.** *Every holomorphic function on a complete Reinhardt domain D has a unique power series representation valid on the interior D' of the intersection of all logarithmically convex complete Reinhardt domains containing D. Clearly, D' is the smallest logarithmically convex complete Reinhardt domain containing D.*

Proof. By what we have shown so far in this section any holomorphic function f on D has a global power series representation whose domain of convergence D_f is a logarithmically convex complete Reinhardt domain containing D, so $D_f \supseteq D'$. ■

Example: The smallest logarithmically convex complete Reinhardt domain containing $\{(z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1|, |z_2|\} < 2, \min\{|z_1|, |z_2|\} < 1\}$ is $\{\max\{|z_1|, |z_2|, |z_1 z_2|\} < 2\}$.



Theorem 43 (Hartogs extension theorem; connected Reinhardt domains version)**.** *Let $D \subseteq \mathbb{C}^m$ be a connected Reinhardt domain such that for every $j = 1, \ldots, m$ there is a point in D with j-th coordinate zero. (For example this happens if D contains the origin, or D is Hartogs H (Theorem 1), or it is a spherical shell $r < |z|_2 < R$.) Then for every holomorphic function f on D there exists a power series which holomorphically extends f to the smallest complete Reinhardt domain which contains D, namely the set of points $w \in \mathbb{C}^m$ such that there exists $z \in D$ with $|w_j| \leq |z_j|$ for all j.*

Proof. We use the Laurent series expansion of holomorphic functions of several variables, given in the first exercise in page 27. Fix $f \in \mathcal{O}(D)$. Since D is Reinhardt it can be covered by closed polyannuli of the form $A_{r,R} := \{r_j \leq |\zeta_j| \leq R_j, \forall j = 1, \dots, m\}$, $0 \leq r_j < R_j$. Consider the Laurent expansion $\sum a_\alpha z^\alpha$ of f on $A_{r,R}$. Laurent coefficients are given by

$$a_\alpha = \frac{1}{(2\pi\sqrt{-1})^m} \int_{|\zeta_1|=\rho_1,\dots,|\zeta_m|=\rho_m} \frac{f(\zeta_1, \dots, \zeta_m)}{\zeta_1^{\alpha_1+1} \cdots \zeta_m^{\alpha_m+1}} d\zeta_1 \wedge \cdots \wedge d\zeta_m,$$

where $r_j < \rho_j \leq R_j$. A standard connectedness argument shows that Laurent coefficients does not depend on the chosen annulus, so f has a global Laurent series expansion valid on whole D. If $p \in A_{r,R}$ is a point with j-th coordinate zero then $r_j = 0$, so $f(\zeta_1, \dots, \zeta_m)$ is holomorphic with respect to ζ_j on the disc $|\zeta_j| < R_j$, and the integral formula above vanishes for $\alpha_j < 0$. This shows that our Laurent series is in fact a Taylor series. ■

3.4 Domains of holomorphy

After Hartogs' discovery of domains which all holomorphic functions on them can be holomorphically extended to larger ones, understanding the domains which violate this phenomenon became a central theme in SCV. The following theorem gives an intrinsic characterization of such domains, and the immediate Example 45 shows that such domains has something to do with convexity.

Theorem 44 (Cartan-Thullen)**.** *For an open $D \subseteq \mathbb{C}^m$ the followings are equivalent:*
(1) D is a **weak domain of holomorphy** *in the sense that there is no part of the boundary across which every holomorphic on D can be extended holomorphically, more precisely, it is impossible to find connected open D_2 not contained in D and nonempty open $D_1 \subseteq D \cap D_2$ such that for every $f \in \mathcal{O}(D)$ there is $f_2 \in \mathcal{O}(D_2)$ such that $f = f_2$ on D_1. Equivalently, for every $p \in \partial D$ there is $f_p \in \mathcal{O}(D)$ such that there exists no function holomorphic on a connected neighborhood U of p which agrees with f on some component of $U \cap D$.*

(2) $\mathrm{dist}(K, \partial D) = \mathrm{dist}(\hat{K}_{\mathcal{O}(D)}, \partial D)$ *for every compact $K \subseteq D$, where $\hat{K}_{\mathcal{O}(D)} := \{z \in D : |f(z)| \leq \|f\|_K, \forall f \in \mathcal{O}(D)\}$ denotes the* ***holomorphically convex hull of*** K ***in*** *D.*

(3) D is **holomorphically convex** *in the sense that for every compact $K \subseteq D$ its holomorphically convex hull $\hat{K}_{\mathcal{O}(D)}$ is compactly supported in D. (Note that by the continuity of holomorphic functions $\hat{K}$ is closed in D. Also testing the definition of $\hat{K}$ for coordinate functions $z_j \in \mathcal{O}(D)$ shows that $\hat{K}$ is bounded. Based on these two facts the expression "$\hat{K}$ is compactly supported in D" can be replaced by "$\hat{K}$ is compact" or 'by 'if a sequence of points in $\hat{K}$ converges $p \in \mathbb{C}^m$ then $p \in D$".)*

(3′) D has an exhaustion by **holomorphically convex** *compacts K_j in the sense that D is the union of a sequence K_j of compacts such that each K_j equals $\hat{K}_j$ and is contained in the interior of K_{j+1}.*

(4) For any sequence of points p_j in D which does not accumulate at any point of D there exists a holomorphic function f on D such that $\sup_j |f(p_j)| = \infty$*. (Also see Theorem 86.)*

(5) D is a **domain of holomorphy** *in the sense that there exists a holomorphic function f on D which can not be extended holomorphically across any boundary point, more precisely, it is impossible to find connected open D_2 not contained in D and nonempty open $D_1 \subseteq D \cap D_2$ and $f_2 \in \mathcal{O}(D_2)$ such that $f = f_2$ on D_1.*

Exercise: Give an open $D \subseteq \mathbb{C}$, a point p on the boundary of D, and a neighborhood U of p in $\mathbb{C}$ such that $D \cap U$ has infinitely many components.

Example 45. (1) Every open subset of the complex plane is a weak domain of holomorphy, because for every boundary point p the function $f_p(z) = (z-p)^{-1}$ is holomorphic everywhere except p but can not be extended holomorphically across p. (2) The unit open ball $|z|_2 < 1$ in $\mathbb{C}^m$ is a weak domain of holomorphy, because for every boundary point p the function $f_p(z_1, \ldots, z_m) = (1 - \sum \overline{p}_j z_j)^{-1}$ is holomorphic on the unit ball (because $|\sum \overline{p}_j z_j| < 1$ by the Cauchy-Schwarz inequality) but can not be extended holomorphically across p just because it blows up at p. More generally, every convex open $D \subseteq \mathbb{C}^m$ is a weak domain of holomorphy. To show this assume $p \in \partial D$. By convexity find [Rud-FA, 3.4] a real-valued $\mathbb{R}$-multilinear function $l(z) = \operatorname{Re} \sum a_j z_j$ on $\mathbb{C}^m \cong \mathbb{R}^{2m}$ such that the hyperplane $l(z) = l(p)$ separates D and p in $\mathbb{R}^{2m}$, in the sense that $l(z) < l(p)$ for all $z \in D$. Then $(z_1, \ldots, z_m) \mapsto (\sum a_j(z_j - p_j))^{-1}$ is a holomorphic function on D that can not be extended holomorphically across p just because it blows up at p. ■

Example: $\sum_{n \geq 0} z^{2^n}$ is a holomorphic function on the unit disk of the complex plane which can not be extended continuously, setting aside holomorphically, to any larger open. This is because all but finitely many terms of the series equal 1 for every z which is some 2^m-th, $m \in \mathbb{N}$, roots of unity, and the set of such z is dense in the boundary of the unit disk. Therefore the unit disk is a domain of holomorphy. More generally, the so-called Hadamard's gap theorem [Rud-RCA, 16.6] says that every power series of the form $\sum c_j z^{n_j}$ where $n_{j+1} > r n_j$ for some $r > 1$ and every j is holomorphic on the unit disc and can not be extended holomorphically across any point of the unit circle. Recall we showed in page 8 that every open subset of the complex plane is a domain of holomorphy.

Example: In Theorem 14.(2) we showed that every open subset of the complex plane is holomorphically convex.

Example: Typical example of opens which are not domains of holomorphy are Hartogs H (Theorem 1) and Hartogs $G \setminus K$ (Theorem 24).

Example: Consider the spherical shell $D = \{1 < |z|_2 < 3\} \subseteq \mathbb{C}^m$ and $K = \{|z|_2 = 2\}$. If $m = 1$ then $\hat{K}_{\mathcal{O}(D)} = K$ by Runge's approximation theorem because K adds no hole to D. If $m > 1$ by Hartogs extension theorem every $f \in \mathcal{O}(D)$ has holomorphic extension to $\{|z|_2 < 3\}$, so by the maximum principle $\{1 < |z|_2 \leq 2\} \subseteq \hat{K}_{\mathcal{O}(D)}$. In fact testing the definition of holomorphically convex hull with all $\exp \sum a_j z_j$, $a = (a_j) \in \mathbb{C}^m$, shows that $\hat{K}_{\mathcal{O}(D)}$ is contained in the usual convex hull of K, therefore $\hat{K}_{\mathcal{O}(D)} = \{1 < |z|_2 \leq 2\}$. This shows that D is holomorphically convex only when $m = 1$.

Exercise: If D is the Hartogs H_m (Theorem 1), find the holomorphically convex hull of $K = \{(z_1, 3/4, \ldots, 3/4) \in D : |z_1| = 3/4\}$ in D.

Exercise: (1) Show that $\{|z_1| < |z_2| < 1\} \subseteq \mathbb{C}^2$ is a weak domain of holomorphy. (2) Show that $\{0 < |z_1| < |z_2| < 1\} \subseteq \mathbb{C}^2$ is a weak domain of holomorphy. (*Hint.* Consider

$1/(z_1 - \exp(\sqrt{-1}\theta)z_2)$.)

Proof of Theorem 44. (1⇒2) This is the deepest part of the proof, and we need to introduce a new concept. For an open polydisc P centered at the origin with multi-radii $r = (r_1, \ldots, r_m)$, let $\operatorname{dist}_r(a, \partial D)$ denote the P**-distance** of $a \in D$ to the boundary, namely $\sup\{t > 0 : a + tP \subseteq D\}$. Note that the usual Euclidean distance $\operatorname{dist}(a, \partial D) = \sup\{t > 0 : a + tB \subseteq D\}$, B the unit open ball $\{|z|_2 < 1\}$ around the origin, equals the infimum of $\operatorname{dist}_r(a, \partial D)$ over $|r|_2 = 1$, $r_j > 0$. We now start the proof. Since $\hat{K}$ contains K we have $\geq$, so assume by contradiction that this inequality is strict. It means that there is a point $a \in \hat{K}$ such that $\operatorname{dist}(a, \partial D) < \eta := \operatorname{dist}(K, \partial D)$. The translation of these Euclidean distance inequalities to polydisc ones is: there exists an open polydisc P around the origin with multi-radii $r = (r_j)$, $|r|_2 = 1$, such that $\operatorname{dist}_r(a, \partial D) < \eta \leq \operatorname{dist}_r(z, \partial D)$ for every $z \in K$. The first part of this inequality says $a + \eta P \not\subseteq D$, so we are done by proving the assertion that the power series representation $f(z) = \sum c_\alpha (z - a)^\alpha$ of an arbitrary $f \in \mathcal{O}(D)$ around a converges on $a + \eta P$. Since $\eta \leq \operatorname{dist}^r(z, \partial D)$ for every $z \in K$, the closure L of $K + t\eta P$, $0 < t < 1$, taken in $\mathbb{C}^m$ is a compact subset of D, so we have the Cauchy estimate $\|f^{(\alpha)}\|_K \leq C\alpha!/(t\eta r)^\alpha$, $C := \|f\|_L$, for the α-th complex derivative of f. Since $a \in \hat{K}$ we have $|c_\alpha| = |f^{(\alpha)}(a)|/\alpha! \leq C/(t\eta r)^\alpha$. By the Abel lemma the power series $\sum c_\alpha (z - a)^\alpha$ converges on every $a + t' t\eta P$, $0 < t' < 1$, hence on $a + \eta P$.

(2⇒3) Trivial.

(3⇔3′) Assume (3). Let L_j be an exhaustion of D by compacts [Lee, A.60]. Set $K_1 := \hat{L}_1$. By the hypothesis K_1 is compact, and it is a formal exercise to show that $K_1 = \hat{K}_1$. Assuming that $K_1, \ldots, K_{j-1}$ has been defined, choose some $n_j > j$ such that K_{j-1} is contained in the interior of L_{n_j} and set $K_j := \hat{L}_{n_j}$. Now assume (3′). Each compact K of D is contained in some K_j, and then $\hat{K} \subseteq \hat{K}_j = K_j$.

(3′⇒4) Since p_j do not accumulate, after passing to a subsequence of p_j and a subsequence of K_j, we can assume $p_j \in K_{j+1} \setminus K_j$. Inductively construct $f_j \in \mathcal{O}(D)$ with two properties:

$$\|f_j\|_{K_j} < 2^{-j}, \quad |f_j(p_j)| > j + \sum_{k<j} |f_k(p_j)|.$$

Momentarily assuming this sequence of functions, the function $f := \sum f_j$ converges uniformly on compacts of D, so represents a holomorphic function on D. Furthermore, $\lim |f(p_j)| = \infty$ because

$$|f(p_j)| \geq |f_j(p_j)| - \sum_{k \neq j} |f_k(p_j)| > j - \sum_{k>j} 2^{-k} \geq j - 1.$$

To construct f_j first set $f_1 = 0$. Suppose $f_1, \ldots, f_{j-1}$ is constructed. Since $p_j \notin K_j = \hat{K}_j$, by the very definition of the holomorphically convex hull there exists $f_j \in \mathcal{O}(D)$ such that $|f_j(p_j)| < \|f_j\|_{K_j}$. Replacing f_j by a suitable multiple of it we can assume $|f_j(p_j)| < 1 < \|f_j\|_{K_j}$. Replacing f_j by a suitable positive integer power of it completes the construction.

(4⇒3) Trivial: If p_j be a sequence of points of $\hat{K}$ which converges $p \in \mathbb{C}^m$ then $p \in D$ because otherwise the contradiction $\infty = \sup |f(p_j)| \leq \|f\|_K < \infty$ happens for $f \in \mathcal{O}(D)$ given in the statement of (4).

($3'{\Rightarrow}5$) We modify the proof ($3'{\Rightarrow}4$). Find a dense sequence q_j of points in D such that every term appears infinitely many times in the sequence. Let B_j denote the largest open ball centered at q_j which is contained in D. Since B_j is not contained in any compact of D we can find a point p_j in the nonempty $(B_j \setminus K_{n_j}) \cup K_{n_j+1}$ for some integer n_j. The proof ($3'{\Rightarrow}4$) gives $f \in \mathcal{O}(D)$ such that $|f(p_j)| \to \infty$. This functions can not be holomorphically extended across any part of the boundary, because otherwise $|f|$ must be bounded on the closure of some B_j, but we know that this closure contains infinitely many of $p_1, p_2, \dots$. *Another proof.* Choose an exhaustion of D by compacts K_j, and a dense sequence ζ_j of points in D. Since $\hat{K}_j$ is compactly supported in D one can find $z_j \in D \setminus \hat{K}_j$ which is closer to ζ_j than ∂D, and then $f_j \in \mathcal{O}(D)$ such that $f_j(z_j) = 1$ and $\|f_j\|_{K_j} < 2^{-j}$. If needed refresh f_j such that it is not identically 1 in any component of D. The infinite product $f := \prod(1 - f_j)^j$ converges uniformly on compacts of D because $\sum j 2^{-j} < \infty$. (Recall that $\prod(1 + g_j)$ converges absolutely and uniformly if and only if a tail of $\sum |g_j|$ converges uniformly [Ahl, page 193].) Therefore $f \in \mathcal{O}(D)$. Note that f is not identically zero in any component of D. Also note that all derivatives of order less than j of f vanish at z_j. We assert that $f \in \mathcal{O}(D)$ can not be holomorphically extended across any point p in the boundary of D. If g is such an extension, find a subsequence ζ_{n_j} approaching p; then $z_{n_j} \to p$, so all derivatives of g vanish at p, and this makes f identically zero on some component of D.

($5{\Rightarrow}1$) Trivial. ■

Remark 46 (Thullen's lemma)**.** (1) The proof ($1{\Rightarrow}2$) above shows that, if an open $D \subseteq \mathbb{C}^m$ is not a domain of holomorphy then there exist a point $a \in D$ and an open polydisc P centered at a not contained in D such that the power series representation around a of every holomorphic function on D converges throughout P. (2) The beautiful argument used in the proof ($1{\Rightarrow}2$) above is called Thullen's lemma. The idea behind can be described in words: *The derivatives of a holomorphic function $f \in \mathcal{O}(D)$ satisfy the same bounds on $\hat{K}$ as on K, hence the power series of f centered around a point $a \in \hat{K}$ has the same domain of convergence as for points in K.* (3) The proof of Thullen's lemma also gives the following generalization: *Assume open $D \subseteq \mathbb{C}^m$, compact $K \subseteq D$, $a \in \hat{K}_{\mathcal{O}(D)}$, $f, g \in \mathcal{O}(D)$, and open polydisc $P \subseteq \mathbb{C}^m$ around the origin with multiradii r. If $\operatorname{dist}_r(z, \partial D) \geq |g(z)|$ for all $z \in K$ then the power series representation of f at a converges on the polydisc $a + |g(a)|P$.* Here is the proof. We need to show that $|f^{(\alpha)}(a)| \leq C\alpha!/(|g(a)|tr)^\alpha$ for some $C > 0$ and every $0 < t < 1$. Since $a \in \hat{K}$ and $f^{(\alpha)} g^{|\alpha|} \in \mathcal{O}(D)$ we reduce to showing $|f^{(\alpha)}(z)| \leq C\alpha!/(|g(z)|tr)^\alpha$ for every $z \in K$. But this is Cauchy's estimate for $f \in \mathcal{O}(P_z)$ on the open polydisc $P_z := z + |g(z)|tP \subseteq D$, so taking C to be the maximum of $|f|$ on the closure of $\bigcup_{z \in K} P_z$ works. ■

The following corollary of Cartan-Thullen theorem lists some ways to get new domains of holomorphy out of old, and provides a large collection of examples for domains of holomorphy besides Example 45 and Theorem 48.

Corollary 47. *(1) The class of domains of holomorphy is closed under finite products as well as taking interiors of arbitrary intersections.*

(2) Finite products of opens of the complex plane are domains of holomorphy.

(3) Let $U \subseteq \mathbb{C}^m$ be an open and let $f_1, \dots, f_k$ be finitely many holomorphic functions on U. Then $D := \{z \in U : |f_j(z)| < 1, \forall j\}$ is a domain of holomorphy if either U

is a domain of holomorphy or if the closure of D in $\mathbb{C}^m$ is contained in U. (If any of these conditions happen then D is called a **holomorphic polyhedron** *defined by the* **frame** *$f_1, \dots, f_k$. Specially, if $U = \mathbb{C}^m$ and f_j are polynomials then D is called a* **polynomial polyhedron**.*)*

(4) If $F : D \to \mathbb{C}^n$ is a holomorphic map defined on domain of holomorphy $D \subseteq \mathbb{C}^m$ then for any domain of holomorphy $D' \subseteq \mathbb{C}^n$, the set of points $z \in D$ such that $F(z) \in D'$ is a domain of holomorphy.

(5) For any open $B \subseteq \mathbb{R}^m$ the set $T = \{z \in \mathbb{C}^m : \mathrm{Re} z \in B\}$ (called the **tube** *over base B) is a domain of holomorphy if and only if all of its connected components are convex.*

Proof. (1) For the first statement it suffices to show that for every finite list D_j of holomorphically convex opens and every compacts $K_j \subseteq D_j$, the holomorphically convex hull L of $\prod K_j$ is contained in $\prod \hat{K}_j$. Assuming any point (z_j) in L and any $f_j \in \mathcal{O}(D_j)$, since $z \in \prod D_j$ mapped to $f_j(z_j)$ is holomorphic it follows that $|f_j(z_j)| \leq \|f_j\|_{K_j}$, hence $z_j \in \hat{K}_j$. For the second statement assume a family $\{D_\alpha\}$ of holomorphically convex domains whose intersection has nonempty interior D, and let K be a compact subset of D. Since $\hat{K}_{\mathcal{O}(D)} \subseteq \hat{K}_{\mathcal{O}(D_\alpha)}$ and $\mathbb{C}^m \setminus D \supseteq \mathbb{C}^m \setminus D_\alpha$ it follows that

$$\delta := \mathrm{dist}\,(K, \partial D) = \mathrm{dist}\,(K, \mathbb{C}^m \setminus D) \leq \mathrm{dist}\,(K, \mathbb{C}^m \setminus D_\alpha) = \\ \mathrm{dist}\,(K, \partial D_\alpha) = \mathrm{dist}\left(\hat{K}_{\mathcal{O}(D_\alpha)}, \partial D_\alpha\right) \leq \mathrm{dist}\left(\hat{K}_{\mathcal{O}(D)}, \partial D_\alpha\right).$$

Since this is true for every α we can deduce that $\mathrm{dist}(\hat{K}_{\mathcal{O}(D)}, \partial D) \geq \delta$. The reverse inequality is trivial.

(2) In Example 45 we showed that opens of the complex plane are weak domains of holomorphy. Now use part (1).

(3) Let $K \subseteq D$ be compact. We have

$$\hat{K}_{\mathcal{O}(D)} \subseteq \hat{K}_{\mathcal{O}(U)} \subseteq \{z \in U : |f_j(z)| \leq \|f_j\|_K < 1, \forall j\} \subseteq D.$$

If U is a domain of holomorphy then $\hat{K}_{\mathcal{O}(U)}$ is compact, and the inclusion above shows that $\hat{K}_{\mathcal{O}(D_1)}$ is compactly supported in D. If $\overline{D} \subseteq U$ then the right hand side of the inclusion

$$\hat{K}_{\mathcal{O}(D)} \subseteq \{z \in U : |f_j(z)| \leq \|f_j\|_K < 1, \forall j\},$$

is compactly supported in D. *Another proof.* For any $p \in \partial D$ some f_j must be of modulus 1 at p, and then $z \mapsto (f_j(z) - f_j(p))^{-1}$ can not be extended holomorphically across p.

(4) Set $D_F := \{z \in D : F(z) \in D'\}$. Considering compact $K \subseteq D_F$ and a sequence p_j of points in $\hat{K}_{\mathcal{O}(D_F)}$ which converges to $p \in \mathbb{C}^m$ we need to show that $p \in D_F$. Since $\hat{K}_{\mathcal{O}(D_F)} \subseteq \hat{K}_{\mathcal{O}(D)}$ and D is a domain of holomorphy we have $p \in D$, so we are left to show that $F(p) \in D'$. Since $F(p_j) \to F(p)$ and D' is a domain of holomorphy we need to show $F(p_j) \in \widehat{F(K)}_{\mathcal{O}(D')}$. This is clear because for any $f \in \mathcal{O}(D')$ we have

$$|f(F(p_j))| = |(f \circ F)(p_j)| \leq \|f \circ F\|_K = \|f\|_{F(K)}.$$

(5) One can easily reduce to the case which B is connected. If B is convex then T is convex, so a weak domain of holomorphy by Example 45. For the other direction Bochner

[BM, page 90–102][Hör, 2.5.10][Gun, volume I, page 20] proved the stronger statement that holomorphic functions on T can be holomorphically extended to the tube over the (linear) convex hull of B. Refer Example 63 for another proof. ■

Exercise: Show that every domain of holomorphy $D \subseteq \mathbb{C}^m$ has an exhaustion by compacts $K_j = \overline{D}_j$ with each D_j a holomorphic polyhedron defined by a frame of functions holomorphic on D. (*Hint.* First show that any holomorphically convex compact K of D has a neighborhood basis consisting of holomorphic polyhedra, namely for any neighborhood $U \subseteq \mathbb{C}^m$ of K one can find a holomorphic polydehron sandwiched between K and U.)

Theorem 48 (Characterization of domains of convergence)**.** *A complete Reinhardt domain in $\mathbb{C}^m$ is a domain of holomorphy if and only if it is the domain of convergence of a power series if and only if it is logarithmically convex.*

Proof. Let $D \subseteq \mathbb{C}^m$ be a complete Reinhardt domain. We already proved that the domain of convergence of power series are logarithmically convex (page 44). If D is a domain of holomorphy, assuming a holomorphic function f on D which can not be extended holomorphically across any boundary point, the global power series representation of f obtained in Theorem 41 has exactly D as its domain of convergence. (Note that D is complete Reinhardt.) It remains to assume logarithmic convexity and deduce holomorphic convexity because Cartan-Thullen says that holomorphically convex domain are domains of holomorphy. Assuming an arbitrary compact $K \subseteq D$ and a point p in the closure in $\mathbb{C}^m$ of the holomorphically convex hull of K, we need to show that $p \in D$. Since K is compact it is contained in the union of open silhouettes of finitely many points in D all their coordinates nonzero. Let F denote the collection of these finitely many points. Maybe after permuting the coordinate axes we can assume that $p_1, \dots, p_n$ are the only nonzero coordinates of p. By the very definition of the holomorphically convex hulls we have $|\prod_{1 \le i \le n} p_i^{\alpha_i}| \le \max_{\zeta \in F} |\prod_{1 \le i \le n} \zeta_i^{\alpha_i}|$ for all multi-indices. After taking logarithm

$$\sum_{1 \le i \le n} \alpha_i \log |p_i| \le \max_{\zeta \in F} \sum_{1 \le i \le n} \alpha_i \log |\zeta_i|,$$

for all nonnegative integers α_i, hence for all nonnegative reals α_i by the density of rationals in reals. In geometric terms this inequality says that the point $(\log |p_i|) \in \mathbb{R}^n$ lies in the convex hull of the set of all points $(\eta_i) \in \mathbb{R}^n$ such that $\eta_i \le \log |\zeta_i|$, $i = 1, \dots, n$, for some $\zeta \in F$. Since the projection of $D \subseteq \mathbb{C}^m$ into the first n coordinates is complete Reinhardt and logarithmically convex, there exists a point $q \in D$ such that $|p_i| = |q_i|$ for every $i = 1, \dots, n$. Therefore p lies in the closed silhouette of q, hence $p \in D$. ■

Example: $\{(z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1|, |z_2|\} < 1, \min\{|z_1|, |z_2|\} < 1/2\}$ is a complete Reinhardt domain which is not holomorphically convex .

3.5 Some geometric conditions that domains of holomorphy must satisfy

Early investigators of SCV found several geometric properties satisfied by all domains of holomorphy. We discuss some of them in this section. A deep fact, called **Levi's**

problem, says that these necessary conditions are also sufficient for opens satisfying some mild smoothness condition on their boundary. We start this section by reviewing some related differential geometry notions.

3.5.1 Preliminaries: Domains with smooth boundary, Complex submanifolds of $\mathbb{C}^m$

Consider an open $D \subseteq \mathbb{C}^m$ and the usual identification $\mathbb{C}^m \cong \mathbb{R}^{2m}$ via $z_j = x_j + \sqrt{-1}x_{j+m}$. We say D is **of class** C^k **at** $p \in \partial D$, $k \in \{1, \ldots, \infty, \omega\}$, if there is a C^k function $r : U \to \mathbb{R}$ defined on some neighborhood $U \subseteq \mathbb{R}^{2m}$ of p such that $D \cap U = \{x \in U : r(x) < 0\}$ and the gradient dr does not vanish at p. (C^ω stands for real analytic functions.) If so, r is called a C^k **local defining function** for D at p, and we have $\partial D \cap U = \{x \in U : r(x) = 0\}$ and $U \setminus \overline{D} = \{x \in U : r(x) > 0\}$. If ρ is another defining function then it is straightforward to show that $r = f\rho$ where f is a positive-valued C^{k-1} function [Ran, page 51]. After a linear change of coordinates one can assume $dr_p = dx_1$ (namely $\partial r / \partial x_j(p)$ equals 1 for $j = 1$ and 0 otherwise.), so by the implicit function theorem we can find another defining function of the form $\rho(x) = x_1 - g(x_2, \ldots, x_{2m})$ where g is a C^k function satisfying $dg_p = 0$. Therefore, under the local change of coordinates $(x_1, \ldots, x_{2m}) \mapsto (x_1 - g, x_2, \ldots, x_{2m})$ the open D is locally represented by $\{x_1 < 0\}$. In the definition of the local defining function if U contains ∂D then r is called a C^k **global defining function** for D, and D is said to be **of class** C^k. This happens exactly when ∂D is an embedded C^k submanifold of $\mathbb{R}^{2m}$, and we sometimes express this situation by saying that D **has** C^k **boundary**. If this happens then by the smooth partition of unity one can assume $U = \mathbb{R}^{2m}$ [Ran, page 52]. Another fact which follows from the implicit function theorem is that if D is of class C^k, $k \in \{2, 3, \ldots, \infty\}$, then

$$r(z) := \begin{cases} -\mathrm{dist}(z, \partial D), & z \in D \\ \mathrm{dist}(z, \partial D), & z \in \mathbb{C}^m \setminus D \end{cases},$$

is a C^k defining function on some neighborhood of the boundary [Foo][Gun, volume I, page 139][Lee, pages 138–140]. (Example: (1) If D is the unit ball then $\mathrm{dist}(z, \partial D)$ is not smooth at the origin; (2) The distance function for the C^1 curve $y = |x|^{3/2}$ in $\mathbb{R}^2$ is not C^1 at any point on the y-axis.)

To every boundary point p one can associate two **real and complex tangent spaces** in the following way:

$$T_p(\partial D) := \left\{ t \in \mathbb{R}^{2m} \,\middle|\, dr_p(t) := \sum_{j=1}^{2m} \frac{\partial r}{\partial x_j}(p) t_j = 0 \right\} = \mathrm{Ker} dr_p,$$

$$T_p^{\mathbb{C}}(\partial D) := \left\{ t \in \mathbb{C}^m \,\middle|\, \partial r_p(t) := \sum_{j=1}^{m} \frac{\partial r}{\partial z_j}(p) t_j = 0 \right\} = \mathrm{Ker} \partial r_p$$

If one naturally considers $\mathbb{R}^{2m} \cong \mathbb{C}^m$ as a complex vector space, then $T_p^{\mathbb{C}}$ is the set of all $t \in \mathbb{C}^m$ such that t and $\sqrt{-1}t$ both belong to T_p. This follows from the trivial observations that dr is the real part of ∂r, and $\partial r(t)$ is complex linear with respect to t:

$$dr(t) = \mathrm{Re}(\partial r(t)), \quad dr(\sqrt{-1}t) = \mathrm{Re}(\partial r(\sqrt{-1}t)) = \mathrm{Re}(\sqrt{-1}\partial r(t)) = -\mathrm{Im}(\partial r(t)).$$

In other words $T_p^{\mathbb{C}}$ is the largest complex vector subspace of $\mathbb{C}^m$ that lives inside T_p. It is of complex dimension $m-1$, so comes into the play only when $m>1$.

Exercise: Show that the tangent spaces does not depend on the defining functions used in their definition.

A subset $M \subseteq \mathbb{C}^m$ is called a k-dimensional (embedded) **complex submanifold of** $\mathbb{C}^m$ if it satisfies any of the following equivalent conditions:

1. M can be locally holomorphically flattened in the sense that every point of M has a neighborhood $U \subseteq \mathbb{C}^m$ and a biholomorphic map F from some open $V_1 \times V_2$ of $\mathbb{C}^k \times \mathbb{C}^{m-k}$ to U such that $M \cap U$ is given by the image of $V_1 \times \{0\}^{m-k}$ under F. Equivalently, every point of M has a neighborhood $U \subseteq \mathbb{C}^m$ and a biholomorphic map G from U to some open of $\mathbb{C}^m$ such that $M \cap U$ is given by $\{z \in U : G_j(z) = 0 \text{ for } j = k+1, \dots, m\}$.

2. M can be locally holomorphically parametrized in the sense that every point p of M has a neighborhood $U \subseteq \mathbb{C}^m$ and a holomorphic map $f : V \to \mathbb{C}^m$ defined on some open $V \subseteq \mathbb{C}^k$ such that $f(V) = M \cap U$ and the complex Jacobian of f is of maximal rank k at a point in $f^{-1}(p)$.

3. M is given locally by independent holomorphic equations in the sense that every point of M has a neighborhood $U \subseteq \mathbb{C}^m$ and a holomorphic function map $g : U \to \mathbb{C}^{m-k}$ such that $M \cap U$ is given by $\{z \in U : g(z) = 0\}$ and the complex Jacobian of g is of maximal rank $m-k$ at p.

The proof that these conditions are equivalent. We need the holomorphic version of the classical implicit function theorem stated in Theorem 49. (1⇒2,3) Trivial. (2⇒1) After permuting coordinates one can assume that the first k rows $\partial f_j/\partial z$, $j = 1, \dots, k$, of the complex Jacobian of f evaluated at p are independent. Consider the map $F(z) := f(z_1, \dots, z_k) + z_{k+1}e_{k+1} + \cdots + z_m e_m$ defined on $V \times \mathbb{C}^{m-k} \subseteq \mathbb{C}^m$, where $e_l \in \mathbb{C}^m$ has 1 in l-th place and zero elsewhere. By the holomorphic inverse function theorem F is a local biholomorphism around $(q, 0, \dots, 0)$, and we have the first statement in (1) after shrinking V. (3⇒2) After permuting coordinates one can assume that the last $m-k$ columns $\partial g/\partial z_j$, $j = k+1, \dots, m$, of the complex Jacobian of g evaluated at p are independent. Consider the map $G(z) := (z_1, \dots, z_k, g(z))$ defined on U with values in $\mathbb{C}^m$. By the holomorphic inverse function theorem G is a local biholomorphism around p, and we have the second statement in (1) after shrinking U. ■

Theorem 49 (Holomorphic inverse and implicit function theorems)**.** *(1) Assuming a holomorphic map $F : D \to \mathbb{C}^m$ defined on an open $D \subseteq \mathbb{C}^m$, if the $m \times m$ complex Jacobian of F is invertible at a point $p \in D$ then F is a biholomorphism on some neighborhood of p. In words: $w = F(z)$ is a* **holomorphic change of coordinates around** p.

(2) Assuming the natural splitting $\mathbb{C}^{m+n} = \mathbb{C}^m \times \mathbb{C}^n$ coordinated by (z, w), a holomorphic function $F : D \to \mathbb{C}^n$ defined on some open of $\mathbb{C}^{m+n}$ and a point $(p, q) \in D$ such that $F(p, q) = 0$, if the $n \times n$ complex Jacobian $\partial F/\partial w$ is invertible at (p, q) then there are neighborhoods $U \subseteq \mathbb{C}^m$ around p and $V \subseteq \mathbb{C}^n$ around q such that for every $z \in U$ there exists a unique $w = f(z) \in V$ such that $F(z, w) = 0$, and furthermore $f : U \to V$ is holomorphic. In words: Locally around (p, q) the implicit equation $F(z, w) = 0$ is solved explicitly by $w = f(z)$ for some holomorphic function f.

Proof. (1) Coordinate the domain and codomain of F by $w = F(z)$, $w = (w_j)$, $z = (z_j)$, $w_j = u_j + \sqrt{-1}v_j$, $z_j = x_j + \sqrt{-1}y_j$. The Jacobian determinant of the real version of F, $(x, y) \mapsto (u, v)$, is related to the complex Jacobian determinant of F via

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = \det \begin{bmatrix} u_x + \sqrt{-1}v_x & -v_x + \sqrt{-1}u_x \\ v_x & u_x \end{bmatrix} =$$
$$\det \begin{bmatrix} u_x + \sqrt{-1}v_x & 0 \\ v_x & u_x - \sqrt{-1}v_x \end{bmatrix} = \left|\det(u_x + \sqrt{-1}v_x)\right|^2 = |\det(w_z)|^2,$$

so it is nonzero at $z = p$. By the classical inverse function theorem F is locally a smooth function with smooth inverse [Apo, 13.16][Rud-PMA, 9.24]. It remain to check that the local inverse $z = G(w)$ satisfies Cauchy-Riemann equations. Differentiating the identity $z = GF(z)$ with respect to $\overline{z}$ gives

$$0 = G_w F_{\overline{z}} + G_{\overline{w}} \overline{F}_{\overline{z}} = G_{\overline{w}} \overline{F}_{\overline{z}}.$$

Since $\overline{F}_{\overline{z}}$ being the conjugate of F_z is invertible at p it follows that $G_{\overline{w}} = 0$ at $F(p)$.

(2) Apply (1) to $(z, w) \mapsto (z, F(z, w))$. ■

For other proofs of the holomorphic implicit function theorem refer [KP, 2.3.1] (by Cauchy majorant method) or [Gun, volume I, page 19] (by Rouche's theorem and induction).

3.5.2 Condition I: Hartogs pseudoconvexity

Consider the following compact subsets of $\mathbb{C}^m$:

$$\Gamma := \{|z_j| = 0 \text{ for } j < m, |z_m| \leq 1\} \cup \{|z_j| = 0 \text{ for } j < m-1, |z_{m-1}| \leq 1, |z_m| = 1\},$$

$$\hat{\Gamma} := \{|z_j| = 0 \text{ for } j < m-1, |z_{m-1}| \leq 1, |z_m| \leq 1\}.$$

Note that the two edges of Γ are of real dimensions 2 and 3. By Hartogs extension theorem (Theorem 1) every holomorphic function on Γ can be holomorphically extended to $\hat{\Gamma}$. Note that $\hat{\Gamma}$ is the holomorphically convex hull of Γ in $\mathbb{C}^m$, and equals Γ only when $m = 1$. The pair $(\Gamma, \hat{\Gamma})$ is called the **Hartogs frame** in $\mathbb{C}^m$. Any biholomorphic image of the Hartogs frame is called a **Hartogs figure**, namely a pair of compacts $(\Gamma', \hat{\Gamma}')$ in $\mathbb{C}^m$ equipped with a biholomorphic map $F : \hat{\Gamma} \to \hat{\Gamma}'$ such that $F(\Gamma) = \Gamma'$. Clearly, every Hartogs figure also satisfies Hartogs extension theorem.

Theorem 50. *Every domain of holomorphy $D \subseteq \mathbb{C}^m$ is* **Hartogs pseudoconvex** *in the sense that for every Hartogs figure $(\Gamma', \hat{\Gamma}')$ in $\mathbb{C}^m$, if D contains Γ' it should also contain $\hat{\Gamma}'$.*

Proof. By contradiction let $(\Gamma', \hat{\Gamma}')$ be a Hartogs figure such that $\Gamma' \subseteq D \not\supseteq \hat{\Gamma}'$. By Hartogs extension theorem, every function holomorphic on D can be extended holomorphically across any point $p \in \hat{\Gamma}' \cap \partial D$. ■

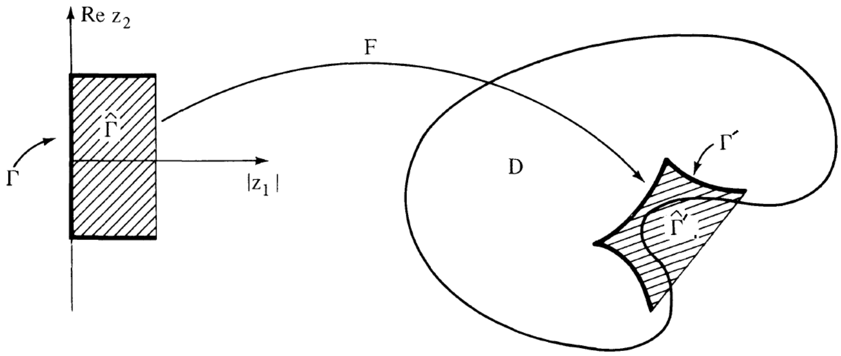


Figure 3: D is not Hartogs pseudoconvex [Ran, page 50].

To motivate the next geometric property of domains of holomorphy, note that an open $D \subseteq \mathbb{R}^n$ is convex if and and only if $\operatorname{dist}(L, \partial D) = \operatorname{dist}(\partial L, \partial D)$ for line segment L in D. By a **holomorphic disc** S in D we mean the range of a D-valued continuous map defined on the closed unit disc of the complex plane which is holomorphic on the open unit disc. The image of the unit circle under the defining map is called the boundary of S and denoted by ∂S.

Theorem 51. *Every domain of holomorphy $D \subseteq \mathbb{C}^m$ has the property that* $\operatorname{dist}(S, \partial D) = \operatorname{dist}(\partial S, \partial D)$ *for every holomorphic disc S in D.*

Proof. By the maximum principle S is contained in the holomorphically convex hull $\widehat{\partial S}$ of ∂S in D, so $\operatorname{dist}(S, \partial D) \geq \operatorname{dist}(\widehat{\partial S}, \partial D)$. Combined with Cartan-Thullen we have $\operatorname{dist}(S, \partial D) \geq \operatorname{dist}(\partial S, \partial D)$. The other direction is trivial. ■

3.5.3 Condition II: Convexity with respect to holomorphic curves

To motivate the second geometric property of domains of holomorphy, note that every convex open subset D of $\mathbb{R}^n$ has the property that for any point p in the boundary of D there is no line segment L containing p such that $L \setminus \{p\} \subseteq D$. Here is a holomorphic analog of this observation:

Theorem 52. *Every Hartogs pseudoconvex open $D \subseteq \mathbb{C}^m$ with C^1 boundary is* **convex with respect to holomorphic curves** *in the sense that for any point p in the boundary of D there is no complex one-dimensional manifold L containing p such that $L \setminus \{p\} \subseteq D$. (Therefore, by Theorem 50 every domain of holomorphy in $\mathbb{C}^m$ with C^1 boundary is convex with respect to holomorphic curves.)*

Proof. Set the trivial case $m = 1$ aside. Assuming that such an L exists we find a Hartogs figure $(\Gamma', \hat{\Gamma}')$ such that $\Gamma' \subseteq D \not\supseteq \hat{\Gamma}'$. Let r be a C^1 local defining function around p. After a holomorphic change of coordinates one can assume that L is given by $\{(z_1, 0, \ldots, 0) : |z_1| < 2\delta\}$ around $p = 0$. Next we are going to find a holomorphic change of coordinates around p that simplifies r without changing L. Since under a holomorphic change of coordinates $z = z(w)$ around p the components of ∂r transform by $\partial r/\partial z_j = \partial w_\alpha/\partial z_j \partial r/\partial w_\alpha$, it follows that by choosing any $m \times m$ invertible matrix

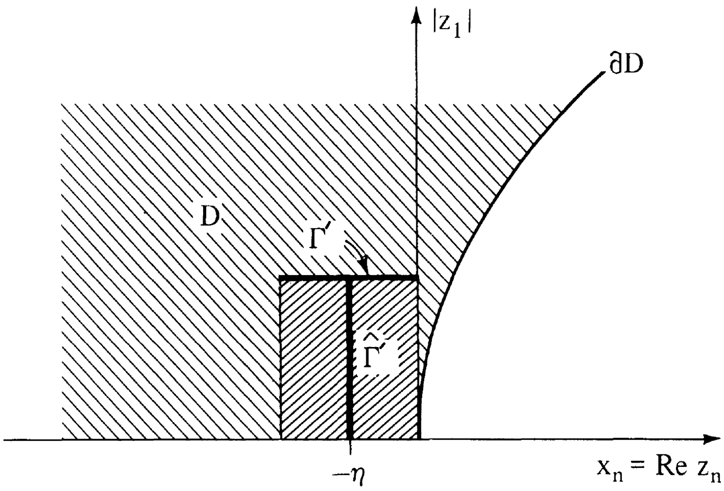


Figure 4: Proof of Proposition 52 [Ran, page 54].

A with first row $(1, 0, \dots, 0)$ and last row $\frac{1}{2}\partial r(p)/\partial z_j$, the complex linear change of coordinates $z \mapsto Az$ does not change L but we can now assume $dr_p = dx_m$. By the implicit function theorem there is a neighborhood $U \subseteq \mathbb{C}^m$ of p such that

$$L \cap U = \{(z_1, 0, \dots, 0) \in U : |z_1| < 2\delta\}, \quad r(z) = x_m - f(z_1, \dots, z_{m-1}, y_m),$$

where $x_m = \mathrm{Re} z_m$, $y_m = \mathrm{Im} z_m$, and f is a C^1 function around p such that $df_p = 0$. Note that since $(L \cap U) \setminus \{p\} \subseteq D$ it follows that $r(z_1, 0, \dots, 0) < 0$ for $0 < |z_1| < 2\delta$. For positive parameter $\eta > 0$ consider:

$$K_1 := \{z \in \mathbb{C}^m : |z_1| = \delta, |z_m + \eta| \leq \eta, z_j = 0 \text{ for } 1 < j < m\},$$

$$K_2 := \{z \in \mathbb{C}^m : |z_1| \leq \delta, z_m = -\eta, , z_j = 0 \text{ for } 1 < j < m\},$$

$$\Gamma' := K_1 \cup K_2,$$

$$\hat{\Gamma}' := \{z \in \mathbb{C}^m : |z_1| \leq \delta, |z_m + \eta| \leq \eta, z_j = 0 \text{ for } 1 < j < m\}.$$

Since r is continuous, taking η sufficiently small, K_1, K_2, Γ' are contained in $D \cap U$, but $\hat{\Gamma}' \not\subseteq D$ because it contains p. Finally note that the assignment

$$w \mapsto z, \quad z_1 = \delta w_m, \quad z_m = \eta w_{m-1} - \eta, \quad z_j = w_{j-1} \text{ for } j = 2, \dots, m-1,$$

maps $(\Gamma', \hat{\Gamma}')$ biholomorphically to the Hartogs frame. ■

3.5.4 Condition III: Levi pseudoconvexity

Consider open $D \subseteq \mathbb{C}^m$ with C^2 defining function r. As usual we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ via $z_j = x_j + \sqrt{-1}x_{j+m}$. It is intuitively clear that D is convex if and only if all the principal curvatures of the boundary hypersurface ∂D are nonnegative. (Proved in [Hör-Conv, 2.1.28][Spi, volume III].) On the other hand it is famous [Tay, volume II, page 564] that the principal curvatures of a hypersurface $r = 0$ at a point p are given by the eigenvalues of the expression $\sum \partial^2 r/\partial x_j x_k(p) t_j t_k$ as a quadratic form over tangent vectors t.[6] Therefore, D is convex if and only if the **real Hessian** of r is nonnegative at the boundary points with respect to the real tangent vectors, namely

$$\sum_{j,k=1}^{2m} \frac{\partial^2 r}{\partial x_j x_k}(p) t_j t_k \geq 0, \quad \forall p \in \partial D \ \forall t \in T_p(\partial D).$$

If the inequality is strict for $t \in T_p(\partial D) \setminus \{0\}$ then D is called **strongly convex**. Geometrically this means that principal curvatures of the boundary are all strictly positive. The open D is called **Levi pseudoconvex** if the **complex Hessian** (or the **Levi form**) of r is nonnegative at the boundary points with respect to the complex tangent vectors, namely

$$L_p(r;t) := \sum_{j,k=1}^{m} \frac{\partial^2 r}{\partial z_j \bar{z}_k}(p) t_j \bar{t}_k \geq 0, \quad \forall p \in \partial D \ \forall t \in T_p^{\mathbb{C}}(\partial D).$$

If the inequality is strict for $t \in T_p^{\mathbb{C}}(\partial D) \setminus \{0\}$ then D is called **strongly Levi pseudoconvex**. Using local defining functions all these notions can be formulated for each points on the boundary. Here are some facts:

- The notion of (strongly) Levi pseudoconvex does not depend on the defining function used, because assuming another defining function fr where f is a smooth positive function, by the chain rule

  $$L_p(fr;t) = f(p) L_p(r;t) + 2\mathrm{Re}\, \langle \partial f_p(t), \partial r_p(t) \rangle, \quad \forall t \in \mathbb{C}^m,$$

  where $\langle a, b \rangle = \sum a_j \bar{b}_j$ is the standard inner product on $\mathbb{C}^m$.

- The notion of (strongly) Levi pseudoconvex is invariant under biholomorphic maps. This is because the implication

  $$\sum r_{z_j} t_j = 0 \Rightarrow \sum r_{z_j \bar{z}_k} t_j \bar{t}_k \geq 0,$$

  under the holomorphic change of coordinates $z = z(w)$ translates into

  $$\sum r_{w_\alpha} \frac{\partial w_\alpha}{\partial z_j} t_j \Rightarrow \sum r_{w_\alpha \overline{w}_\beta} \frac{\partial w_\alpha}{\partial z_j} t_j \frac{\partial \overline{w}_\beta}{\partial \bar{z}_k} \bar{t}_k \geq 0,$$

  which is Levi pseudoconvexity in w-coordinates with respect to the tangent vectors $t' = (t'_\alpha)$, $t'_\alpha = \sum \partial w_\alpha/\partial z_j t_j$. More conceptually, under a biholomorphic map F defined locally around $p \in \partial D$ we have

  $$\partial r_p(t) = \partial \left(r \circ F^{-1}\right)_{F(p)} (F'(p)t), \quad L_p(r;t) = L_{F(p)} \left(r \circ F^{-1}; F'(p)t\right), \tag{3.13}$$

  for all $t \in \mathbb{C}^m$, where F' is the complex Jacobian matrix of F.

[6] More concretely, after translating and rotating coordinates we can assume p is the origin, $x_1 = 0$ is the tangent space at p, and $r = 0$ is given locally by $x_1 = f(x_2, \ldots, x_{2m})$. Then the principal curvatures are the eigenvalues of $\sum_{2 \leq j,k \leq 2m} \partial^2 f/\partial x_j x_k(p) t_j t_k$ as a quadratic form over $t \in \mathbb{R}^{2m-1}$.

- Convexity implies Levi pseudoconvexity. This is also true for the strong version. Here is the reason. By the chain rule the real and complex Hessians are related by

$$\underbrace{\sum_{j,k=1}^{2m} \frac{\partial^2 r}{\partial x_j x_k} dx_j dx_k}_{\text{real Hessian } R(r;dx)} = 2 \underbrace{\sum_{j,k=1}^{m} \frac{\partial^2 r}{\partial z_j \overline{z}_k} dz_j d\overline{z}_k}_{\text{complex Hessian } L(r;dz)} + 2\mathrm{Re} \underbrace{\sum_{j,k=1}^{m} \frac{\partial^2 r}{\partial z_j z_k} dz_j dz_k}_{\text{holomorphic quadratic } Q(r;dz)}, \tag{3.14}$$

  where $dz_j = dx_j + \sqrt{-1} dx_{j+m}$. Convexity means R is nonnegative at boundary points p with respect to real tangent vectors $dx \in T_p$. If this happens then for any complex tangent vectors $dz \in T_p^{\mathbb{C}}$, since dz and $\sqrt{-1}dz$ both belong to T_p it follows that $2L + 2\mathrm{Re}Q$ and $2L - 2\mathrm{Re}Q$ are both nonnegative, so $2L \geq 0$.

Note that based on the representation (3.14) the Taylor expansion of r around the point p is given by

$$r(z) = r(p) + 2\mathrm{Re}(\partial r_p(t-p)) + \mathrm{Re}Q_p(r; t-p) + L_p(r; t-p) + o\left(|z-p|_2^2\right).$$

Theorem 53. *Every C^2 open $D \subseteq \mathbb{C}^m$ which is convex with respect to holomorphic curves is Levi pseudoconvex. (Therefore, by Theorem 52 every domain of holomorphy in $\mathbb{C}^m$ with C^2 boundary is Levi pseudoconvex.)*

Proof. By contraction let $r(z)$ be a C^2 local defining function around the boundary point $p = 0$ such that the complex Hessian $L_p(r;t)$ is negative for some complex tangent vector $t \in \mathrm{Ker}\partial r_p$. Since $\partial r(t)$ and $L(r;t)$ transform similarly under holomorphic change of coordinates (equations (3.13) above), after a complex linear change of coordinates around p one can assume that

$$t = (1, 0, \ldots, 0) \quad \text{or equivalently} \quad \frac{\partial r}{\partial z_1}(p) = 0 \text{ and } A := -\frac{\partial^2 r}{\partial z_1 \partial \overline{z}_1}(p) > 0. \tag{3.15}$$

We are going to find another holomorphic change of coordinates around p that simplifies r without changing (3.15). Since under a holomorphic change of coordinates $z = z(w)$ around p the components of ∂r transform by $\partial r/\partial z_j = \partial w_\alpha / \partial z_j \partial r / \partial w_\alpha$, it follows that by choosing any $m \times m$ invertible matrix A with first row $(1, 0, \ldots, 0)$ and second row $\partial r(p)/\partial z_j$, the complex linear change of coordinates $z \mapsto Az$ does not change (3.15) but we can now assume

$$\partial r_p = dz_2.$$

By the Taylor theorem (also recall (3.14))

$$r(z_1, z_2, 0, \ldots, 0) = 2\mathrm{Re}\zeta - A|z_1|^2 + O\left(|z_1||z_2| + |z_2|^2\right) + o\left(|z_1|^2 + |z_2|^2\right), \ (z_1, z_2) \to (0,0),$$

where $\zeta = \zeta(z_1, z_2) = z_2 + 2Q(r; (z_1, z_2, \ldots, 0))$ is the sum of z_2 with the holomorphic quadratic appearing in the real Hessian of r. Since $\partial \zeta / \partial z_2(0) = 1$ is nonzero the holomorphic implicit function theorem says that the equation $\zeta = 0$ describes around p a holomorphic curve L parametrized by a holomorphic function $z_2 = z_2(z_1)$. On this curve z_2 equals a quadratic expression in terms of z_1 and z_2, so $r(z) = -A|z_1|^2 + o(|z_1|^2)$ and z_1 never vanishes on $L \setminus \{p\}$ maybe after shrinking L. This implies $r(z) < 0$ on L, namely $L \setminus \{p\} \subseteq D$. ∎

Exercise: Consider the open

$$D := \left\{ z \in \mathbb{C}^m \middle| r(z) := \sum_{j=1}^{m} |z_j|^{p_j} - 1 < 0 \right\},$$

where $p_j > 2$ are constants. (1) Show that D is Levi pseudoconvex. (2) Find the points on the boundary of ∂D which are not strongly Levi pseudoconvex.

3.5.5 Condition IV: Existence of complete Kähler metrics

Theorem 54. *(1) Every domain of holomorphy $D \subseteq \mathbb{C}^m$ has a real analytic strictly plurisubharmonic exhaustion function*[7]*. (2) Every domain of holomorphy $D \subseteq \mathbb{C}^m$ has a complete real analytic Kähler metric.*

Proof. The argument is a variation of the proof (3⇒4) in Cartan-Thullen. Exhaust D by holomorphically convex compacts K_j. Since D is holomorphically convex and ∂K_{j+1} is compact one can find finitely many holomorphic functions $f_{j1}, \ldots, f_{jk_j}$ on D such that $\|f_{jk}\|_{K_j} < 1 < |f_{jk}(z)|$ for $z \in \partial K_{j+1}$. Raising to appropriate positive integer powers one can assume that $F_j := \sum |f_{jk}|^2$ satisfies $\|F_j\|_{K_j} < 2^{-j}$ and $|F_{jk}(z)| > j$ for $z \in \partial K_{j+1}$. We assert that $F := \sum F_j = \sum |f_{jk}|^2$ is a real analytic plurisubharmonic exhaustion function for D. To see this consider the polarization $\mathcal{F}(z,w) := \sum f_{jk}(z)\overline{f_{jk}(\overline{w})}$ of F on $D' := \{(z, \overline{w}) \in \mathbb{C}^m \times \mathbb{C}^m : z, w \in D\}$. By the Cauchy-Schwarz inequality the infinite series defining $\mathcal{F}$ converges uniformly on compacts, so $\mathcal{F}$ is holomorphic on D', hence $F(z) = \mathcal{F}(z, \overline{z})$ is real analytic. The complex Hessian of F is given by $L(F;t) = \sum |\sum_l \partial f_{jk}/\partial z_l t_l|^2$, so F is plurisubharmonic. Therefore $\varphi := F + |z|_2^2$ works as our desired function.

(2) Since φ (constructed in the previous part) is real analytic and strictly plurisubharmonic so it works as a (global) Kähler potential, namely $h_{\alpha\overline{\beta}} := \partial^2\varphi/\partial z_\alpha \partial \overline{z}_\beta$ gives a Kähler metric $h = \sum h_{\alpha\overline{\beta}} dz_\alpha d\overline{z}_\beta$. To prove the completeness of the metric we show that the length of a smooth curve $\gamma : [0, \infty) \to D$, $\gamma(t) \to \partial D$, measured with respect to the metric h is infinity. The length of γ equals

$$\int_0^\infty \sqrt{\sum_{j,k} \left(\left| \sum_\alpha \frac{\partial f_{jk}}{\partial z_\alpha} \frac{d\gamma_\alpha}{dt}(t) \right|^2 + \left| \sum_\alpha \frac{d\gamma_\alpha}{dt}(t) \right|^2 \right)}\, dt,$$

so is no smaller than

$$\int_0^\infty \left| \sum_\alpha \frac{\partial f_{jk}}{\partial z_\alpha} \frac{d\gamma_\alpha}{dt}(t) \right| dt \geq |f_{jk} \circ \gamma(\infty) - f_{jk} \circ \gamma(0)|.$$

[7]A function $f : U \to \mathbb{C}$ defined on an open $U \subseteq \mathbb{R}^m$ is called **real analytic** if for every $\xi \in U$ there exists a power series $\sum_{\alpha \in \mathbb{N}^m} a_\alpha (x - \xi)^\alpha$ absolutely convergent on some nonempty neighborhood of x and the values of the power series coincides with $f(x)$. Equivalently, $f \in C^\infty(U)$ and for every $\xi \in U$ there exists $C, R > 0$ and neighborhood $\xi \in V \subseteq \mathbb{R}^m$ such that $|f^{(\alpha)}| \leq C\alpha! R^{-|\alpha|}$ on V for all multi-index $\alpha \in \mathbb{N}^m$. Embedding $\mathbb{R}^m$ into $\mathbb{C}^m$ by $x \mapsto x + \sqrt{-1}0$, it can be shown that there is a neighborhood D of U in $\mathbb{C}^m$ and a holomorphic function F on D, called the **complexification** of f, such that $f(x) = F(x)$ for all $x \in U$. For their basic properties refer [BM, pages 33–35]. A C^2 function $f : D \to \mathbb{C}^m$ defined on an open $D \subseteq \mathbb{C}^m$ is called **strictly plurisubharmonic** if its complex Hessian $\sum \partial^2 f/\partial z_j \partial \overline{z}_k(z) t_j \overline{t_k}$ is strictly positive for every $z \in D$ and $t \in \mathbb{C}^m \setminus \{0\}$.

This latter quantity can be made infinitely large by choosing sufficiently large j, and picking that f_{jk} which makes the largest contribution in the summation $F_j = \sum |f_{jk}|^2$. ■

There are many opens in $\mathbb{C}^m$ which are not domains of holomorphy but carry real analytic complete Kähler metrics, for example this happens if one deletes a complex submanifold from a domain of holomorphy. However we prove later (Theorem 88) that any C^1 open in $\mathbb{C}^m$ which carries a complete smooth Kähler metric is necessarily a domain of holomorphy. For references refer [JP, section 1.19].

3.6 Pseudoconvexity

We have seen several times so far that being a domain of holomorphy has something to do with "convexity" (Theorems 44, 52, 53, Example 45). Cartan-Thullen theorem characterizes domains of holomorphy by convexity with respect to the absolute values of holomorphic functions. A simple application of the Cauchy integral formula shows that the absolute value of a holomorphic function of one complex variable satisfies the submean value property: $|f(z)| \le \int_{|\zeta - z| = \epsilon} |f(\zeta)| d\mu(\zeta)$, where μ is the normalized Lebesgue measure on the circle $|\zeta - z| = \epsilon$. This property defines *subharmonic functions.* Since the restriction of holomorphic functions of several variables to any complex line is a holomorphic function of one variable, it follows that the absolute values of holomorphic functions of several variables satisfy the submean value property when restricted to complex lines. This property defines *plurisubharmonic functions.* They are more flexible to work with compared to the more rigid notion of holomorphic functions. A deep fact is that domains of holomorphy are exactly those opens which are convex with respect to plurisubharmonic functions (Theorems 58 and 70 combined). This property defines *pseudoconvex* opens, and gives a geometric characterization of domains of holomorphy as exactly those opens of $\mathbb{C}^m$ which satisfy the property mentioned Theorem 51. Here is the fundamental analogy to have in mind:

> *The real analysis interaction between convex opens and linear functions is in analogy with the complex analysis interaction between pseudoconvex opens and plurisubharmonic functions.*

We will start this section by reviewing basic properties of subharmonic functions.

3.6.1 Subharmonic functions

The theory of subharmonic functions is an important chapter in function theory [HK], with numerous applications to potential theory (namely solving the Dirichlet, Neumann, etc. boundary value problems associated to $\Delta u = f$) [Ahl, chapter 6][Con, chapter 10, 19,21][AG][Helm][Med], complex analysis [Oka][Hör-Conv] and operator theory [Bro]. We recall some familiar notions to motivate their definition. Harmonic functions on $\mathbb{R}^n$ are C^2 solutions of the Laplace equation $\sum \partial^2 h/\partial x_j^2 = 0$, or equivalently [Ahl, page 242][AD, C.5.3] (or the proof of Remark 57) continuous functions satisfying the mean value property: $h(x) = \int_{|y-x|_2 = r} h(y) dS(y)$ for any $r > 0$ where dS is the normalized Riemannian surface element on the sphere of radius r around x. When $n = 1$ harmonic functions are nothing but linear functions $x \mapsto ax + b$, and convex functions are those which on each interval are everywhere below the line connecting the end points. Substituting "linear functions" in the definition of convex functions with "harmonic functions" leads to the notion of subharmonic functions, although there are some extra technicalities in order to have a richer theory.

Exercise: Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is convex (in the sense that $f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2)$ for any $x_1, x_2 \in \mathbb{R}$ and any nonnegative $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfying $\lambda_1 + \lambda_2 = 1$) if and only if $\sup_K(f - L) = \sup_{\partial K}(f - L)$ for any compact interval $K \subseteq \mathbb{R}$ and any affine linear function $L(x) = ax + b$, $a, b \in \mathbb{R}$. (*Hint.* The second condition needs only be checked for linear functions L satisfying $L = f - C$ on ∂K for constants C.)

An upper semicontinuous function $f : D \to [-\infty, \infty)$ defined[8] on an open $D \subseteq \mathbb{C}$ is called **subharmonic** if it satisfies any of the following equivalent properties:

1. f minus every harmonic function satisfies the **weak maximum principle** in the sense that for every compact $K \subseteq D$ and function h continuous on K and harmonic in the interior of K such that $f \le h$ on ∂K we have $f \le h$ on K.

2. For every compact disc $K \subseteq D$ and holomorphic polynomial $P \in \mathbb{C}[z]$ such that $f \le \mathrm{Re}P$ on ∂K we have $f \le \mathrm{Re}P$ at the center of K.

3. f has the **strong submean value property** in the sense that for every $a \in D$ and $0 < r < \mathrm{dist}(a, \partial D)$ we have $f(a) \le \mathcal{M}_a(f; r)$, where
$$\mathcal{M}_a(f;r) := \int_0^{2\pi} f\left(a + re^{\sqrt{-1}\theta}\right) \frac{d\theta}{2\pi}.$$

4. f has the **weak submean value property** in the sense that every $a \in D$ has $r_a > 0$ such that for every $0 < r < r_a$ we have $f(a) \le \mathcal{M}_a(f; r)$.

5. f minus every harmonic function satisfies the **strong maximum principle** in the sense that for every harmonic function h on open disc $D' \subseteq D$ the function $f - h$ can not have a local maximum unless being constant.

The proof that these conditions are equivalent. (1⇒2) Trivial.

(2⇒3) Fix a and r as in the statement of (2), and let K denote the closed disc of radius r centered at a. Since the integral of an upper semicontinuous function on a compact (in our case the integral of f on ∂K) equals the infimum of the integral of its continuous majorants, it suffices to show that $f(a) \le \int_{\partial K} F$ for any continuous function F on ∂K

[8]Upper semicontinuity (USC) means $\{z \in D : f(z) < c\}$ is open for every $c \in \mathbb{R}$. Equivalently, $\mathrm{limsup}_{z\to a} \le f(a)$. For example the characteristic functions of closed subsets are USC. Here are some standard facts about them: (1) The pointwise infimum of families of USC functions is USC. (2) An USC function is bounded above on every compact and achieve its supremum there; (3) An USC function which is bounded above is the pointwise limit of a sequence of decreasing continuous functions [AG, 3.2.1]. (4) The integral of an USC function with respect to a regular Borel measure on a compact $K \subseteq \mathbb{C}$ is given by the infimum of the integral of its continuous majorants on K [Fol, 7.13]. Notice that (4) follows from (2) and (3) via Lebesgue's monotone convergence theorem. Exercise: Find a family of USC functions whose pointwise is everywhere finite but not USC.

such that $f \le F$ on ∂K. (Here the integral is with respect to the normalized Lebesgue measure on the circle ∂K.) After replacing F by its Poisson extension we can assume that F is continuous on K and harmonic on the interior of K [Ahl, page 169]. Consider the one-parameter family of functions $F_t(z) := F(a + t(z - a))$, $0 < t < 1$. (Without loss of generality one can assume a is the origin; then $F_t(z) = F(tz)$ is obtained simply by a dilation in F.) Each F_t is harmonic on a neighborhood K and converges uniformly to F on K as $t \to 1-$. Each F_t is the real part of a holomorphic function, so by truncating the Taylor series of holomorphic functions we can find for every $\epsilon > 0$ a holomorphic polynomial P such that $\|F - \mathrm{Re} P\|_K \le \epsilon$. Since $f \le \mathrm{Re}(P + \epsilon)$ on ∂K the hypothesis in (2) gives

$$f(a) \le \mathrm{Re}(P(a) + \epsilon) = \int_{\partial K} (\mathrm{Re} P + \epsilon) \le \int_{\partial K} F + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that $f(a) \le \int_{\partial K} F$.

(3⇒4) Trivial.

(4⇒5) $g := f - h$ is upper semicontinuous and satisfies the submean value property, and every such function satisfies the strong maximum principle according to the proof of Theorem 11.

(5⇒1) The maximum M of $f - h$ on K must be attained at some point on the boundary (otherwise $f - h$ is constant), however $M \le 0$ by the hypothesis in (1). This means $f \le h$ on K. ■

A C^2 function $f : D \to (-\infty, \infty)$ defined on an open $D \subseteq \mathbb{C}$ is called **strictly subharmonic** if the Laplacian Δf is strictly positive at every point of D.

Theorem 55 (First properties of subharmonic functions)**.** *(0) If f is subharmonic on D then $\limsup_{z \to a} f(z) = f(a)$ for every $a \in D$.*

(1) The pointwise supremum of a family of subharmonic functions is subharmonic if it is upper semicontinuous. Specially this is the case if the family is finite.

(2) The limit of a decreasing sequence of subharmonic functions is subharmonic.

(3) Subharmonicity is a local property in the sense that a function is subharmonic if and only if it is such on a neighborhood of every point.

(4) Subharmonicity is preserved under finite summation and multiplication by non-negative real constants.

(5) If f is holomorphic on open $D \subseteq \mathbb{C}$ then $\log |f|$ and $|f|^\alpha$, $\alpha > 0$, are subharmonic on D if one sets $\log 0 := -\infty$. More generally, if f is subharmonic on D and φ is convex and increasing on $\mathbb{R}$ then $\varphi \circ f$ is subharmonic on D, where we set $\varphi(-\infty) := \lim_{t \to -\infty} \varphi(t)$.

(6) Let f be a subharmonic function on connected open $D \subseteq \mathbb{C}$ which is not identically $-\infty$. Then f is integrable on every compact subset of D. Specially, $f^{-1}(-\infty)$ has Lebesgue measure zero.

(7) Set

$$\mathcal{A}_a(f; r) := \int_{|z-a|<r} f\,(a + z)\, d\mu(z),$$

where μ is the normalized Lebesgue measure on the disc $|z - a| < r$. If f is subharmonic on a neighborhood of $a \in \mathbb{C}$ then both $\mathcal{M}_a(f; r)$ and $\mathcal{A}_a(f; r)$ decrease to $f(a)$ as r decreases *to* 0.

(7′) Two subharmonic functions which are equal almost everywhere with respect to the Lebesgue measure are equal everywhere.

(8; Regularization) Every subharmonic function is the pointwise limit of a decreasing sequence of smooth subharmonic functions on compacts. More precisely, let f be a subharmonic function on open $D \subseteq \mathbb{C}$ which is not identically $-\infty$ on any component of D. Choose a radial nonnegative smooth bump function ψ compactly supported in the open unit disc and normalized to have integral 1 *(for example, $\psi(z/2) = C \exp\left(1/(|z|^2 - 1)\right) 1_{\{|z|<1\}}$) and consider the* **approximate identity** *$\psi_\epsilon(z) := \epsilon^{-2}\psi(z/\epsilon)$. Then the* **mollification**

$$f_\epsilon(z) := f * \psi_\epsilon(z) = \int_{\mathbb{C}} f(z')\psi_\epsilon(z - z')d\mu(z'), \quad z \in D_\epsilon := \{z \in D : \operatorname{dist}(z, \partial D) > \epsilon\},$$

is a finite-valued smooth plurisubharmonic function on D_ϵ and $f_\epsilon(z)$ decreases to $f(z)$ as $\epsilon \to 0$ for every $z \in D$. Here μ is the Lebesgue measure on $\mathbb{C}$. Also, $u_\epsilon(z) := \epsilon|z|^2 + \int_{D_\epsilon} f(z')\psi_\epsilon(z-z')d\mu(z')$ is a finite-valued smooth function on D, strictly plurisubharmonic on D_ϵ, $u_\epsilon \geq u_\delta$ on D_δ if $\delta < \epsilon$, and $u_\epsilon(z) \to f(z)$ as $\epsilon \to 0$ for every $z \in D$. If f is also continuous on D then both convergences is uniform on compact subsets of D.

(9) A C^2 function f is subharmonic on $D \subseteq \mathbb{C}$ if and only if its Laplacian $\Delta f = f_{xx} + f_{yy} = 4f_{z\bar{z}}$ is nonnegative on D.

(9′) A function $f : D \to [-\infty, \infty)$ defined on a connected open $D \subseteq \mathbb{C}$ which is not identically $-\infty$ is subharmonic if and only if it is locally integrable and $\Delta f \geq 0$ in the distributional sense namely $\int_D f\Delta\varphi d\mu \geq 0$ for the Lebesgue measure μ on $\mathbb{C}$ and every nonnegative smooth function φ compactly supported in D. In the only if case in fact there is a unique subharmonic function on D which equals f almost everywhere.

(10) Subharmonicity is preserved under composition with holomorphic maps.

Proof. (0) $\leq$ is by upper semicontinuity. If this inequality is strict then $f(z) < f(a)$ on some deleted neighborhood of a. This contradicts the weak submean value property.

(1) Trivial from the definition.

(2) The limit function f is upper semicontinuous because $\{f < c\} = \bigcup\{f_j < c\}$ is open. Let h be a continuous function on compact $K \subseteq D$ which is harmonic on the interior of K and majorizes f on ∂K. Fix an arbitrary $\epsilon > 0$. Since ∂K is compact some f_j is majorized by $h + \epsilon$ on ∂K, hence also on K. Then f is majorized by $h + \epsilon$ on K. *Alternative argument.* Since upper semicontinuous functions are bounded above on compacts one can find a common upper bound for all f_j on a compact disc $K \subseteq D$, and then Lebesgue monotone convergence theorem (or Fatou's lemma) is applicable to deduce the submean value property of f from that of f_j.

(3,4) Immediate from the last two definitions of subharmonicity.

(5) If $\log|f|$ is majorized on the boundary ∂K of a compact $K \subseteq D$ by the real part of a holomorphic polynomial P then $|f \exp(-P)| \leq 1$ on ∂K, so the same inequality holds on K by the maximum modulus principle for holomorphic functions. *Another argument.* If f is not zero at $a \in D$ then $\log|f|$ is locally around a the real part of the holomorphic function $\log f$, so it is harmonic. At the points where f vanishes the submean value property trivially holds. This shows that $\log|f|$ is subharmonic. That $|f|$ satisfies the submean value property is immediate from the Cauchy integral formula. If f is not zero

at $a \in D$ then f can be expressed as $g^{1/\alpha}$ for some holomorphic function around a, so $|g| = |f^\alpha| = |f|^\alpha$ is subharmonic. *Another argument.* Since $|f|^\alpha = \exp(\alpha \log |f|)$ it suffices to prove the last statement about subharmonicity of compositions with increasing convex functions φ. Convex functions φ satisfy the Jensen inequality $\varphi(\int_X g d\mu) \le \int_X \varphi \circ g d\mu$ for every probability space (X, μ) and real-valued L^1 function g on X. The proof of this inequality is just one line: for every $t_0 \in \mathbb{R}$ find a slope $s \in \mathbb{R}$ such that $\varphi(t) \ge \varphi(t_0) + s(t - t_0)$, set $t := g(x)$, integrate, and set $t_0 := \int g d\mu$. Q.E.D. For every $a \in D$ and $0 < r < \operatorname{dist}(a, \partial D)$ we have $\varphi(f(a)) \le \varphi(\mathcal{M}_a(f; r)) \le \mathcal{M}_a(\varphi \circ f; r)$, so $\varphi \circ f$ has the strong submean value property. It is also upper semicontinuous because convex functions (on open intervals) are continuous [Rud-RCA, 3.2].

(6) Note that if $f(a) \ne -\infty$ for some $a \in D$ then by the submean value property f is integrable on every compact disc around a which is contained in D. Therefore the set S of points in D such that f in integrable on some compact disc around them is nonempty S is clearly open. To show that S is closed assume $a \in D \setminus S$ and find a compact disc $a + \Delta$ around a which is contained in D. For every $b \in a + \Delta/2$, since $b + \Delta/2$ is a compact disc around a which is contained in D by our initial notice $f(b) = -\infty$. Therefore $a + \Delta/2 \subseteq D \setminus S$. This shows that S is both open and closed, so $S = D$. Since every compact $K \subseteq D$ can be covered by finitely many compact discs so f is integrable on K. If $f^{-1}(-\infty)$ is not null then by the interior regularity of the Lebesgue measure some compact subset K of it is not null. But then f is not integrable on K.

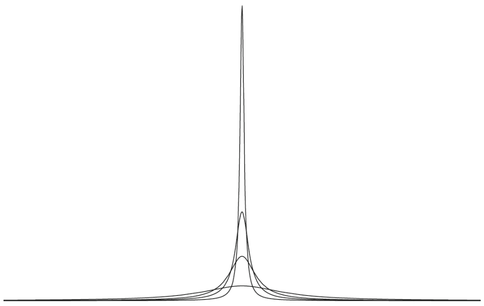
(7) Fix $0 < r_1 < r_2 < \operatorname{dist}(a, \partial D)$, and let K be the compact disc around around a of radius r_2. Since the integral of an upper semicontinuous function on a compact equals the infimum of the integral of its continuous majorants, it suffices to show that $\mathcal{M}_a(f; r_1) \le \mathcal{M}_a(F; r_2)$ for every continuous function F on ∂K which majorizes f. After replacing F by its Poisson extension we can assume that F is continuous on K and harmonic on the interior of K. F majorizes f on K because f is subharmonic. Therefore $\mathcal{M}_a(f; r_1) \le \mathcal{M}_a(F; r_1) = F(a) = \mathcal{M}_a(F; r_2)$. This proves monotonicity. The submean value property of f combined with its upper semicontinuity shows that for every $\epsilon > 0$ we have $f(a) \le \mathcal{M}_a(f; r) < f(a) + \delta$ for sufficiently small r, therefore $\mathcal{M}_a(f; r) \to f(a)$ as $r \to 0+$. The analog statements for $\mathcal{A}$ are proved similarly.

(7′) Immediate from $f(a) = \lim_{r \to 0} \mathcal{A}_a(f; r)$ proved in (7).

(8) We start by developing some intuition about approximate identity and mollification. ψ is a radial bump of area 1 centered at the origin. Accordingly, ψ_ϵ is a family of radial bumps of area 1 centered at the origin which becomes sharper and sharper as $\epsilon \to 0$. In the sense of distributions, ψ_ϵ approaches the idealized bump namely the Dirac unit mass (impulse) distribution $\delta(z)$ at the origin [Fol, 9.1]. The characteristic property of the Dirac distribution is that it is the identity element of the convolution operation namely $f * \delta = f$. Intuitively, the process of convolving data f with approximate identity ψ_ϵ generates a family of nice smooth functions f_ϵ which "converges" f in some sense. Since ψ is nonnegative and with area 1 it follows that the mollification $f * \psi_\epsilon = \int f \psi_\epsilon / \int \psi_\epsilon$ is an averaging process, and the big intuition is that: *Averaging fixes disorders.* We now start the proof. The mollification

$$f_\epsilon(z) = f * \psi_\epsilon(z) = \int_{\mathbb{C}} f(z') \epsilon^{-2} \psi((z - z')/\epsilon) d\mu(z')$$

is a finite-valued smooth function on D_ϵ because for every $z \in D_\epsilon$ the integration is



actually on a compact disc in D and we know from (6) that f is locally integrable on D. The representation

$$f_\epsilon(z) = \int_{\mathbb{C}} f(z - \epsilon z')\psi(z')d\mu(z') \tag{3.16}$$

shows that f_ϵ has the submean value property on D_ϵ, because for $z \in D_\epsilon$ and $r > 0$ sufficiently small we have

$$\int f_\epsilon\left(z + re^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi} = \iint f\left(z + re^{\sqrt{-1}\theta} - \epsilon z'\right)\psi(z')\frac{d\theta}{2\pi}d\mu(z') \geq$$
$$\int f\left(z - \epsilon z'\right)\psi(z')d\mu(z') = f_\epsilon(z).$$

Decomposing the Lebesgue measure into radial and angular parts gives the representation

$$f_\epsilon(z) = \int_0^1 \psi(r)\mathcal{M}_z(f;\epsilon r)2\pi dr, \quad z \in D_\epsilon. \tag{3.17}$$

This representation combined with (7) shows that f_ϵ majorizes f and decreases as $\epsilon \to 0+$. This combined with the upper semicontinuity of f shows that for every $z \in D$ and $\delta > 0$ we have $f(z) \leq f_\epsilon(z) < f(z) + \delta$ for sufficiently small ϵ, therefore $f_\epsilon(z) \to f(z)$. The assertions about u_ϵ is proved similarly.

(9) If $g := f - h$, for some harmonic function h, has a local maximum at $a \in D$ then the Taylor expansion of g around a shows that the real Hessian $R_a(g;t) = g_{xx}t_1^2 + 2g_{xy}t_1t_2 + g_{yy}t_2^2$ of g at a is identically zero or negative definite, therefore $\Delta g = \Delta f \leq 0$ at a. This argument shows that if $\Delta f > 0$ on D then f is subharmonic. If $\Delta f \geq 0$ on D then $f_j := f + j^{-1}x^2$ is a decreasing sequence of subharmonic functions (because $\Delta f_j = \Delta f + 2j^{-1} > 0$), so $f = \lim f_j$ is also subharmonic. For the other direction suppose f is subharmonic but $\Delta f < 0$ at some $a \in D$. By continuity $\Delta f < 0$ on some neighborhood $U \subseteq \mathbb{C}$ of a. By the previous argument $-f$ is subharmonic on U. Since f is already subharmonic we deduce that f satisfies the mean value property on U, hence harmonic, so the contradiction $\Delta f = 0$ on U. *Another argument.* Fix $f \in C^2(D)$, $a \in D$, and consider $\mathcal{M}_a(f;r) = \int_0^{2\pi} f(a + r\exp(\sqrt{-1})\theta)d\theta/2\pi$ for $0 < r < \operatorname{dist}(a, \partial D)$. By Stokes' theorem we have

$$\frac{d\mathcal{M}_a(f;r)}{dr} = \frac{1}{2\pi r}\int_{|z-a|=r} -\frac{\partial f}{\partial x}(z)dx + \frac{\partial f}{\partial y}(z)dy = \int_{|z-a|\leq r} \Delta f(z)\frac{dxdy}{2\pi r},$$

where $z = a + r\exp(\sqrt{-1}\theta) = x + \sqrt{-1}y$. Since $\mathcal{M}_a(f;0) = f(a)$ it follows that

$$\mathcal{M}_a(f;r) - f(a) = \int_{r\in[0,\rho]}\int_{|z-a|\le\rho} \Delta f(z)\frac{dxdyd\rho}{2\pi\rho}.$$

From this formula we can deduce that $\Delta f(a) \geq 0$ if and only if the submean value inequality $f(a) \leq \mathcal{M}_a(f;r)$ holds for sufficiently small positive r. *Yet another argument.* Since the Laplacian Δf is given by $r^{-2}(rf_r)_r + f_{\theta\theta}$ in polar coordinates it follows

$$\frac{d}{dr}\left(r\frac{d\mathcal{M}_a(f;r)}{dr}\right) = \int r^2\Delta f\left(a + re^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi} \geq 0.$$

Therefore, $rd\mathcal{M}_z(f;r)/dr$ is an increasing function of r. Since this latter function tends to 0 as $r \to 0$ it follows that $\mathcal{M}_a(f;r)$ is an increasing function of r, and we get the same result as before.

(9′) First of all note that doing integration by parts twice shows that $\int_D f\Delta\varphi = \int_D \varphi\Delta f$ for every $f \in C^2(D)$ and $\varphi \in C_c^\infty(D)$, so (9′) is a generalization of (9). *Only if part.* For every $0 < r < \mathrm{dist}(\mathrm{Supp}\varphi, \partial D)$ we have

$$f(z)\varphi(z) \leq \int_0^{2\pi} f\left(z + re^{\sqrt{-1}\theta}\right)\varphi(z)\frac{d\theta}{2\pi}, \quad z \in \mathbb{C}.$$

Here we are extending by zero wherever the argument of f exist D. Integration against Lebesgue measure gives

$$\int_D f(z)\left(\int_0^{2\pi}\varphi\left(z - re^{\sqrt{-1}\theta}\right)\frac{d\theta}{2\pi} - \varphi(z)\right)d\mu(z) \geq 0.$$

A straightforward computation shows that the Taylor expansion of the left hand side equals $\frac{1}{4}\int f\Delta\varphi d\mu(z)r^2 + O(r^3)$ as $r \to 0+$, and the result follows. *If part.* We use the setting of the proof of (8). Just because f is locally integrable f_ϵ converges f in $L^1_{\mathrm{loc}}(D)$ [Fol, 8.14]. Since $\Delta f \geq 0$ in the distributional sense it follows that $\Delta f_\epsilon \geq 0$ in the distributional sense, so integration by parts shows that $\Delta f_\epsilon \geq 0$ in the usual sense, hence f_ϵ is subharmonic by (9). As in the proof of (8) the subharmonicity of f_ϵ implies that $(f_\epsilon)_\delta = \int_0^1 \psi(r)\mathcal{M}(f_\epsilon;\delta r)2\pi dr$ decreases pointwisely to f_ϵ as $\delta \to 0+$. Since $(f_\epsilon)_\delta = f * \psi_\epsilon * \psi_\delta = (f_\delta)_\epsilon$ it follows that f_ϵ decreases pointwisely as $\epsilon \to 0+$ to some limit function g, which is subharmonic by (2). Since $f_\epsilon \to f$ in L^1_{loc} it follows that $f = g$ almost everywhere on D. The uniqueness assertion comes from (7′) but we give another proof. If f equals almost everywhere to two subharmonic functions F and G then $F = \lim F_\epsilon$ equals $G = \lim G_\epsilon$ because $F_\epsilon = G_\epsilon$.

(10) For smooth subharmonic functions the assertion is immediate from the nonnegative Laplacian criterion in (7). The general case is immediate from this special case applied to a regularization. ■

Exercise: Directly prove that a continuous function $f : D \to \mathbb{R}$ defined on an open $D \subseteq \mathbb{C}$ is subharmonic if and only if f is majorized on every compact disc $K \subseteq D$ by the Poisson extension of $f|_{\partial K}$.

Example: Consider $f(z) := \sum_{j\geq 2} 2^{-j}\log|z-1/j|$ for $z\in\mathbb{C}$. Note that f is $-\infty$ at $z=1/j$ and finite elsewhere. f is subharmonic on the unit disc because for every $|z|<1$ all but finitely many summands are negative and f is the decreasing limit of its subharmonic partial sums. f is discontinuous at the origin. $g := \exp(f)$ is an example of a non-continuous subharmonic function with finite values. One can check that f is subharmonic on $\mathbb{C}$.

Example 56. The function $f(z) := -\log\operatorname{dist}(z,\partial D)$ is subharmonic for every open $D\subseteq\mathbb{C}$. Here is the reason. If $D=\mathbb{C}$ then then f is identically $-\infty$, so assume $D\neq\mathbb{C}$. Then ∂D is nonempty and f is finite-valued. For any $z,w\in D$ and $\zeta\in\partial D$ we have

$$\operatorname{dist}(z,\partial D)\leq\operatorname{dist}(z,\zeta)\leq\operatorname{dist}(z,w)+\operatorname{dist}(w,\zeta)\leq\operatorname{dist}(z,w)+\operatorname{dist}(w,\partial D),$$

which implies the continuity of f. On the other hand $f(z)=\sup_{\zeta\in\partial D}-\log|z-\zeta|$ is the pointwise supremum of harmonic (hence subharmonic) functions. The same argument shows that $\varphi(\operatorname{dist}(z,\partial D))$ is also subharmonic for every decreasing convex function $\varphi:\mathbb{R}\to\mathbb{R}$. ■

Remark 57 (Weyl's lemma)**.** As another application of mollification we show: *A locally integrable function $f\in L^1_{\mathrm{loc}}(D)$ on open $D\subseteq\mathbb{C}^m$ satisfying the distributional Cauchy-Riemann equations is holomorphic after correction on a null set (page 7).* Recall the setting of the proof of Theorem 55.(8,9′), where now ψ is a radial nonnegative smooth bump function compactly supported in the open unit ball of $\mathbb{C}^m$ and normalized to have integral 1. The mollification $f_\epsilon=f*\psi_\epsilon$ is smooth in D_ϵ and satisfies the usual Cauchy-Riemann equations just because f satisfies the distributional Cauchy-Riemann equations:

$$\frac{\partial f_\epsilon}{\partial\overline{z}_j}=\frac{\partial}{\partial\overline{z}_j}\int f(z')\psi_\epsilon(z-z')d\mu(z')=\int f(z')\frac{\partial}{\partial\overline{z}_j}\psi_\epsilon(z-z')d\mu(z')=0.$$

Bochner-Martinelli integral formula (Theorem 33) together with Theorem 32 shows that holomorphic functions (in our case f_ϵ) satisfy the mean value property: Their value at a point equal their average on spheres centered at that point. This combined with the fact that ψ is radial implies that $(f_\epsilon)_\delta=f_\epsilon$ on $D_{\epsilon+\delta}$. Since $(f_\epsilon)_\delta=(f_\delta)_\epsilon$ anyway it follows that $f_\epsilon=f_\delta$ on $D_{\epsilon+\delta}$. Since $f_\epsilon\to f$ in L^1_{loc} it follows that $f=f_\epsilon$ almost everywhere on D_ϵ. *Other arguments.* Note that it suffices to show that f is C^2, because then integration by parts shows that f satisfies the usual Cauchy-Riemann equations. Here are several ways to prove that $f\in C^2$: (1) Theorem 77.(2) combined with Theorem 76.(3). (2) Since each differential operator $\partial/\partial\overline{z}_j$ is elliptic the smoothness of f is a special case of the "interior regularity of linear elliptic equations" proved in [Rud-FA, 8.12][Fol, 8.14][Jos, 23.7,23.11] or every standard PDE textbook [Tay, volume I, page 442][Fol-PDE, 6.34][Hör-PDE, 4.1.7][Eva, 6.3.1][Bre, 9.25][Jos-PDE, 9.3.2]. ■

3.6.2 Plurisubharmonic functions

Here is an important generalization of subharmonic functions to SCV first introduced by Oka under the name of **pseudoconvex functions** in order to solve Levi's problem [Nog, page 300]. An upper semicontinuous function $f:D\to[-\infty,\infty)$ defined on open $D\subseteq\mathbb{C}^m$

is called **plurisubharmonic** if each of its slices on complex lines are subharmonic, namely for any $z, w \in \mathbb{C}^m$ the function $\lambda \mapsto f(z+w\lambda)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

Example: The function $-\log \operatorname{dist}(z, \partial D)$ is not plurisubharmonic for $D = \mathbb{C}^2 \backslash \{(0,0)\}$, because on the complex line passing through $a = (1,0)$ with direction $b = (0,1)$ the function equals $-\log \sqrt{1+|\lambda|^2}$, which has a local maximum at $\lambda = 0$.

Exercise: Show that if f is holomorphic in a connected open $D \subseteq \mathbb{C}^m$ which is not identically zero then the analytic variety $V = \{z \in D : f(z) = 0\}$ defined by f is null with respect to the $2m$-dimensional Lebesgue measure. (*Hint.* First write D as a union of countably many open balls centered at the points of V, and then use submean value property of $\log |f|$.)

All the properties of subharmonic functions in Theorem 55 inherit to plurisubharmonic functions. Proofs are similar, but have the following notes in mind:

- About (0,6,7): Since the Lebesgue measure is rotation-invariant it follows that a plurisubharmonic function $f : D \to \mathbb{C}^m$ is subharmonic as a function on $D \subseteq \mathbb{R}^{2m}$ in the sense that it is upper semicontinuous and satisfies the submean value property: For every $a \in D$ and sufficiently small $r > 0$ we have:

$$\int_{|z-a|_2 \leq r} f(a+z)\, d\mu(z) = \int_{|z-a|_2 \leq r} \int_{\theta \in [0,2\pi]} f\left(a + z e^{\sqrt{-1}\theta}\right) \frac{d\theta}{2\pi} d\mu(z) \geq f(a)$$

  where μ is normalized the Lebesgue measure on $|z-a|_2 < r$.

- About (7): $\mathcal{M}_a(f;r)$ and $\mathcal{A}_a(f;r)$ are now averages over the sphere $|z-a|_2 = r$ and the ball $|z-a|_2 < r$ respectively. In the proof one needs to consider the Poisson extension operator of the unit ball in $\mathbb{R}^{2m}$ given by the kernel $(|z'|_2^2 - |z|_2^2)/(|z'-z|_2^2)$ [AG, section 1.3].

- About (8): ψ is now a radial nonnegative smooth bump function compactly supported in the open unit ball of $\mathbb{C}^m$ and normalized to have integral 1. The approximate identity is $\psi_\epsilon(z) = \epsilon^{-2m}\psi(z/\epsilon)$ whose integral is 1. If f is plurisubharmonic then the mollification $f_\epsilon = f * \psi_\epsilon$ satisfies the the submean value property on each complex line (hence plurisubharmonic) because for every $z \in D_\epsilon$, $w \in \mathbb{C}^m$ and $r > 0$ sufficiently small we have:

$$\int f_\epsilon\left(z + wre^{\sqrt{-1}\theta}\right) \frac{d\theta}{2\pi} = \iint f\left(z + wre^{\sqrt{-1}\theta} - z'\right) \psi_\epsilon(z') \frac{d\theta}{2\pi} d\mu(z') \geq \int f(z-z')\, \psi_\epsilon(z') d\mu(z') = f_\epsilon(z).$$

  That f_ϵ majorizes f and decreases as $\epsilon \to 0$ is immediate from the following representation:

$$f_\epsilon(z) = \int f(z-\epsilon z')\psi(z')d\mu(z') = \int \int_0^{2\pi} f\left(z - \epsilon e^{\sqrt{-1}\theta} z'\right) \frac{d\theta}{2\pi} \psi(z') d\mu(z').$$

- About (9): The statement is now: *A function $f \in C^2(D)$ on open $D \subseteq \mathbb{C}^m$ is plurisubharmonic if and only if the complex Hessian of f is positive semidefinite with respect to all complex vectors.* It is proved by applying the corresponding Laplacian criterion for subharmonicity to the formula $\partial^2 f(z+w\lambda)/\partial\lambda\partial\overline{\lambda} = L_{z+w\lambda}(f;w)$.

- About (9′): The statement is now: *A function $f : D \to [-\infty, \infty)$ defined on a connected open $D \subseteq \mathbb{C}^m$ which is not identically $-\infty$ is plurisubharmonic if and only if it is locally integrable and $\int_D f(z)L_z(\varphi;t)d\mu(z) \geq 0$ for the Lebesgue measure μ on $\mathbb{C}^m$, every $t \in \mathbb{C}^m$ and every nonnegative smooth function φ compactly supported in D.*

A C^2 function $f : D \to \mathbb{R}$ on open $D \subseteq \mathbb{C}^m$ whose complex Hessian is positive definite with respect to all complex vectors is called **strictly plurisubharmonic**. Examples are $\sum |z_j|^2$, $\sum |\mathrm{Re} z_j|^2$ and $\sum |\mathrm{Im} z_j|^2$ on $\mathbb{C}^m$.

3.6.3 Pseudoconvex domains

Theorem 58 (Hartogs-Oka). *For every open $D \subseteq \mathbb{C}^m$ the followings are equivalent:*

(1) D has a continuous plurisubharmonic exhaustion[9] function.

(1′) D has a smooth strictly plurisubharmonic exhaustion function.

(1″) D has a real analytic strictly plurisubharmonic exhaustion function.

(1‴) D has a plurisubharmonic exhaustion function.

(2) D is **plurisubharmonic convex** *in the sense that for every compact $K \subseteq D$ its plurisubharmonic convex hull $\hat{K}_{PS(D)} = \{z \in D : f(z) \leq \sup_K f, \forall f \in PS(D)\}$ is compactly supported in D.*[10]

(3) D satisfies the **continuity principle** *in the sense that if the union of the boundaries of a collection of holomorphic discs in D is compactly supported in D then the union of those holomorphic discs is also compactly supported in D. (Holomorphic discs are defined before Theorem 51.)*

(4) D is **Hartogs pseudoconvex**. *(Defined before Theorem 50.)*

(5) The function $-\log \mathrm{dist}(z, \partial D)$ is plurisubharmonic on D.

(5′) The function $1/\mathrm{dist}(z, \partial D)$ is plurisubharmonic on D.

(6) $\mathrm{dist}(S, \partial D) = \mathrm{dist}(\partial S, \partial D)$ for any every holomorphic disc S in D.

(7) One of the conditions in (5), (5′) or (6) holds when the Euclidean distance $\mathrm{dist}(z, \partial D)$ is replaced by $\sup\{t \in \mathbb{R} : z + tU \subseteq D\}$ for some (or all) open neighborhood $U \subseteq \mathbb{C}^m$ of the origin with the property that $tU \subseteq U$ for all $0 < t < 1$.

In case any of these conditions happens then D is called **pseudoconvex**.

Proof. (1⇔1′) We are going to regularize some fixed continuous plurisubharmonic exhaustion function u. Consider opens $D_j := \{z \in D : u < j\}$, $j \geq 0$, compactly supported in D. By Theorem 55.(8) find $u_j \in C^\infty(D)$ such that

$$u_j \text{ is strictly plurisubharmonic on } D_{j+2},$$

[9] An exhaustion function for D is a real-valued function f defined on D with all pre-level sets $\{z \in D : f(z) < c\}$, $c > 0$, compactly supported in D. Therefore, $f(z) \to \infty$ as $\mathrm{dist}(z, \partial D) \to 0$; this is also sufficient if D is bounded.

[10] It is a deep fact (Theorem 100) that the expression "$\hat{K}$ is compactly supported in D" can be replaced by "$\hat{K}$ is compact".)

$$u < u_0 < u+1 \text{ on } \overline{D_1}, \quad u < u_j < u+1 \text{ on } D_j, \quad j \geq 1.$$

Therefore

$$u_j - j + 1 < 0 \text{ on } D_{j-2}, \quad u_j - j + 1 > 0 \text{ on } \overline{D_j} \setminus D_{j-1}.$$

Choose $\chi \in C^\infty(\mathbb{R})$ which vanishes on $(-\infty, 0]$ but $\chi, \chi', \chi'' > 0$ on $(0, \infty)$, for example $\chi(t) := \exp(-1/t)1_{[0,\infty)}$. Set $\chi_j := \chi \circ (u_j - j + 1)$. Then

$$\chi_j = 0 \text{ on } D_{j-2}, \quad \chi_j \geq 0 \text{ on } D.$$

Since the complex Hessian $L(\chi_j; t)$ of χ_j is given by

$$L(u_j - j + 1; t)\chi''(u_j - j + 1) + \left| \sum_{k=1}^{m} \frac{\partial u_j}{\partial z_k} t_k \right|^2 \chi'(u_j - j + 1),$$

it follows that

χ_j is plurisubharmonic on D_{j+2}, $\quad \chi_j$ is strictly plurisubharmonic and > 0 on $\overline{D_j} \setminus D_{j-1}$.

Therefore one can inductively find positive integers m_j such that

$$\varphi_l := u_0 + \sum_{2 \leq j \leq l} m_j \chi_j \text{ is strictly plurisubharmonic on } D_j, \quad l \geq 2.$$

It follows that

$$\varphi_l = u_0 \text{ on } D_0, \quad \varphi_l \geq u, \quad \varphi_l = \varphi_{l-2} \text{ on } D_{l-2}.$$

Therefore φ_∞ is a smooth strictly plurisubharmonic exhaustion function.

(1$\Leftrightarrow$1$''$) Combine Theorem 54 with Levi's problem (Theorem 70).

(1$\Rightarrow$1$'''$) Trivial.

(1$'''$$\Rightarrow$2) Trivial, because the plurisubharmonically convex hull of a compact $K \subseteq D$ is contained in the pre-level set $\{f(z) < \|f\|_K + 1\}$ of any chosen plurisubharmonic exhaustion function f.

(2$\Rightarrow$3) Since plurisubharmonic functions satisfy the maximum principle it follows that every holomorphic disc is contained in the plurisubharmonic convex hull of it boundary. Therefore, assuming holomorphic discs S_α in D such that $\bigcup \partial S_\alpha$ is compactly supported in D we have

$$\bigcup S_\alpha \subseteq \bigcup \widehat{\partial S_\alpha} \subseteq \widehat{\overline{\bigcup \partial S_\alpha}} \subset\subset D,$$

where all the convex hulls are taken with respect to plurisubharmonic functions.

(3$\Rightarrow$4) Assume a Hartogs figure $(\Gamma', \hat{\Gamma}')$ in $\mathbb{C}^m$ defined by the biholomorphic map $F : \hat{\Gamma} \to \hat{\Gamma}'$, and let $\Gamma' \subseteq D$. For every complex number α in the closure of the open unit disc $\Delta \subseteq \mathbb{C}$, let S_α be the holomorphic disc in D defined by the map $z \in \overline{\Delta} \mapsto F(0, \ldots, 0, \alpha, z)$. The assumption $\Gamma' = F(\Gamma) \subseteq D$ translates into $S_0 \subseteq D$ and $\partial S_\alpha \subseteq D$ for $\alpha \in \overline{\Delta}$. That $\hat{\Gamma}' = F(\hat{\Gamma})$ is contained in D is equivalent to $A := \{\alpha \in \overline{\Delta} : S_\alpha \subseteq D\}$ being equal to $\overline{\Delta}$. Since A is nonempty ($0 \in A$) and open in $\overline{\Delta}$, it remains to show that it is closed in $\overline{\Delta}$. If α_j is a sequence of points in A which converges to $\alpha \in \overline{\Delta}$ then $\bigcup \partial S_{\alpha_j} \subseteq \Gamma'$ is compactly supported in D, so by the continuity principle $S_\alpha \subseteq \overline{\bigcup S_{\alpha_j}} \subseteq D$, namely $\alpha \in A$.

(4⇒5) We need to introduce a new concept. For any vector $u \in \mathbb{C}^m$ of unit length $|u|_2 = 1$, let $\operatorname{dist}_u(a, \partial D)$ denote the **u-directional distance** of $a \in D$ to the boundary, namely the supremum $t > 0$ such that $a + \eta u \in D$ for all complex numbers η satisfying $|\eta| \leq t$. It is straightforward to show that the usual Euclidean distance $\operatorname{dist}(a, \partial D) = \sup\{t > 0 : a + tz \in D, \forall z \in \mathbb{C}^m, |z|_2 < 1\}$ equals the infimum of $\operatorname{dist}_u(a, \partial D)$ over $|u|_2 = 1$, so we need to show that $-\log \operatorname{dist}_u(z, \partial D)$ is plurisubharmonic. It is upper semicontinuous because it is a pointwise infimum of continuous functions. Fixing $a \in D$, unit length $u \in \mathbb{C}^m$ and $w \in \mathbb{C}^m$, we need to check the subharmonicity of $\lambda \mapsto -\log \operatorname{dist}_u(a + \lambda w)$ on $D' := \{\lambda \in \mathbb{C} : a + \lambda w \in D\}$. If u and w are linearly dependent then $\operatorname{dist}_u(a+\lambda w)$ measures, up to a factor $|w|_2$, the usual Euclidean distance of λ to the boundary of D', so we are done by Example 56. So assume u and w are part of a linear basis $\{u, w, u_1, \ldots, u_{m-2}\}$ of $\mathbb{C}^m$. Fixing $r > 0$ and holomorphic polynomial $g(\lambda) \in \mathbb{C}[\lambda]$ such that

$$\{a + \lambda w : |\lambda| \leq r\} \subseteq D \tag{3.18}$$

and

$$-\log \operatorname{dist}_u(a + \lambda w) \leq \operatorname{Re} g(\lambda), \tag{3.19}$$

for $|\lambda| = r$, we need to deduce the same inequality (3.19) for $|\lambda| \leq r$. Equivalently, assuming

$$\left\{a + \lambda w + \eta e^{-g(\lambda)} u : |\eta| \leq \tau\right\} \subseteq D, \quad \forall \tau \in (0, 1), \tag{3.20}$$

for $|\lambda| = r$, we need to deduce the same inclusion (3.20) for $|\lambda| \leq r$. Consider the biholomorphic map

$$F : \mathbb{C}^m \to \mathbb{C}^m, \quad (z_1, \ldots, z_m) \mapsto a + r z_m w + \tau z_{m-1} e^{-g(r z_m)} u + z_{m-2} u_{m-2} + \cdots + z_1 u_1.$$

Then (3.18) together with (3.20) to hold for all $|\lambda| = r$ translates into $\Gamma' := F(\Gamma) \subseteq D$, so by Hartogs pseudoconvexity $\hat{\Gamma}' := F(\hat{\Gamma}) \subseteq D$, which says (3.20) holds for all $|\lambda| \leq r$.

(5⇒1) $f(z) := -\log \operatorname{dist}(z, \partial D)$ is an exhaustion function if D is bounded. For an arbitrary open $\max\{|z|_2^2, f(z)\}$ works.

(5′⇔others) If (5) holds then $1/\operatorname{dist}(z, \partial D) = \exp(-\log \operatorname{dist}(z, \partial D))$ is plurisubharmonic. If (5′) holds then $\max\{|z|_2^2, 1/\operatorname{dist}(z, \partial D)\}$ is a continuous plurisubharmonic exhaustion function.

(6⇔others) (6) readily implies the continuum principle (3). Assuming (5) then the maximum principle applied to the plurisubharmonic function $-\log \operatorname{dist}(\varphi(z), \partial D)$ (where $\varphi : \{|z| \leq 1\} \to D$ is the function which defines S) gives the nontrivial direction $\operatorname{dist}(S, \partial D) \geq \operatorname{dist}(\partial S, \partial D)$ in the desired equality of (6).

(7⇔others) The Euclidean distance function $\operatorname{dist}(z, \partial D)$ corresponds to U being the unit ball, and in the whole proof so far we have only used those properties of the unit ball which is abstracted in U. ■

Exercise: Show that the unit open ball is pseudoconvex. (*Hint.* By computing the complex Hessian show that $1/|z|_2^2$ is a plurisubharmonic exhaustion function.)

Exercise: Consider open $D = \{|z_1| < 1, |z_2| < 2\} \cup \{|z_1| < 2, |z_2| < 1\}$ in $\mathbb{C}^2$ and unit vector $u = (0, 1) \in \mathbb{C}^2$. Find all the points of discontinuity of the u-directional distance function $z \in D \mapsto \operatorname{dist}_u(z, \partial D)$ introduced in the proof above. *Hint.* It is discontinuous if $|z_1| = 1$.

Remark 59. This remark is adapted from [Gun, volume I, page 130–131]. The existence of plurisubharmonic exhaustion functions on plurisubharmonic convex opens shows the greater flexibility of subharmonic functions compared with holomorphic functions, because one can shows that holomorphically convex opens never admit exhaustion functions of the form of the absolute values of holomorphic functions. Here is the reason. Let the modulus of $f \in \mathcal{O}(D)$ exhaust D, a holomorphically convex open of $\mathbb{C}^m$. Then f is nonzero outside a compact $K \subseteq D$, so $1/f$ is holomorphic on $D \setminus K$. If $m > 1$ by Hartogs extension theorem $1/f$ can be holomorphically extended to D. This extension is constant by the maximum principle, so $|f|$ can not exhaust D. If $m = 1$ after dividing f by a suitable polynomial one may assume that f has no zeros in K, so again $1/f$ can be defined. ■

Theorem 60. *The class of pseudoconvex opens are closed under taking finite products, interior of arbitrary intersections, and union of increasing sequences.*

Proof. For opens $D_1 \subseteq \mathbb{C}^{m_1}$ and $D_2 \subseteq \mathbb{C}^{m_2}$, since $\partial(D_1 \times D_2) = \partial D_1 \times \overline{D_2} \cup \overline{D_1} \times \partial D_2$ it follows that

$$1/\operatorname{dist}((z_1, z_2), \partial(D_1 \times D_2)) = \max\{1/\operatorname{dist}(z_1, \partial D_1), 1/\operatorname{dist}(z_2, \partial D_2)\}.$$

If $D_\alpha \subseteq \mathbb{C}^m$, $\alpha \in A$, is a family of opens and D is the interior of $\bigcap D_\alpha$ then

$$-\log \operatorname{dist}(z, \partial D) = \sup \{-\log \operatorname{dist}(z, \partial D_\alpha) : \alpha \in A\}.$$

Finally, the union of every increasing sequence of Hartogs pseudoconvex opens is again Hartogs pseudoconvex. (Note that Hartogs Γ is compact.) ■

The proof of the next fundamental theorem reveals the great power of plurisubharmonic functions.

Theorem 61. *Pseudoconvexity is a local property of the boundary in the sense that an open $D \subseteq \mathbb{C}^m$ is pseudoconvex if and only if every $p \in \partial D$ has a neighborhood $U \subseteq \mathbb{C}^m$ such that $D \cap U$ is pseudoconvex.*

Proof. Taking open balls for U gives the only if part because finite intersections of pseudoconvex opens are pseudoconvex. Set $f(z) := -\log \operatorname{dist}(z, \partial D)$. Since $f(z)$ equals $-\log \operatorname{dist}(z, \partial(U \cap D))$ for $z \in D$ sufficiently close to p it follows that f is plurisubharmonic on $D \cap V$ for some neighborhood $V \subseteq \mathbb{C}^m$ of ∂D. If D is bounded then $\max\{f(z), \|f\|_{D \setminus V} + 1\}$ is a plurisubharmonic exhaustion function for D, so D is pseudoconvex. If D is unbounded then the intersection D_j of D with the open ball $|z|_2 < j$, $j = 1, 2, \ldots$, is pseudoconvex by the previous case, so $D = \bigcup D_j$ being the union of an increasing sequence of pseudoconvex opens is again pseudoconvex. *Another proof.* We find a smooth increasing convex function $\chi : \mathbb{R} \to \mathbb{R}$ such that $g(z) := \max\{|z|_2^2, f(z), \chi(|z|_2^2)\}$ is a plurisubharmonic exhaustion function for D. Since $f(z)$ blows up at ∂D and $|z|_2^2$ blows up at ∞, we only need to make $\chi(|z|_2^2) > f(z)$ on $D \setminus V$. Since χ is increasing it is enough for χ to satisfy

$$\chi(t) \geq \sup \left\{ f(z) : z \in D \setminus V, |z|_2^2 \leq t \right\}, \quad \forall t \in \mathbb{R}.$$

One can achieve this just because the function on the right hand side is a finite-valued increasing function $\mathbb{R} \to \mathbb{R}$ which vanish on $(-\infty, 0)$. (Refer Lemma 75.(4) for the proof.) ■

Theorem 62. *Those opens of $\mathbb{C}^m$ which are domains of holomorphy are pseudococonvex.*

Proof. We give several proofs. (1) Domains of holomorphy are holomorphically convex (Cartan-Thullen), so also plurisubharmonically convex because the modulus of holomorphic functions are plurisubharmonic. (2) In Theorem 50 we proved that domains of holomorphy are Hartogs pseudoconvex. (3) In Theorem 51 we proved that domains of holomorphy D satisfy $\operatorname{dist}(S, \partial D) = \operatorname{dist}(\partial S, \partial D)$ for holomorphic discs S in them. (4) In Theorem 54 we proved that domains of holomorphy have real analytic exhaustion functions. (5) Let us assume $D \subseteq \mathbb{C}^m$ is a domain holomorphy and prove directly that $-\log \operatorname{dist}_r(z, \partial D)$ is plurisubharmonic. Here r is the multi-radii for a fixed open polydisc around the origin. (dist_r is defined in the course of proving Cartan-Thullen.) Fix $z_0 \in D$, $w \in \mathbb{C}^m \setminus \{0\}$ and $r > 0$ so small such that $\Delta := \{z_0 + \lambda w : |\lambda| < r\}$ is contained in D. Assuming

$$-\log \operatorname{dist}_r(z_0 + \lambda w, \partial D) \leq \operatorname{Re} f(\lambda) \quad \text{for} \quad |\lambda| = r, \tag{3.21}$$

for some holomorphic polynomial f, we need to show that the same inequality holds for $|\lambda| \leq r$. Choose a holomorphic polynomial F in m variables such that $F(z_0 + \lambda w) = f(\lambda)$. Then (3.21) becomes

$$|\exp(-F(z))| \leq \operatorname{dist}_r(z) \quad \text{for} \quad z \in \partial\Delta.$$

From this we infer

$$|\exp(-F(z))| \leq \operatorname{dist}_r(z) \quad \text{for} \quad z \in \widehat{\partial\Delta}_{\mathcal{O}(D)}. \tag{3.22}$$

because if (3.22) is violated at $z = a$ then by Thullen's lemma (Remark 46) the power series representation of every holomorphic function in D around a is convergent up to the polydisc $P' := a + |\exp(-F(a))|P \not\subseteq D$. By the maximum principle $\widehat{\partial\Delta}$ contains D, so we have proved (3.21) for $|\lambda| \leq r$. ■

Example 63. We can now give a proof for the hard part of Corollary 47.(5): Connected tubes which are domain of holomorphy are convex. By contradiction assume a connected tube T over base B which is a domain of holomorphy but not convex. Since B is not convex there is a closed real line segment in B such that the function $\operatorname{dist}(-, \partial B)$ restricted to that segment attains its minimum at an internal point, or equivalently there is $a \in B$ and $b \in \mathbb{R}^m$ such that the function $f : [-1, 1] \to \mathbb{R}$, $t \mapsto -\log \operatorname{dist}(a + tb, \partial B)$, attains its maximum at some $-1 < t < 1$. Consider the function $F(\lambda) = -\log \operatorname{dist}(a + \lambda b, \partial T)$ defined in the closed unit disc of the complex plane. Note that F is subharmonic (because T is pseudoconvex by Theorem 62), and $F(\lambda) = f(\operatorname{Re}\lambda)$, so F attains a maximum inside T, so F is constant, so also f. This is a contradiction. ■

3.6.4 Strongly pseudoconvex domains

We have already defined a strong notion of pseudoconvexity for C^2 opens. This notion can be generalized to arbitrary opens in the following way. An open $D \subseteq \mathbb{C}^m$ is called

strongly pseudoconvex if there is a neighborhood $U \subseteq \mathbb{C}^m$ of ∂D and a strictly plurisubharmonic function ρ on U such that $D \cap U = \{z \in U : \rho(z) < 0\}$. Note that a strongly pseudoconvex domain does not necessarily have a C^2 boundary (we did not require $d\rho_p \neq 0$ for $p \in \partial D$) as the example $D = \{x + \sqrt{-1}y \in \mathbb{C} : 2x^2 - y^2 < 0\}$ shows.

Theorem 64. *(1) Every bounded strongly Levi pseudoconvex open of $\mathbb{C}^m$ has a strictly plurisubharmonic defining function. (1′) Every strongly Levi pseudoconvex boundary point of an open of $\mathbb{C}^m$ has a strictly plurisubharmonic local defining function. (2) A bounded open of $\mathbb{C}^m$ with C^2 boundary is strongly pseudoconvex if and only if it is strongly Levi pseudoconvex. (3) Strongly pseudoconvex opens of $\mathbb{C}^m$ are pseudoconvex.*

Proof. (1) Assume bounded strongly Levi pseudoconvex open $D \subseteq \mathbb{C}^m$ with C^2 defining function r. We will prove that for sufficiently large constant $C > 0$ the function $\rho := \exp(Cr) - 1$ is a strictly plurisubharmonic defining function. ρ is a C^2 defining function because $\rho < 0$ is equivalent to $r < 0$, and $d\rho = C\exp(Cr)dr$ does not vanish on the boundary. A simple computation shows that

$$L_p(\rho; t) = C\exp(Cr)\left(C|\partial r_p(t)|^2 + L_p(r; t)\right).$$

The strong Levi pseudoconvexity condition exactly says that at least one of the summands of $L_p(\rho; t)$ are strictly positive for each $p \in \partial D$ and $t \in \mathbb{C}^m \setminus \{0\}$, so choosing C large enough makes ρ strictly plurisubharmonic on some neighborhood of ∂D. To see this consider compacts

$$K_1 := \partial D \times \{t \in \mathbb{C}^m : |t|_2 = 1\}, \quad K_2 := \{(p, t) \in K_1 : L_p(r; t) \leq 0\},$$

and constants

$$C_1 := \min\{|\partial r_p(t)|^2 : (p, t) \in K_2\} > 0, \quad C_2 := \min\{L_p(r; t) : (p, t) \in K_1\}.$$

Choosing any C with $C_3 := CC_1 + C_2 > 0$ gives $L_p(\rho; t) > C_4C_3$ on K_1, where $C_4 > 0$ is the minimum of $C\exp(Cr)$ on ∂D. By homogeneity $L_p(\rho; t) \geq C_4C_3|t|_2^2$ for every $p \in \partial D$ and $t \in \mathbb{C}^m$. By continuity the same inequality holds on some neighborhood of ∂D. For the converse direction,

(1′) The same proof of (1).

(2) The if part is immediate from (1). For the other direction, assuming a neighborhood $U \subseteq \mathbb{C}^m$ of ∂D and strictly plurisubharmonic function $\rho : U \to \mathbb{R}$ such that $D \cap U = \{z \in U : \rho(z) < 0\}$, it suffices to show that $d\rho$ never vanishes on ∂D. Let $r : \mathbb{C}^m \to \mathbb{R}$ be a C^2 defining function for D. Note that both ρ and r are negative on $D \cap U$, zero ∂D, and nonnegative on $U \setminus \overline{D}$. By the fundamental theorem of calculus one can write $\rho = fr$ for some $f \in C^1(U)$. By contradiction assume $p \in \partial D$ such that $d\rho = 0$ at p. Since $d\rho = rdf$ at p it follows that $f(p) = 0$. Since f is nonnegative on U it follows that $df = 0$ at p. But then all second-order derivatives of ρ vanish at p, and this contradicts strict plurisubharmonicity.

(3) Let r be a C^2 plurisubharmonic function on a neighborhood $U \subseteq \mathbb{C}^m$ of ∂D such that that $D \cap U = \{z \in U : r(z) < 0\}$. (We do not need strict pseudoconvexity.) By Theorem 61 we need only check that $D \cap B$ is pseudoconvex for every open ball

$B \subseteq \mathbb{C}^m$ which is contained in U. Indeed, the plurisubharmonic convex hull $\hat{K}_{PS(D\cap B)}$ of every compact $K \subseteq D \cap B$ is compactly supported in both B (because every open ball is pseudoconvex) and $D \cap U$ (because $r < 0$ on K but $r \geq 0$ on $U \setminus D$.). *Another argument when D is bounded.* By a partition of unity we can assume $r \in C^2(\overline{D})$ and $r < 0$ on D. Clearly, $\rho := -\log(-r)$ is a plurisubharmonic exhaustion function for $D \cap U$. Then $-\log(-r) + (C+1)|z|^2$ where $C = \inf\{L_z(\rho; t) : z \in D \setminus U, |t|_2 = 1\} > -\infty$ is a plurisubharmonic exhaustion function for D. ■

Example: (1) Open polydiscs in $\mathbb{C}^m$ are pseudoconvex (because they are products of open subsets of the plane), but not strongly if $m > 1$.??? (2) Every C^2 open subset of the complex plane is strongly Levi pseudoconvex for the trivial reason that the complex tangent spaces are all complex dimension 0.

3.6.5 Pseudoconvex domains with smooth boundary

Theorem 65. *A C^2 open $D \subseteq \mathbb{C}^m$ is pseudoconvex if and only if it is Levi pseudoconvex.*

Proof. First of all since D has C^2 boundary it follows from the implicit function theorem that

$$r(z) = -\delta(z) := \begin{cases} -\mathrm{dist}(z, \partial D), & z \in D \\ \mathrm{dist}(z, D), & z \in \mathbb{C}^m \setminus D \end{cases}, \tag{3.23}$$

is C^2 on a neighborhood of the boundary (page 52). Theorems 52 and 53 combined show that Hartogs pseudoconvex opens are Levi pseudoconvex. *Another argument.* Assuming that $-\log\delta = -\log(-r)$ is plurisubharmonic on D we have

$$\sum -r^{-1}\frac{\partial^2 r}{\partial z_\alpha \partial \overline{z}_\beta} t_\alpha \overline{t}_\beta + \sum r^{-2}\frac{\partial r}{\partial z_\alpha}\frac{\partial r}{\partial \overline{z}_\beta} t_\alpha \overline{t}_\beta \geq 0, \quad \forall t \in \mathbb{C}^m,$$

for points in D sufficiently close to the boundary, so also for the points on the boundary by continuity. Specializing to complex tangent vectors t, the second summand vanishes and we have $\sum \partial^2 r/\partial z_\alpha \partial \overline{z}_\beta t_\alpha \overline{t}_\beta \geq 0$.

If part. Assume D is not pseudoconvex. Since pseudoconvexity is a local property of the boundary (Theorem 61) and δ is C^2 near the boundary, there is a point z_0 sufficiently close to the boundary such that

$$A := \frac{\partial^2}{\partial\lambda\partial\overline{\lambda}}\bigg|_{\lambda=0} \log\delta(z + \lambda w) > 0,$$

for some $w \in \mathbb{C}^m$. By Taylor theorem

$$\log\delta(z + \lambda w) = \log\delta(z) + \mathrm{Re}(B\lambda + C\lambda^2) + A|\lambda|^2 + o(|\lambda|^2), \quad \lambda \to 0,$$

therefore

$$\delta(z + \lambda w) \geq \delta(z) e^{\mathrm{Re}(B\lambda + C\lambda^2)} e^{A|\lambda|^2/2}, \tag{3.24}$$

for $|\lambda|$ sufficiently small. Choose $a \in \mathbb{C}^m$ such that $z + a \in \partial D$ and $\delta(z) = |a|_2$, and consider the holomorphic curve

$$z(\lambda) = z + \lambda w + e^{B\lambda + C\lambda^2} a,$$

passing through the boundary point $z(0) = z + a$. This is the point that the Levi pseudoconvexity is violated, as follows. By the triangle inequality and (3.24) we have

$$\delta(z(\lambda)) \geq \delta(z + \lambda w) - \left| a e^{B\lambda + C\lambda^2} \right|_2 \geq |a|_2 e^{B\lambda + C\lambda^2} \left(e^{A|\lambda|^2/2} - 1 \right),$$

for $|\lambda|$ sufficiently small. This shows that the C^2 function $\lambda \mapsto \delta(z(\lambda))$ has a strict local minimum at the origin, so its real Hessian should be positive definite and specially its Laplacian should be strictly positive:

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} \delta(z(\lambda)) = 0, \quad \left.\frac{\partial^2}{\partial \lambda \partial \overline{\lambda}}\right|_{\lambda=0} \delta(z(\lambda)) > 0.$$

This means that $t := \partial z/\partial \lambda(0) \in T^{\mathbb{C}}_{z+a}$ is a complex tangent vector which violates the Levi pseudoconvexity condition at $z + a \in \partial D$. *Another proof.* The idea is to reduce to the strongly pseudoconvex case. Let r be a local defining function for $p \in \partial D$. Consider

$$D_{p,\epsilon} := \left\{ z \in D : r(z) + \epsilon |z - p|_2^2 < 0 \right\}, \quad \epsilon > 0.$$

Find a neighborhood $U \subseteq \mathbb{C}^m$ of p and $\epsilon_0 > 0$ such that firstly for every $z \in U$ and $\epsilon < \epsilon_0$ there exists $w \in \partial D_{p,\epsilon} \cap U$ such that $\operatorname{dist}(z, \partial D_{p,\epsilon}) = |z - w|_2$, and secondly for every $\epsilon < \epsilon_0$ and $q \in \partial D_{p,\epsilon} \cap U$ the Levi form of $r + \epsilon |z - p|_2^2$ is positive definite at q. Since every point of $\partial D_{p,\epsilon} \cap U$ is a strongly pseudoconvex boundary point of $D_{p,\epsilon}$ it follows that $-\log \operatorname{dist}(z, \partial D_{p,\epsilon})$ is plurisubharmonic on $D_{p,\epsilon} \cap U$. Since $-\log \operatorname{dist}(z, \partial D_{p,\epsilon})$ decreases to $-\log \operatorname{dist}(z, \partial D)$ as $\epsilon \to 0$ it follows that $-\log \operatorname{dist}(z, \partial D)$ is plurisubharmonic on $D \cap U$, and we are done because pseudoconvexity is a local property of the boundary. *Yet another proof.* Here is another proof for the only if part under extra conditions that D is bounded and with C^3 (instead of C^2) boundary [Ran, page 63]. Let r be a C^3 defining function. Since $dr_z = 2\mathrm{Re}\partial r_z \neq 0$ on the boundary there exists a neighborhood $U \subseteq \mathbb{C}^m$ of the boundary such that $\partial r_z \neq 0$ for $z \in U$. Every $t \in \mathbb{C}^m$ has the canonical orthogonal decomposition

$$t = t'_z + t''_z, \quad t'_z \in T^{\mathbb{C}}_z, \quad t''_z = \frac{\langle t, r_{\overline{z}} \rangle}{\langle r_{\overline{z}}, r_{\overline{z}} \rangle} r_{\overline{z}} \in \left(T^{\mathbb{C}}_z \right)^{\perp}, \quad z \in U,$$

where $r_{\overline{z}}$ denotes $(\partial r/\partial \overline{z}_j(z)) \in \mathbb{C}^m$. Note that this decomposition varies C^2 with respect to z. Levy pseudoconvexity says that $L_z(r; t'_z) \geq 0$ for $z \in \partial D$ and $t \in \mathbb{C}^m$. Since the left hand side of this inequality is a C^1 function in z, comparing the Taylor expansions of it with that of $r(z)$ around boundary points shows that every $z \in \partial D \cap U$ has neighborhood $V_z \subseteq \mathbb{C}^m$ and $C_z > 0$ such that

$$L_z(r; t'_z) \geq -C_z |r(z)| \quad \text{for} \quad (z, t) \in V_z \times \{ t \in \mathbb{C}^m : |t'_z| = 1 \}.$$

Since ∂D is compact there exists $C > 0$ such that after shrinking U we have

$$L_z(r; t'_z) \geq -C |r(z)| |t'_z|^2 \quad \text{for} \quad (z, t) \in (D \cap U) \times \mathbb{C}^m.$$

Since $L_z(r; t'_z)$ is quadratic in t'_z, and $t''_z = O(|\partial r_z(t)|)$ (note that $\partial r_z(t)$ is the notation for the inner product $\langle t, \partial r/\partial \overline{z} \rangle = \sum \partial r/\partial \overline{z}_j(z) t_j$) we have

$$L_z(r; t) = L_z(r; t'_z) + O(|t'_z||t''_z|) + O(|t''_z|^2) \geq -C |r(z)| |t|^2 - C_1 |\partial r_z(t)| |t|.$$

Switching to $\rho := -\log(-r)$ we get

$$\begin{aligned}
L_z(\rho;t) &= -r^{-1}L_z(r;t) + r^{-2}|\partial r_z(t)|^2 \\
&\geq -C|t|^2 - \underbrace{\frac{|\partial r_z(t)|}{|r(z)|}}_{A}\underbrace{C_1|t|}_{B} + \frac{|\partial_z(t)|^2}{|r(z)|^2} \\
&\geq -C_2|t|^2. && AB \leq A^2 + (B/2)^2
\end{aligned}$$

Finally, as in the proof of Theorem 64.(3), for sufficiently large $C_2 > 0$ one can make $\rho + C_2|z|^2$ a strictly plurisubharmonic exhaustion function for D. ■

Theorem 66 (Narasimhan)**.** *A C^2 open $D \subseteq \mathbb{C}^m$ is strongly Levi pseudoconvex if and only if it is locally biholomorphically equivalent to strongly convex opens, namely for every $p \in \partial D$ there is a holomorphic change of coordinates on some neighborhood $U \subseteq \mathbb{C}^m$ of p such that $D \cap U$ is strongly convex.*

Proof. Recall that (strong) pseudoconvexity is a biholomorphic invariant notion weaker than (strong) convexity (page 58). That gives the if part. For the other direction fix $p \in \partial D$ and a C^2 local defining functions $r(z)$ at p. Replacing r by $\exp(Cr) - 1$ for sufficiently large $C > 0$, Theorem 64.(1′) shows that we can assume r to be strictly plurisubharmonic on a neighborhood of p. (Note that D might be unbounded.) After a holomorphic change of coordinates we can assume $p = 0$ and $dr_p = dx_1$ where $x_1 = \mathrm{Re} z_1$. The Taylor expansion of r around p is given by

$$r(z) = \mathrm{Re}\left(z_1 + \sum_{1\leq j,k\leq m} \frac{\partial^2 r}{\partial z_j \partial z_k}(0) z_j z_k\right) + L_p(r;z) + o\left(|z|_2^2\right), \quad z \to 0.$$

Under the holomorphic change of coordinates

$$w_1 = z_1 + \sum_{1\leq j,k\leq m} \frac{\partial^2 r}{\partial z_j \partial z_k}(0) z_j z_k, \quad w_2 = z_2, \quad \ldots, w_m = z_m,$$

our defining function have the following simple Taylor expansion:

$$r(w) = \mathrm{Re} w_1 + L_0(r(w);w) + o\left(|w|_2^2\right), \quad w \to 0.$$

The real Hessian is now given by a strictly plurisubharmonic function on some neighborhood U of the origin, so $D \cap U$ is strongly convex. ■

The following example shows that the theorem above fails if we drop "strongly".

Example 67 (Kohn-Nirenberg [KN])**.** Consider the following open subset of $\mathbb{C}^2$ with real-analytic boundary:

$$D := \left\{ z \in \mathbb{C}^2 \;\middle|\; r := \mathrm{Re} z_2 + |z_1|^8 + \frac{15}{7} z_1^2 \,\mathrm{Re} z_1^6 < 0 \right\}.$$

The complex Hessian of r equals $(16|z_1|^6 + 15\mathrm{Re} z_1^6)\, dz_1 d\bar{z}_1$, so is nonnegative Also, the complex tangents are of the form $(a, -2a\partial r/\partial z_1)$. Therefore all the points of the boundary of D are strongly pseudoconvex except for $\{z_1 = \mathrm{Re} z_2 = 0\}$. Specially, the origin is among these exceptional boundary points. It can be shown that for any holomorphic function f on a neighborhood U of the origin such that $f(0) = 0$, the zero set of f meets both $U \cap D$ and $U \setminus \overline{D}$. This shows that there is no holomorphic change of coordinates in any neighborhood U of the origin such that $U \cap D$ is convex. ??? ■

Theorem 58.(1′) combined with Sard's lemma gives:

Theorem 68. *Pseudoconvex opens of $\mathbb{C}^m$ can be exhausted by strongly pseudoconvex open with smooth boundaries.*

We end this chapter with a motivation to the later materials.

Example 69 (Levi's construction)**.** Suppose an open $D \subsetneq \mathbb{C}^m$ and fix an arbitrary boundary point. If D is convex then we constructed in Example 45 a holomorphic function on D which blows up at p; this shows that convex opens are (weak) domain of holomorphy. This time let us assume that D is strongly Levi pseudoconvex with smooth boundary and try to find again a function $f_p \in \mathcal{O}(D)$ which blows up at p. For the simplicity of presentation assume $p = 0$. Let r be a strictly plurisubharmonic local defining function (Theorem 64.(1′)). The Taylor expansion of r around p gives

$$\mathrm{Re} F(z) = r(z) - L_p(r; z) + o(|z|_2^2), \quad z \to 0,$$

where

$$F(z) := 2 \sum_{1 \le j \le m} \frac{\partial r}{\partial z_j}(0) z_j + \sum_{1 \le j,k \le m} \frac{\partial^2 r}{\partial z_j z_k}(0) z_j z_k.$$

The quadratic expression $L_p(r; z)$ dominates $o(|z|_2^2)$ on some sufficiently small neighborhood $U \subseteq \mathbb{C}^m$ of p, so $\mathrm{Re} F$ never vanishes on $\overline{D} \cap U$, hence the function

$$g_p := \frac{1}{F} \in \mathcal{O}(D \cap U) \cap C^\infty(\overline{D} \cap U \setminus \{p\})$$

blows up at p. To prove that D is a (weak) domain of holomorphy we need to find a modification f_p of g_p which is holomorphic on whole D and still blows up at p. Assuming a smooth bump function ψ compactly supported in U then $g := \overline{\partial}(\psi g_p) = g_p \overline{\partial} \psi$ can be seen as a $\overline{\partial}$-closed $(0,1)$-form living in $C^\infty_{0,1}(\overline{D})$. If it is true that

For any $g \in C^\infty_{0,1}(\overline{D})$ with $\overline{\partial} g = 0$ there exists $u \in C^\infty(\overline{D})$ such that $\overline{\partial} u = g$.

then

$$f_p := \psi g_p - u \in \mathcal{O}(D) \cap C^\infty(\overline{D} \setminus \{p\})$$

works as our desired holomorphic function on D which blows at p. The statement in italics above is a deep fact that will be proved in Theorem 110. So far we have shown that D is a weak domain of holomorphy. Let us show directly (without using Cartan-Thullen) show that D is a domain of holomorphy by constructing a function $f \in \mathcal{O}(D)$ which can not be extended holomorphically across any boundary point. Choose a sequence of points p_j which is dense in ∂D, mutually disjoint line segments γ_j normal to the boundary at

p_j, and let $f_j := f_{p_j}\mathcal{O}(D)$ be the blow up function constructed above. Choose nonzero constants c_j small enough such that

$$|c_j f_j(z)| < 2^{-j} \quad \text{for} \quad z \in \{z \in \mathbb{C}^m : \operatorname{dist}(z, \partial D) \geq 2^{-j}\} \cup \bigcup_{k<j} \gamma_k.$$

Then $f := \sum c_j f_j$ converges uniformly on compact of D, so represents a holomorphic function on D. On the other hand since

$$\left|\sum_{k \neq j} c_k f_k(z)\right| < 2^{-j} + \sum_{k>j} 2^{-k} < 2^{-j} + 1, \quad z \in \gamma_j,$$

it follows that $|f(z)|$ blows up as z approaches p_j along γ_j. ■

Chapter 4

d-bar problem on pseudoconvex domains (via PDE methods) with applications

References: [Hör, chapter 4][Ohs, chapters 2,4,5]

The d-bar problem was introduced in Section 3.2.1, and we have seen so far numerous applications of it in solving important problems of SCV, for example Hartogs extension phenomenon (Theorem 24), Cousin problems (Theorems 19, 20) and interpolation problem (Theorem 21). In the first two section of this chapter we solve the d-bar problem on pseudoconvex domains and discuss some of its applications. Later we use more sophisticated d-bar techniques to solve some important problems in the function theory and functional analysis of holomorphic functions of several variables including the division and interpolation problems.

4.1 d-bar problem

We want to prove the following fundamental theorem:

Theorem 70 (d-bar problem; smooth solutions. Levi's problem)**.** *For an open $D \subseteq \mathbb{C}^m$ the followings are equivalent:*

(1) D is a domain of holomorphy.

(2) D is pseudoconvex.

(3) For every smooth $(p, q+1)$-form f on D with $\overline{\partial} f = 0$ there exists a smooth (p, q)-form u on D such that $\overline{\partial} u = f$.[1]

(4) For every smooth $(0, q+1)$-form f on D with $\overline{\partial} f = 0$ and $q \leq m-2$ there exists a smooth $(0, q)$-form u on D such that $\overline{\partial} u = f$.[2] *This is called* **Serre's condition***.*

[1] In cohomological terms: Dolbeault cohomologies $H^{p,q}_{\mathrm{Db}}(D)$ vanish for $1 \leq q \leq m$ and $0 \leq p \leq m$.

[2] In cohomological terms: Dolbeault cohomologies $H^{0,q}_{\mathrm{Db}}(D)$ vanish for $1 \leq q \leq m-1$.

To solve $\overline{\partial} u = f$ in the smooth category we use a method originally developed by Hilbert to solve the Poisson equation $\Delta u = f$ in potential theory (Δ denotes the usual Laplacian acting on functions), and later generalized by Hodge and Weyl to solve the corresponding problem for the Hodge Laplacian acting on differential forms on oriented Riemannian manifolds (the so-called Hodge decomposition theorem) [Don-GA, pages 4–15][Don-RS, chapter 9][War, chapter 6][Tay, chapter 5][GH, pages 80–100][Wey1, Wey2]. This strategy is sometimes called the **direct method of the calculus of variations** [Mor][Dac][GM], and is a two-step procedure:

1. *Step I: To solve the problem in some Hilbert space sense.* The initial function spaces involved in the statement of the d-bar problem, namely $C^\infty_{p,q}(D)$, are merely Frechet spaces [Rud-FA, chapters 1-4], hence lack a rich geometric theory (besides convexity) contrary to Hilbert spaces. To fix we bring Hilbert spaces into the scene: For every open $D \subseteq \mathbb{C}^m$ and $f \in C^\infty_{p,q+1}(D)$ one can find a large variety of smooth functions φ such that f belongs to the Hilbert space $L^2_{p,q+1,\varphi}(D)$ of measurable forms square-integrable with respect to the Lebesgue measure multiplied by the weight factor $\exp(-\varphi)$. We now can look at $\overline{\partial} u = f$ as a linear problem on Hilbert spaces. Integration by parts is used to obtain an energy estimate (also called apriori estimate); this is an inequality which controls some L^2 norm of f by the L^2 norms of some differential operators applied to f. This energy estimate combined with some standard geometric techniques of the theory of Hilbert spaces (orthogonal projection, Riesz representation theorem, etc.) gives some solution u of $\overline{\partial} u = f$.

2. *Step II: Regularization.* The energy estimate combined with standard regularity techniques (mollification, Sobolev embedding, difference quotients, etc.) shows that (all or some canonical) Hilbert space solution obtained in Step I is in fact a smooth differential form, maybe after correction on some null set.

The long proof of Theorem 70 will be completed in page 94. We start by developing some basic function theory (distributions) and functional analysis (unbounded operators) needed for the proof.

4.1.1 Preliminaries: Distributions, Sobolev spaces

We review some basic notions and examples from the theory of distributions [Rud-FA, chapter 6][Fol, chapter 9][Hör-PDE, chapter 1]. The fundamental theme here is that sometimes in mathematics the dual notions, although more abstract at the first sight, are easier to work with compared to original notions: Differential forms compared to vector fields, cohomology compared to homology, and now distributions compared to integrable functions. Let $U \subseteq \mathbb{R}^n$ be an open. As usual $\mathcal{D}(U)$ denotes the space of smooth functions compactly supported in U. A **distribution** on U is a $\mathbb{C}$-linear functional $F : \mathcal{D}(U) \to \mathbb{C}$ which is continuous in the following sense: For every sequence $\psi_j \in \mathcal{D}(U)$ compactly supported in a compact $K \subseteq U$ such that $\sup_U |\partial^\alpha(\psi_j - \psi)| \to 0$ for all multi-indices $\alpha \in \mathbb{N}^n$ and some $\psi \in \mathcal{D}(U)$ (we then write "$\psi_j \to \psi$ in $\mathcal{D}(U)$"; necessarily, ψ is compactly supported in K.) we have that $F(\psi_j) \to F(\psi)$. Here ∂^α is an abbreviation for the differential operator $\partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $x_1, \ldots, x_n$

are the standard coordinates of $\mathbb{R}^n$. Later we will see that it is more useful to use the pairing notation $\langle F, \psi \rangle$ for the action of distributions on functions with compact support. The space of all distributions on U is denoted by $\mathcal{D}'(U)$.

- A linear functional $F : \mathcal{D}(U) \to \mathbb{C}$ is a distribution if and only if for every compact $K \subseteq U$ there exist $C > 0$ and integer k such that $|\langle F, \psi \rangle| \leq C \sum_{|\alpha| \leq k} \sup_U |\partial^\alpha \psi|$ for every smooth function ψ which is compactly supported in K. (*Proof.* The if part is trivial. If the only if part is wrong then there exists a compact $K \subseteq U$ and a sequence $(\psi_j)_{j \geq 1}$ of smooth functions compactly supported in K such that $\langle F, \psi_j \rangle = 1$ but $\sup_U |\partial^\alpha \psi| < j^{-1}$ for every positive integer j and multi-index α. Then $\psi_j \to 0$ however $F(\psi_j) \to 1$.)

- Distributions are generalized functions in the sense that $L^1_{\mathrm{loc}}(U)$ can be canonically injected into $\mathcal{D}'(U)$ by sending $f \in L^1_{\mathrm{loc}}(U)$ to the functional $F : \mathcal{D}(U) \to \mathbb{C}$ given by $\langle F, \psi \rangle = \int_U f \psi d\mu$, where μ is the Lebesgue measure. That this is an injection follows from the Lebesgue differentiation theorem [Fol, 3.18, 8.15].[3] If f is mapped to F as above then F is said to be **represented by** f, and denoted again by f. Example: The distribution δ given by $\langle \delta, \psi \rangle = \psi(0)$ is called the **Dirac unit mass distribution**, and can not be represented by any L^1_{loc} function.

- Given a smooth function φ and distribution F on U, the **product** φF is the distribution given by the pairing $\langle \varphi F, \psi \rangle = \langle F, \varphi \psi \rangle$. Clearly, if F is represented by a smooth function then this definition reduces to the usual poitwise multiplication of functions.

- Given a distribution F on U and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, the **distributional derivative** $\partial^\alpha F$ of F is the distribution on U given by the pairing $\langle \partial^\alpha F, \psi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \psi \rangle$, where $|\alpha| = \sum \alpha_j$ The sign factor is there so that if F is represented by a smooth function then this definition reduces (after applying integration by parts $\sum \alpha_j$ times) to the usual differentiation of functions. If F and $\partial^\alpha F$ are represented respectively by functions f and g then g is called the **weak derivative** of f. Example: The absolute value function $| \; | : \mathbb{R} \to \mathbb{R}$ has the weak derivative $d|x|/dx = 2H(x) - 1$, where $H = 1_{[0,\infty)}$ is the unit step function. The distributional derivative of the unit step function is the Dirac unit mass distribution δ.

- The Leintiz rule holds for differentiation of the product:

$$\partial^\alpha (\varphi F) = \sum \frac{\alpha!}{\beta!(\alpha - \beta)!} \left(\partial^\beta \varphi\right) \left(\partial^{\alpha - \beta} F\right), \quad \forall \alpha \in \mathbb{N}^n \; \forall F \in \mathcal{D}'(U) \; \forall \varphi \in C^\infty(U),$$

  where the summation is over all multi-indices $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ such that $\beta_j \leq \alpha_j$ for all j.

- Given $s \in \{0, 1, \ldots, \infty\}$, the **Sobolev space** $W^{2,s}(U)$ consists of all Borel measurable functions on U whose all distributional derivatives of total order $\leq s$ are represented

[3] Here is a direct argument. Assume $f \in L^1_{\mathrm{loc}}(U)$ such that $\int_U f \psi d\mu = 0$ for every $\psi \in \mathcal{D}(U)$. Let η be a smooth bump function compactly supported in the open unit ball of $\mathbb{R}^n$ and normalized to have integral 1. Then the mollification $f_\epsilon(x) = f(x) * \epsilon^{-n} \eta(x/\epsilon)$ vanishes identically on $\{x \in U : \mathrm{dist}(x, \partial U) > \epsilon\}$, and it is famous that $f_\epsilon \to f$ in $L^1_{\mathrm{loc}}(U)$ [Fol, 8.14].

by $L^2(U)$ function. Note that $W^{2,0}(U) = L^2(U)$. More generally, $W^{2,s}_{\mathrm{loc}}(U)$ consists of all Boreal measurable functions on U such that $\psi f \in W^{2,s}(U)$ for every smooth bump function ψ compactly supported in U.

4.1.2 Preliminaries: Unbounded operators

We need some basic notions and facts from the theory of unbounded operators [Rud-FA, chapter 13][Wei, chapters 4-5][dOl, chapters 1-2][Bre, section 2.6]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over $\mathbb{C}$. By an **unbounded operator** $A : \mathcal{H}_1 \to \mathcal{H}_2$ we just mean a $\mathbb{C}$-linear map $A : \mathrm{Dom}_A \to \mathcal{H}_2$ defined on some linear subspace $\mathrm{Dom}_A \subseteq \mathcal{H}_1$. (So every bounded (= continuous) operator is an unbounded operator in this terminology!) A is called **densely defined** if Dom_A is dense in $\mathcal{H}_1$.

- A is called **closed** if the graph $\mathcal{G}_A = \{(f, Af) : f \in \mathrm{Dom}_A\}$ of A is closed in $\mathcal{H}_1 \times \mathcal{H}_2$. Equivalently, for every sequence f_j in Dom_A such that f_j converges to $f \in \mathcal{H}_1$ and Af_j converges to g we must have $f \in \mathrm{Dom}_A$ and $Af = g$. (The closed graph theorem says that an unbounded operator defined on whole $\mathcal{H}_1$ is closed if and only it is continuous, but when $\mathrm{Dom}_A \neq \mathcal{H}_1$ neither of the notions of closedness and continuity implies the other.)

- If A is densely defined then the **adjoint** of A, denoted by A^*, is the unbounded operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ defined as follows: Dom_{A^*} consists of all $g \in \mathcal{H}_2$ such that $\langle Ah, g\rangle$ is continuous with respect to $h \in \mathrm{Dom}_A$, namely $|\langle Ah, g\rangle| \leq C\|h\|$ for some positive constant C. If so then the functional $\mathrm{Dom}_A \to \mathbb{C}$ mapping h to $\langle Ah, g\rangle$ has a unique continuous extension to $\mathcal{H}_1$ by Hahn-Banach theorem, so by Riesz representation theorem there exists a unique $f \in \mathcal{H}_1$ such that $\langle Ah, g\rangle = \langle h, f\rangle$, and we set $A^*g = f$. Equivalently, A^* can be characterized by $\mathcal{G}_{A^*} = (J\mathcal{G}_A)^\perp$ where $J(f, g) = (g, -f)$.

- If A is densely defined then A^* is closed. (*Proof.* $\mathcal{G}_{A^*} = (J\mathcal{G}_A)^\perp$ and the orthogonal complement of every subset of a Hilbert space is closed.)

- If A is densely defined and closed then so is A^*, and we have $A^{**} = A$. (*Proof.* Since $J^2 = -\mathrm{id}$ and J commutes with the operations of closure and orthogonal complement when applied to subspaces it follows that $J\mathcal{G}_{A^*}^\perp = -\overline{\mathcal{G}_A} = \mathcal{G}_A$. To show that A^* is densely defined assume $g \in \mathrm{Dom}_{A^*}^\perp$. Since $(0, g) \in J\mathcal{G}_{A^*}^\perp = \mathcal{G}_A$ it follows that $g = 0$. This shows that Dom_{A^*} is dense in $\mathcal{H}_2$. Finally, $\mathcal{G}_{A^{**}} = J\mathcal{G}_{A^*}^\perp = JJ\mathcal{G}_A^{\perp\perp} = -\overline{\mathcal{G}_A} = \mathcal{G}_A$ shows that $A^{**} = A$.)

- If A is densely defined then $\mathrm{Ran}_A^\perp = \mathrm{Ker}_{A^*}$. If A is densely defined and closed then $\mathrm{Ran}_{A^*}^\perp = \mathrm{Ker}_A$, so Ker_A is closed. (*Proof.* The first assertion is immediate from definition. Replacing A by A^* gives the second.)

- Staying in the framework of Zermelo-Frankel set theory (not using the axiom of choice) one can not construct a noncontinuous unbounded operator $\mathcal{H}_1 \to \mathcal{H}_2$ which is defined on whole $\mathcal{H}_1$ [Wri][Fol, page 179]. In other words, all concrete noncontinuous unbounded operators are partially defined. The most important examples of unbounded operators are differential operators, specially the d-bar operator in our case.

4.1.3 The proof of Theorem 70

Step 0: The basic setting

Our basic setting throughout this section is as follows. Fix open $D \subseteq \mathbb{C}^m$ and integers $p, q \in \{0, \ldots, m\}$. Consider continuous functions $\varphi_1, \varphi_2, \varphi_3$ on D. We will put more restrictions on φ_j during the proof. The d-bar operator $\overline{\partial} : L^2_{p,q,\varphi_j} \to L^2_{p,q+1,\varphi_k}$ is an unbounded differential operator defined on the dense subspace $\mathcal{D}_{p,q}(D) \subseteq L^2_{p,q,\varphi_j}$ of smooth compactly supported (p, q)-forms on D. Here L^2_{p,q,φ_j} denotes the Hilbert space of (p, q)-forms with Borel measurable components which are square integrable with respect to the Lebesgue measure μ multiplied by the weight function $\exp(-\varphi_j)$. We use the notations

$$\langle u, v\rangle = {\sum_{I,K}}' u_{I,K}\overline{v_{I,K}}, \quad |u|^2 = \langle u, u\rangle, \quad (u, v)_{\varphi_j} = \int_D \langle u, v\rangle e^{-\varphi_j} d\mu, \quad \|u\|^2_{\varphi_j} = (u, u)_{\varphi_j},$$

for the pointwise and global inner product and norm of differential forms

$$u = {\sum_{I,K}}' u_{I,K} dz_I \wedge d\overline{z}_K, \;\; v = {\sum_{I,K}}' v_{I,K} dz_I \wedge d\overline{z}_K \in L^2_{p,q,\varphi_j}(D).$$

Let T and S denote the **maximal closed extensions** [BDT] of $\overline{\partial}$ acting on $L^2_{p,q,\varphi_1}(D)$ and $L^2_{p,q+1,\varphi_2}(D)$ respectively:

$$\boxed{L^2_{p,q,\varphi_1}(D) \xrightarrow{T} L^2_{p,q+1,\varphi_2}(D) \xrightarrow{S} L^2_{p,q+2,\varphi_3}(D)}. \tag{4.1}$$

This means that Dom_T consists of all $u \in L^2_{p,q,\varphi_1}$ such that the distributional (or weak) derivative $\overline{\partial} u$ is represented by some $f \in L^2_{p,q+1,\varphi_2}$, namely $(f, \psi)_{\varphi_2} = (u, \overline{\partial}^* \psi)_{\varphi_1}$ for every $\psi \in \mathcal{D}_{p,q+1}(D)$, where $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$ given by formula (4.2) below; in that case we set $Tu = f$. It is a basic fact[4] that T is closed, namely if a sequence $u_j \in \mathrm{Dom}_T$ converges in $L^2_{p,q,\varphi_1}(D)$ to some u and Tu_j converges in $L^2_{p,q+1,\varphi_2}(D)$ to some f then $u \in \mathrm{Dom}_T$ and $Tu = f$. *Proof.* For every $\psi \in \mathcal{D}_{p,q+1}$ we have $(f, \psi)_{\varphi_2} = \lim(Tu_j, \psi)_{\varphi_2} = \lim(u_j, \overline{\partial}^* \psi)_{\varphi_1} = (u, \overline{\partial}^* \psi)_{\varphi_1}$. Q.E.D. Similar remarks hold for S. Since $\overline{\partial}^2 = 0$ it follows that $ST = 0$, or equivalently $\mathrm{Ran}_T \subseteq \mathrm{Ker}_S$.

Lemma 71 (Formula for the action of the adjoint). *Assume* $f = {\sum}' f_{I,J} dz_I \wedge d\overline{z}_J \in \mathrm{Dom}_{T^*}$, *for example* $f \in \mathcal{D}_{p,q+1}(D)$. *Then:*

(1) We have

$$\boxed{T^* f = (-1)^{p-1} {\sum_{|I|=p, |K|=q}}' \sum_{j=1}^{m} e^{\varphi_1} \frac{\partial}{\partial z_j} \left(e^{-\varphi_2} f_{I,jK}\right) dz_I \wedge d\overline{z}_K}, \tag{4.2}$$

where $f_{I,L}$ is defined for all tuples $L \subseteq \{1, \ldots, m\}^{q+1}$ (shuffle or not) by declaring it to be antisymmetric with respect to L. More precisely, $f_{I,L} = f_{I,J} \epsilon^J_L$ where ϵ^J_L equals zero if $L \neq J$ as sets and equals the sign of the permutation making L to shuffle J otherwise.

[4] And one of the motivations for the development of distribution theory

(2) We have

$$T^*f = (-1)^{p-1}e^{\varphi_1}\vartheta\left(e^{-\varphi_2}f\right) = (-1)^{p-1}e^{\varphi_1-\varphi_2}\left(\vartheta + a\right)f,$$

where ϑ acting by

$$\boxed{\vartheta f = \sum_{|I|=p,|K|=q}{}' \sum_{j=1}^{m} \frac{\partial f_{I,jK}}{\partial z_j} dz_I \wedge d\overline{z}_K},$$

is a constant-coefficient differential operator obtained from T^* by assuming $\varphi_1 = \varphi_2 = 0$, and a is a matrix multiplication operator with C^k entries if $\varphi_2 \in C^{k+1}$.

(3) $\vartheta^2 = 0$.

Proof. (1) We test the equation $(T^*f, u)_{\varphi_1} = (f, Tu)_{\varphi_2}$ by $u = \sum' u_{I,K} dz_I \wedge d\overline{z}_K \in \mathcal{D}_{p,q}(D)$. Since

$$Tu = \overline{\partial}u = \sum_{I,K}{}' \frac{\partial u_{I,K}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_K = \sum_{I,K}{}' \sum_{j} \frac{\partial u_{I,K}}{\partial \overline{z}_j} (-1)^p \epsilon^J_{jK} dz_I \wedge d\overline{z}_J,$$

where J is the shuffle made of j and K, the equation $(T^*f, u)_{\varphi_1} = (f, Tu)_{\varphi_2}$ is expanded as

$$\int \sum_{I,K}{}' (T^*f)_{I,K}\overline{u_{I,K}} e^{-\varphi_1} d\mu = (-1)^p \int \sum_{I,K}{}' \sum_{j} f_{I,jK} \frac{\partial \overline{u_{I,K}}}{\partial z_j} e^{-\varphi_2} d\mu =$$
$$(-1)^{p-1} \int \sum_{I,K}{}' \sum_{j} \overline{u_{I,K}} \frac{\partial}{\partial z_j}\left(e^{-\varphi_2} f_{I,jK}\right) d\mu,$$

where we have used integration by parts for the last equality. Since this is true for every u we obtain the formula for $(T^*f)_{I,K}$ given in the statement of the lemma.

(2) Immediate from (1).

(3)

$$\vartheta^2 f = \sum_{|I|=p,|L|=q-1}{}' \sum_{j,k=1}^{m} \frac{\partial^2 f_{I,kjL}}{\partial z_k \partial z_j} dz_I \wedge d\overline{z}_L$$

vanishes because $f_{I,kjL} = -f_{I,jkL}$ and $\partial^2/\partial z_k \partial z_j = \partial^2/\partial z_j \partial z_k$. ■

Step I: Hilbert space solution

To find a smooth solution of $\overline{\partial}u = f$ our plan is to first find a Hilbert space solution: If φ_j are chosen suitably in a pseudoconvex domain D then for every $f \in L^2_{p,q+1,\varphi_2}(D)$ with $Sf = 0$ there exists $u \in L^2_{p,q,\varphi_1}(D)$ with $Tu = f$; equivalently $\mathrm{Ker}_S \subseteq \mathrm{Ran}_T$, or even $\mathrm{Ker}_S = \mathrm{Ran}_T$ because Ker_S always contains Ran_T. To achieve this we will need the second part of the following theorem. To motivate the statement of the theorem recall that in finite dimensional linear analysis (namely linear algebra) we have $\mathrm{Ran}_A = \mathrm{Ker}^{\perp}_{A^*}$ for every matrix A, so that $Au = f$ has a solution if and only if $\langle f, g\rangle = 0$ for every g with $A^*g = 0$. In infinite dimenional linear analysis (namely functional analysis) we have only $\overline{\mathrm{Ran}_A} = \mathrm{Ker}^{\perp}_{A^*}$ for densely defined closed operators A. The following theorem says how to deal with the closure in the left hand, and gives an if and only if condition for the solvability of $Au = f$.

Theorem 72 (Closed range theorem)**.** *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a densely defined closed unbounded operator between Hilbert spaces. Then:*

(1) For every $f \in \mathcal{H}_2$, there exists $u \in \mathcal{H}_1$ with $Au = f$ if and only if $|\langle f, g\rangle| \leq C\|A^*g\|$ for every $g \in \mathrm{Dom}_{A^*}$ and some $C \geq 0$.

(2) For every closed subspace $F \subseteq \mathcal{H}_2$ which $F \supseteq \mathrm{Ran}_A$, we have $F = \mathrm{Ran}_A$ if and only if $\|g\| \leq C\|A^*g\|$ for every $g \in \mathrm{Dom}_{A^*} \cap F$ and some $C \geq 0$.

(3) Ran_A is closed if and only if $\|g\| \leq C\|A^*g\|$ for every $g \in \mathrm{Dom}_{A^*} \cap \overline{\mathrm{Ran}_A}$ and some $C \geq 0$.

(4) Ran_A is closed if and only if Ran_{A^*} is closed.

Proof. (1) We only prove the if part because the other direction trivial. In accordance with the general philosophy of the duality theory in functional analysis (namely understanding a linear space through linear functionals living on it) one observes that our desired u is exactly the element of $\mathcal{H}_1$ which represents the anti-linear functional $\mathrm{Ran}_{A^*} \to \mathbb{C}$ mapping A^*g to $\langle f, g\rangle$. This functional is well-defined and bounded by C according to our hypothesis. By Hahn-Banach theorem it can be extended to a linear functional on whole $\mathcal{H}_1$ with the same bounded operator norm. (Another way: First extend by continuity to $\overline{\mathrm{Ran}_{A^*}}$ and then extend to whole $\mathcal{H}_1$ by declaring the functional to vanish on the orthogonal complement of $\overline{\mathrm{Ran}_{A^*}}$.) If $u \in \mathcal{H}_1$ is the vector that represents this extended functional according to the Riesz representation theorem then $\langle f, g\rangle = \langle u, A^*g\rangle$ for every $g \in \mathrm{Dom}_{A^*}$. It then follows by the very definition of the adjoint that $u \in \mathrm{Dom}_{A^{**}}$ and $A^{**}u = f$. Since A is densely defined and closed it follows that $A^{**} = A$, and we are done.

(2) For the if part, fixing arbitrary $f \in F$ and $g \in \mathrm{Dom}_{A^*}$, according to (1) we need to show that $|\langle f, g\rangle| \leq C\|A^*g\|$ for some $C \geq 0$. Let $g = g' + g''$, $g' \in F$, $g'' \in F^\perp$, be the orthogonal decomposition of g. Since $F \supseteq \mathrm{Ran}_A$ it follows that $F^\perp \subseteq \mathrm{Ran}_A^\perp = \mathrm{Ker}_{A^*}$, hence we deduce $g'' \in \mathrm{Ker}_{A^*}$, $g' \in \mathrm{Dom}_{A^*}$ and $A^*g' = A^*g$. By applying our hypothesis to g' we have

$$|\langle f, g\rangle| = |\langle f, g'\rangle| \leq \|f\|\|g'\| \leq C\|f\|\|A^*g\|.$$

Only if part. If for some $g \in \mathrm{Dom}_{A^*} \cap F$ we have $A^*g = 0$, then $g = Af \in \mathrm{Ran}_A = F$ for some $f \in \mathrm{Dom}_A$ and $A^*Af = 0$, hence $\|g\|^2 = \langle Af, Af\rangle = \langle f, A^*Af\rangle = 0$, therefore $g = 0$. As a result we need only show that

$$G := \{g/\|A^*g\| : g \in \mathrm{Dom}_{A^*} \cap F, A^*g \neq 0\}$$

is bounded as a subset of the Hilbert space F. For every $h = Af \in \mathrm{Ran}_A = F$ the set $\{\langle h, g\rangle : g \in G\}$ is a bounded subset of $\mathbb{C}$ with bound $\|h\|$. This means that G is weakly bounded in F. It is famous that weakly bounded subsets of Hilbert spaces are bounded. (This is immediate from the uniform boundedness principle [Fol, 5.13]. Refer [Jos-RS, page 85] for a direct proof. Compare [Rud-FA, 3.18].)

(3) In (2) let F be the closure of the range of A.

(4) It suffices to prove the only if part because the other direction can be deduced from this one by replacing A with A^*. Let Ran_A be closed. Then (3) gives

$$\|g\| \leq C\|A^*g\|, \quad \forall g \in G, \quad G := \mathrm{Dom}_{A^*} \cap \overline{\mathrm{Ran}_A}. \tag{4.3}$$

This inequality combined with a straightforward Cauchy sequence argument shows that A^* restricted to G has closed range. (Details: Assume a sequence $g_j \in G$ such that $A^* g_j$ converges to $f \in \mathcal{H}_1$. Since $A^* g_j$ is Cauchy it follows from (4.3) that g_j is also Cauchy, hence convergent to some $g \in \mathcal{H}_2$. Since A^* has closed graph it follows that $g \in G$ and $A^* g = f$.) However the range of $A^*|_G$ equals the range of A^* because A^* kills the orthogonal complement of $\overline{\mathrm{Ran}_A}$. This proves the only if part. ■

As mentioned before, in order to solve the d-bar problem in Hilbert spaces we want to apply Theorem 72.(2) to $A := T$, introduced in (4.1), and $F := \mathrm{Ker}_S$. It suffices to show $\|f\|_{\varphi_2} \le C\|T^* f\|_{\varphi_1}$ for every $f \in \mathrm{Dom}_{T^*} \cap F$ and some $C \ge 0$, or the following even stronger **energy estimate**[5]:

$$\boxed{\|f\|_{\varphi_2}^2 \le C\left(\|T^* f\|_{\varphi_1}^2 + \|Sf\|_{\varphi_3}^2\right), \quad \forall f \in \mathrm{Dom}_{T^*} \cap \mathrm{Dom}_S}. \tag{4.4}$$

This is proved conditionally in the following lemma assuming some control over the growth of weight functions φ_j:

Lemma 73 (Energy estimate)**.** *(1) Let K_j be an exhaustion of open $D \subseteq \mathbb{C}^m$ by compacts and for each j let $\eta_j : D \to [0,1]$ be a smooth function on D compactly supported in K_j which equals 1 on some neighborhood of K_{j-1}. Assume continuous functions $\varphi_1, \varphi_2, \varphi_3$ on D such that $\varphi_2 \in C^1(D)$ and*

$$\boxed{e^{-\varphi_2}\left|\bar\partial \eta_j\right|^2 \le e^{-\varphi_1}, \quad e^{-\varphi_3}\left|\bar\partial \eta_j\right|^2 \le e^{-\varphi_2} \quad \text{on } D, \quad j = 1, 2, \dots}. \tag{4.5}$$

Then $\mathcal{D}_{p,q+1}(D)$ is dense in $\mathrm{Dom}_{T^*} \cap \mathrm{Dom}_S$ with respect to the graph norm $f \mapsto \|f\|_{\varphi_2} + \|T^* f\|_{\varphi_1} + \|Sf\|_{\varphi_3}$.

(2) Assume $\psi \in C^1(D)$ and $\varphi \in C^2(D)$ satisfying

$$\boxed{\left|\bar\partial \eta_j\right|^2 \le e^{\psi} \quad \text{on } D, \quad j = 1, 2, \dots}, \tag{4.6}$$

$$\boxed{\sum_{j,k=1}^m \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} t_j \overline{t_k} \ge 2\left(|\bar\partial \psi|^2 + e^{\psi}\right)|t|_2^2 \quad \text{on } D, \quad \forall t \in \mathbb{C}^m}, \tag{4.7}$$

and set

$$\boxed{\varphi_1 := \varphi - 2\psi, \quad \varphi_2 := \varphi - \psi, \quad \varphi_3 := \varphi}. \tag{4.8}$$

Then the energy estimate (4.4) holds, hence $\mathrm{Ran}_T = \mathrm{Ker}_S$.

[5] Integrals of quadratic expressions is called *energy*. Example: The amount of energy dissipated in a unit electric resistance in the time interval $[0,T]$ equals $\int_0^T i^2 dt$, where i is the electric current passing through the resistance. Another example: The electromagnetic energy saved in the volume $V \subseteq \mathbb{R}^3$ in vacuum equals $\frac{1}{2}\int_V \left(\epsilon_0 |E|^2 + \mu_0 |H|^2\right) d\mu$, where ϵ_0, μ_0, E, H and μ are, respectively, the permittivity constant of vacuum, the permeability constant of vacuum, electric field strength, magnetic field strength and Lebesgue measure [Jac, page 259].

Proof. (1) This is done in two steps. *Step I: For every $f \in \mathrm{Dom}_S \cap \mathrm{Dom}_{T^*}$ we have $\eta_j f \in \mathrm{Dom}_S \cap \mathrm{Dom}_{T^*}$ and $\eta_j f \to f$ in the graph norm.* By Lebesgue's dominated convergence theorem

$$\|\eta_j f - f\|_{\varphi_2} \to 0 \quad \text{as} \quad j \to \infty, \quad \forall f \in L^2_{p,q+1,\varphi_1}, \tag{4.9}$$

because $|\eta_j f - f| \le |f|$. For every $f \in \mathrm{Dom}_S$ by Cauchy-Schwarz inequality we have

$$|\overline{\partial}(\eta_j f)|^2 = |\overline{\partial}\eta_j \wedge f + \eta_j \overline{\partial} f|^2 \le 2|\overline{\partial}\eta_j|^2 |f|^2 + 2|\eta_j|^2 |\overline{\partial} f|^2,$$

so using $e^{-\varphi_3}|\overline{\partial}\eta_j|^2 \le e^{-\varphi_2}$ we have

$$\left\|\overline{\partial}(\eta_j f)\right\|^2_{\varphi_3} \le 2\|f\|^2_{\varphi_2} + 2\left\|\overline{\partial} f\right\|^2_{\varphi_3},$$

hence $\eta_j f \in \mathrm{Dom}_S$. Again by Cauchy-Schwarz

$$\|S(\eta_j f) - \eta_j S f\|^2_{\varphi_3} = \|\overline{\partial}\eta_j \wedge f\|^2_{\varphi_3} \le \|f\|^2_{\varphi_2},$$

so

$$\|S(\eta_j f) - \eta_j S f\|_{\varphi_3} \to 0 \quad \text{as} \quad j \to \infty, \quad \forall f \in \mathrm{Dom}_S, \tag{4.10}$$

by the dominated convergence theorem. Combined with $\|\eta_j S f - S f\|_{\varphi_3} \to 0$ we get

$$\|S(\eta_j f) - S f\|_{\varphi_3} \to 0 \quad \text{as} \quad j \to \infty, \quad \forall f \in \mathrm{Dom}_S. \tag{4.11}$$

For every $g \in \mathrm{Dom}_{T^*}$ the identity

$$(\eta_j f, Tg) - (\eta_j T^* f, g) = (f, \overline{\eta_j} Tg - T(\overline{\eta_j} g)) = (f, -\overline{\partial}\overline{\eta_j} \wedge g),$$

combined with the estimate $e^{-\varphi_2}|\overline{\partial}\eta_j|^2 \le e^{-\varphi_1}$ shows that $(\eta_j f, Tg)$ is continuous with respect to g, hence $\eta_j f \in \mathrm{Dom}_{T^*}$. Since $\|\eta_j T^* f - T^* f\|_{\varphi_3} \to 0$ it remains to show that

$$\|T^*(\eta_j f) - \eta_j T^* f\|_{\varphi_1} \to 0 \quad \text{as} \quad j \to \infty, \quad \forall f \in \mathrm{Dom}_{T^*}, \tag{4.12}$$

because then

$$\|T^*(\eta_j f) - T^* f\|_{\varphi_1} \to 0 \quad \text{as} \quad j \to \infty, \quad \forall f \in \mathrm{Dom}_{T^*}, \tag{4.13}$$

and the combination of (4.9), (4.11) and (4.13) is the result we wanted to show. (4.12) can be shown in the same way that (4.10) was proved having the explicit formula for T^* at hand (Lemma 71.(1)). Here is another argument. For every $u \in \mathcal{D}_{p,q}$ we have

$$\begin{aligned}|(T^*(\eta_j f) - \eta_j T^* f, u)_{\varphi_1}| = |(f, \overline{\eta_j} Tu - T(\overline{\eta_j} u))_{\varphi_2}| = \left|(f, -\overline{\partial}\overline{\eta_j} \wedge u)_{\varphi_2}\right| = \\ \le \int |f||u||\partial\eta_j| e^{-\varphi_2} d\mu \le \int |f||u||\overline{\partial}\eta_j| e^{-\frac{\varphi_2}{2}} e^{-\frac{\varphi_1}{2}} d\mu.\end{aligned}$$

Since u was arbitrary it follows that

$$\left|T^*(\eta_j f) - \eta_j T^* f\right|^2 e^{-\varphi_1} \le |f|^2 e^{-\varphi_2},$$

and (4.13) follows by the dominated convergence theorem.

Step II: Regularization of those $f \in \mathrm{Dom}_{T^*} \cap \mathrm{Dom}_S$ *which has compact support.* Choose a smooth bump function ψ compactly supported on the unit ball of $\mathbb{C}^m$, and scale it to have integral 1. Consider the mollification f_ϵ obtained by convolving each component of the form f with the approximate identity $\psi_\epsilon(z) = \epsilon^{-2m}\psi(z/\epsilon)$. Note that f_ϵ is compactly supported in D for sufficiently small ϵ. It is a standard fact that the mollification of every function on $\mathbb{C}^m$ which is L^p summable, $1 \le p < \infty$, with respect to the Lebesgue measure converges to the function in L^p sense as $\epsilon \to 0$ [Fol, 8.14]. Therefore $f_\epsilon \to f$ in $L^2_{p,q+1,\varphi_2}$. For the same reason $Sf_\epsilon = (Sf)_\epsilon \to Sf$ in $L^2_{p,q+2,\varphi_3}$. It remains to show that $T^*f_\epsilon \to T^*f$ in L^2_{p,q,φ_1}. Recall the decomposition $T^* = (-1)^{p-1}e^{\varphi_1-\varphi_2}(\vartheta + a)$ obtained in Lemma 71. We are done by observing that

$$(\vartheta + a)f_\epsilon = (\vartheta f)_\epsilon + af_\epsilon = ((\vartheta + a)f)_\epsilon + af_\epsilon - (af)_\epsilon$$

converges $(\vartheta + a)f + af - af = (\vartheta + a)f$ in $L^2_{p,q}$.

(2) Let d_j denote the differential operator $\partial/\partial\bar{z}_j$. In the course of the proof I, J, L, K are shuffles of $\{1,\dots,m\}$ of lengths p, $q+1$, $q+1$, q respectively, and j,k,l are indices ranging on $\{1,\dots,m\}$. Since (4.6) and (4.8) imply (4.5), the density result of the previous part reduces us to prove the energy estimate for every $f \in \mathcal{D}_{p,q+1}$ assuming (4.7). This is basically integration by parts as follows. Fix $f = \sum' f_{I,J}dz_I \wedge d\bar{z}_J \in \mathcal{D}_{p,q+1}$. Since

$$Sf = \sideset{}{'}\sum_{I,J}\sum_j d_j f_{I,J} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J,$$

it follows that

$$|Sf|^2 = \sideset{}{'}\sum_{I,J,L}\sum_{j,l} d_j f_{I,J}\overline{d_l f_{I,L}}\epsilon^{jJ}_{lL}.$$

We decompose this sum into two. When $j = l$, to get a nonzero summand we must have $J = L$, so these summands sum up to

$$\sideset{}{'}\sum_{I,J}\sum_{j\notin J}|d_j f_{I,J}|^2.$$

When $j \neq l$, to get a nonzero summand, J and L must be obtained from a common shuffle K of length q by adding l and j respectively. These summands sum up to

$$-\sideset{}{'}\sum_{I,K}\sum_{j\neq l} d_j f_{I,lK}\overline{d_k f_{I,jK}}$$

because $\epsilon^{jJ}_{lL} = \epsilon^{jJ}_{jlK}\epsilon^{jlK}_{ljK}\epsilon^{ljK}_{lL} = -\epsilon^{J}_{lK}\epsilon^{jK}_{L}$, where as usual $f_{I,J}$ is defined for all tuples $J \subseteq \{1,\dots,m\}^{q+1}$ by antisymmetry. (More details is given in the statement of Lemma 71.(1).) The whole analysis shows that

$$|Sf|^2 = \sideset{}{'}\sum_{I,J}\sum_j |d_j f_{I,J}|^2 - \sideset{}{'}\sum_{I,K}\sum_{j,k} d_k f_{I,jK}\overline{d_j f_{I,kK}},$$

hence

$$\|Sf\|^2_{\varphi_3} = \int \sum_{I,J}{}' \sum_j |d_j f_{I,J}|^2 \, e^{-\varphi} d\mu - \int \sum_{I,K}{}' \sum_{j,k} d_k f_{I,jK} \overline{d_j f_{I,kK}} e^{-\varphi} d\mu. \tag{4.14}$$

Let $-\delta_j$ be the formal adjoint of $d_j : L^2_\varphi(\mathbb{C}^m) \to L^2_\varphi(\mathbb{C}^m)$ characterized by

$$\int w_1 \overline{d_j w_2} e^{-\varphi} d\mu = -\int (\delta_j w_1)\overline{w_2} e^{-\varphi} d\mu, \quad \forall w_1, w_2 \in \mathcal{D}(\mathbb{C}^m). \tag{4.15}$$

Integration by parts shows that

$$\delta_j w = e^\varphi \frac{\partial}{\partial z_j}\left(w e^{-\varphi}\right) = \frac{\partial w}{\partial z_j} - w\frac{\partial \varphi}{\partial z_j},$$

$$(\delta_j d_k - d_k \delta_j) w = \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} w, \tag{4.16}$$

for every $w \in \mathcal{D}(\mathbb{C}^m)$. Using δ_j, the explicit formula for T^* obtained in Lemma 71.(1) can be written as

$$\sum_{I,K}{}' \sum_j \delta_j f_{I,jK} dz_I \wedge d\overline{z}_K = (-1)^{p-1} e^\psi T^* f - \sum_{I,K}{}' \sum_j f_{I,jK} \frac{\partial \psi}{\partial z_j} dz_I \wedge d\overline{z}_K.$$

Applying $\int_D |-|^2 e^{-\varphi} d\mu$ and using the parallelogram identity $|A-B|^2 \leq |A-B|^2 + |A+B|^2 = 2|A|^2 + 2|B|^2$ gives

$$\int \sum_{I,K}{}' \sum_{j,k} \delta_j f_{I,jK} \overline{\delta_k f_{I,kK}} e^{-\varphi} d\mu \leq 2\|T^* f\|^2_{\varphi_1} + 2\int |f|^2 |\partial \psi|^2 e^{-\varphi} d\mu. \tag{4.17}$$

Estimates (4.14) and (4.17) combined gives

$$\int \sum_{I,K}{}' \sum_{j,k} \left(\delta_j f_{I,jK}\overline{\delta_k f_{I,kK}} - d_k f_{I,jK}\overline{d_j f_{I,kK}}\right) e^{-\varphi} d\mu + \int \sum_{I,J}{}' \sum_j |d_j f_{I,J}|^2 \, e^{-\varphi} d\mu \leq$$

$$2\|T^* f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3} + 2\int |f|^2 |\overline{\partial}\psi|^2 e^{-\varphi} d\mu. \tag{4.18}$$

This combined with (4.15) and (4.16) gives

$$\int \sum_{I,K}{}' \sum_{j,k} f_{I,jK} \overline{f_{I,kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} e^{-\varphi} d\mu + \int \sum_{I,J}{}' \sum_j |d_j f_{I,J}|^2 \, e^{-\varphi} d\mu \leq$$

$$2\|T^* f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3} + 2\int |f|^2 |\overline{\partial}\psi|^2 e^{-\varphi} d\mu. \tag{4.19}$$

Using the hypothesis (4.7) and dropping the second term on the left hand side gives

$$2\|f\|^2_{\varphi_2} \leq 2\|T^* f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3},$$

so the energy estimate holds with $C := 1$. ■

Finally, we show that the hypothesis in the conditional Lemma 73.(2) is fullfilled in pseudoconvex opens:

Theorem 74 (d-bar problem; L^2 solutions)**.** *Let $D \subseteq \mathbb{C}^m$ be a pseudoconvex open. Then for every $f \in L^2_{p,q+1,\mathrm{loc}}(D)$ with $\overline{\partial} f = 0$ there exists $u \in L^2_{p,q,\mathrm{loc}}(D)$ such that $\overline{\partial} u = f$.*

Proof. Fix f as in the statement of the theorem. A straightforward partition of unity argument shows that $L^2_{p,q+1,\mathrm{loc}}(D)$ is the union of all $L^2_{p,q+1,\Phi}(D)$, $\Phi \in C^\infty(D)$. (Refer Lemma 75.(2) for the proof.) Fix some $\Phi \in C^\infty(D)$ such that $f \in L^2_{p,q+1,\Phi}(D)$. Again, by partition of unity construct $\psi \in C^\infty(D)$ satisfying (4.6). (Refer Lemma 75.(3) for the proof.) It suffices to construct $\varphi \in C^\infty(D)$ satisfying $\varphi - \psi \geq \Phi$ and (4.7), because then defining $\varphi_1, \varphi_2, \varphi_3$ as (4.5) and T, S as (4.1), we will have $f \in L^2_{p,q+1,\varphi_2}(D)$ and the energy estimate (4.4) holds by Lemma 73.(2), and then the closed range theorem (Theorem 72) gives $u \in L^2_{p,q,\varphi_1}(D) \subseteq L^2_{p,q,\mathrm{loc}}(D)$ satisfying $\overline{\partial} u = f$. We use the pseudoconvexity of D to construct φ as follows. Fix a strictly plurisubharmonic exhaustion function p for D. Since $L(p;t) > 0$ on D for all $t \in \mathbb{C}^m$ one can find a positive-valued continuous function $c : D \to \mathbb{R}$ such that $L(p;t) \geq c|t|_2^2$ on D for every $t \in \mathbb{C}^m$. Here L denotes the complex Hessian. For every smooth increasing convex function $\chi : \mathbb{R} \to \mathbb{R}$ we have $L(\varphi;t) \geq \chi'(p)c|t|_2^2$ where $\varphi := \chi \circ p$. We need to find χ such that

$$\chi'(p)c \geq 2(|\overline{\partial}\psi|^2 + \exp\psi), \quad \chi(p) \geq \Phi + \psi. \tag{4.20}$$

Since χ is increasing (4.20) is equivalent to

$$\chi'(t) \geq \alpha(t) := \sup_{\{p \leq t\}} 2(|\overline{\partial}\psi|^2 + \exp\psi)/c, \quad \chi(t) \geq \beta(t) := \sup_{\{p \leq t\}} (\Phi + \psi), \quad \forall t \in \mathbb{R}.$$

One can easily find χ satisfying these two conditions just because α and β are finite-valued increasing function $\mathbb{R} \to \mathbb{R}$ which vanish on $(-\infty, \min p)$. p has minimum on D because it is a continuous exhaustion function. (Refer Lemma 75.(4) for the proof.) ■

Lemma 75. *(1) If $\alpha : D \to \mathbb{R}$ is a function on open $D \subseteq \mathbb{C}^m$ which is bounded above on every compact then there exists $\Phi \in C^\infty(D)$ such that $\Phi \geq \alpha$. (2) If $D \subseteq \mathbb{C}^m$ is an open then $L^2_{p,q+1,\mathrm{loc}}(D)$ is the union of all $L^2_{p,q+1,\Phi}(D)$, $\Phi \in C^\infty(D)$. (3) The exists $\psi \in C^\infty(D)$ satisfying (4.6). (4) If $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are functions on $\mathbb{R}$ which are bounded above on compacts and vanish identically on some $(-\infty, t_0)$ then there exists an increasing convex smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi' \geq \alpha$ and $\chi \geq \beta$.*

Proof. (1) Choose an exhaustion K_j of D by compacts, find nonnegative smooth bump function ψ_j equal 1 on $K_j \setminus K_{j-1}$ and equal 0 outside $K_{j+1} \setminus K_{j-2}$. Then $\Phi := \sum c_j \psi_j$, where $c_j = \sup_{K_j \setminus K_{j-1}} \alpha$, works.

(2) Clearly $L^2_{p,q+1,\mathrm{loc}}(D)$ contains all $L^2_{p,q+1,\Phi}(D)$. Fix $f \in L^2_{p,q+1,\mathrm{loc}}(D)$. Choose an exhaustion K_j of D by compacts, set $c_j := \int_{K_j} |f|^2 d\mu$, and by (1) find $\Phi \in C^\infty(D)$ such that $\exp(-\Phi) \leq 1/(2^j c_j)$ on $K_j \setminus K_{j-1}$. Then

$$\int |f|^2 e^{-\Phi} d\mu = \sum_j \int_{K_j \setminus K_{j-1}} |f|^2 e^{-\Phi} d\mu \leq c_1 + \sum_{j \geq 2} 2^{-j} < \infty.$$

(3) By (1) find $\psi \in C^\infty(D)$ such that $\psi \geq |\overline{\partial}\eta_j|^2$ on $K_j \setminus K_{j-1}$. Then $e^\psi > \psi \geq |\overline{\partial}\eta_j|^2$ on D for every j.

(4) One can assume $\alpha = \beta$. For every integer j find nonnegative smooth bump function ψ_j which equals 1 on $[j, j+2]$ and compactly supported in $[j-1, j+3]$, and set

$$\Phi := \sum c_j \psi_j, \quad c_j = \sup_{[j,j+2]} \alpha.$$

Clearly, $\Phi \geq \alpha$. Also, $\int_{-\infty}^t \Phi(x)dx \geq \alpha(t)$ for every $t \in \mathbb{R}$ because

$$\int_{-\infty}^t \Phi(x)dx \geq \int_j^{j+1} c_j\psi_j(x)dx = c_j \geq \alpha(t), \quad \forall j \in \mathbb{Z} \forall t \in [j+1, j+2].$$

Therefore, the primitive $\chi_1(t) := \int_{-\infty}^t \Phi(x)dx$ of Φ satisfies:

$$\chi_1' \geq \alpha, \quad \chi_1 \geq \alpha.$$

Applying the process $\alpha \mapsto \Phi$ above to $|\chi''|$ gives nonnegative smooth function $\gamma : \mathbb{R} \to \mathbb{R}$ supported on some $[t_1, \infty)$ such that $\gamma \geq \chi_1''$. Then $\chi(t) := \int_{-\infty}^t \int_{-\infty}^x \gamma(y)dydx$ is our desired function because

$$\chi(t) \geq \int_{-\infty}^t \int_{-\infty}^x \chi_1''(y)dydx = \chi_1(t) \geq \alpha(t),$$

$$\chi'(t) = \int_{-\infty}^t \gamma(y)dy \geq \int_{-\infty}^t \chi_1''(y)dy = \chi_1'(t) \geq \alpha(t),$$
$$\chi''(t) = \gamma(t) \geq 0.$$

■

Step II: Regularization

Having a L^2 solution of $\overline{\partial}u = f$ at hand we start the second step of our plan to prove that u (or a canonical version of it) is in fact smooth after correction on a null set. The proof is based on the following regularization lemma, the last part being a weak version of the famous Sobolev embedding enough for our purposes [Fol, 9.17][Gri, 6.1].

Lemma 76 (Regularization lemma). *(1) If $u \in L^2(\mathbb{C}^m)$ has compact support and $\partial u/\partial \overline{z}_j \in L^2(\mathbb{C}^m)$ for every j then $u \in W^{2,1}(\mathbb{C}^m)$. (2) If $u \in L^2_{p,q}(\mathbb{C}^m)$ has compact support, $\overline{\partial}u \in L^2_{p,q+1}(\mathbb{C}^m)$ and $\vartheta u \in L^2_{p,q-1}(\mathbb{C}^m)$ then $u \in W^{2,1}_{p,q}(\mathbb{C}^m)$. (3) If $u \in W^{2,s+n}_{p,q,\mathrm{loc}}(U)$ for some open $U \subseteq \mathbb{R}^n$ and $s \in \{0, 1, \ldots, \infty\}$ then $u \in C^s_{p,q}(U)$.*

Proof. (1) We need to show that $\partial u/\partial z_j \in L^2$ for every j. Let u_ϵ be the mollification of u with some smooth bump function compactly supported in the unit ball of $\mathbb{C}^m$ which has been normalized to have integral 1. For every $v \in \mathcal{D}(\mathbb{C}^m)$ doing integration by parts twice shows that $\|\partial v/\partial z_j\|_{L^2} = \|\partial v/\partial \overline{z}_j\|_{L^2}$. Applying this equality to $u_\epsilon - u_\delta$ shows that $\partial u_\epsilon/\partial z_j$ converges in L^2. Since $u_\epsilon \to u$ in L^2 it follows that $\partial u/\partial z_j \in L^2$.

(2) (4.19) for $\varphi = \psi = 0$ gives

$$\int \sum_{I,J}{}' \sum_j \left|\frac{\partial v_{I,J}}{\partial \overline{z}_j}\right|^2 d\mu \leq 2\|\vartheta v\|_0^2 + \|\overline{\partial} v\|_0^2, \quad \forall v \in \mathcal{D}_{p,q}(\mathbb{C}^m). \tag{4.21}$$

Let u_ϵ be the mollification of u with some smooth bump function compactly supported in the unit ball of $\mathbb{C}^m$ which has been normalized to have integral 1. Applying (4.21) to $u_\epsilon - u_\delta$ shows that $\partial(u_{I,J})_\epsilon/\partial\overline{z}_j$ converges in L^2. Since $(u_{I,J})_\epsilon \to u_{I,J}$ in L^2 it follows that $\partial u_{I,J}/\partial\overline{z}_j \in L^2$. By (1) we have $\partial u_{I,J}/\partial z_j \in L^2$, so $u_{I,J} \in W^{2,1}$.

(3) One can assume $p = q = 0$ because the general case follows by applying this special case to components. One can assume $s = 0$ because the general case follows by applying this special case to the weak derivatives of u of total order $\leq s$. One can assume that u is compactly supported in U because the general case follows by applying this special case to u multiplied by smooth bump functions compactly supported in U. For every $v \in \mathcal{D}(U)$ the fundamental theorem of calculus gives

$$v(x) = \int_{(-\infty,x_1]\times\cdots\times(-\infty,x_n]} \frac{\partial^n v(y)}{\partial y_1 \cdots \partial y_n} d\mu(y), \quad \forall x \in U,$$

hence

$$\sup_{x\in U} |v(x)| \leq \left\|\frac{\partial^n v}{\partial y_1 \cdots \partial y_n}\right\|_{L^1(U)} \leq C \left\|\frac{\partial^n v}{\partial y_1 \cdots \partial y_n}\right\|_{L^2(U)}, \tag{4.22}$$

where the constant C is the square root of the volume of U. Let u_ϵ be the mollification of u with some smooth bump function compactly supported in the unit ball which has been normalized to have integral 1. Applying (4.22) to $u_\epsilon - u_\delta$ shows that u_ϵ converges uniformly on compacts of U to some continuous function w. On the other hand $u_\epsilon \to u$ in L^2. Therefore $u = w$ almost everywhere. ∎

Theorem 77 (d-bar problem; L^2 Sobolev solutions). *Suppose an open $D \subseteq \mathbb{C}^m$. (1) If D is pseudoconvex then for every $f \in W^{2,s}_{p,q+1,\mathrm{loc}}(D)$, $s \in \{0, 1, \ldots, \infty\}$, with $\overline{\partial} f = 0$ there exists $u \in W^{2,s+1}_{p,q,\mathrm{loc}}(D)$ such that $\overline{\partial} u = f$. (2) If $q = 0$ then every $u \in L^2_{p,q,\mathrm{loc}}(D)$ solution of $\overline{\partial} u = f \in W^{2,s}_{p,q+1,\mathrm{loc}}(D)$ satisfies $u \in W^{2,s+1}_{p,q,\mathrm{loc}}(D)$.*[6]

Proof. Note that if D is pseudoconvex then we have a solution $u \in W^{2,0}_{p,q,\mathrm{loc}}$ according to Theorem 74.

Case $q = 0$. Assuming a solution $u = \sum' u_I dz_I \in W^{2,\sigma}_{p,q,\mathrm{loc}}$ for some $0 \leq \sigma \leq s$, we need to show that $u \in W^{2,\sigma+1}_{p,q,\mathrm{loc}}$. The equation $\overline{\partial} u = f$ means $\partial u_I/\partial\overline{z}_j = f_{I,j} \in W^{2,s}_{p,1,\mathrm{loc}}$ for $j = 1, \ldots, m$. That $u \in W^{2,\sigma+1}_{p,q,\mathrm{loc}}$ is immediate from the regularization lemma 76.(1) applied to the derivatives of total order σ of ψu_I, where ψ is an arbitrary smooth bump function compactly supported in D.

Case $q > 0$. As Remark 79 shows there might be $u \in L^2_{p,q,\mathrm{loc}}$ solutions of $\overline{\partial} u = f \in W^{2,s}_{p,q+1,\mathrm{loc}}$ which $u \notin W^{2,s+1}_{p,q,\mathrm{loc}}$, so we need to search for a canonical solution. The intuition is

[6] More generally, using Sobolev spaces with negative s one can show that every distribution $u \in \mathcal{D}'_{p,q}(D)$ satisfying of $\overline{\partial} u = f \in W^{2,s}_{p,q+1,\mathrm{loc}}(D)$ satisfies $u \in W^{2,s+1}_{p,q,\mathrm{loc}}(D)$. Refer [Rud-FA, pages 219–22][Fol, pages 307–8].

that when $q > 0$ then the kernel of T contains many non-smooth solutions, so returning back to the proof of Theorem 74 which was based on Theorem 72 we find a solution $Tu = f$ with restriction that $u \in \mathrm{Ker}_T^{\perp} = \overline{\mathrm{Ran}_{T^*}} = \mathrm{Ran}_{T^*}$. (Since $\mathrm{Ran}_T = \mathrm{Ker}_S$ is closed it follows by Theorem 72.(4) that Ran_{T^*} is also closed.) This restriction gives another partial differential equation satisfiesd by u which together with $\overline{\partial} u = f$ forces u to live in $W^{2,s}_{p,q,\mathrm{loc}}$, as follows. Let $u = T^* v$. Then $u = (-1)^{p-1} e^{\varphi_1} \vartheta(e^{-\varphi_2} v)$, so $\vartheta(e^{-\varphi_1} u) = 0$ because $\vartheta^2 = 0$. Therefore $\vartheta u = bu$ for some multiplication matrix b with smooth entries. Assuming that this canonical solution u lives in $W^{2,\sigma}_{p,q,\mathrm{loc}}$ for some $0 \le \sigma \le s$, we need to show that $u \in W^{2,\sigma+1}_{p,q,\mathrm{loc}}$. This is immediate from the Regularization lemma 76.(2) applied to the derivatives of total order σ of ψu, where ψ is an arbitrary smooth bump function compactly supported in D. ■

Step III: The proof

The proof of Theorem 70. (1)⇒(2) Theorem 62.

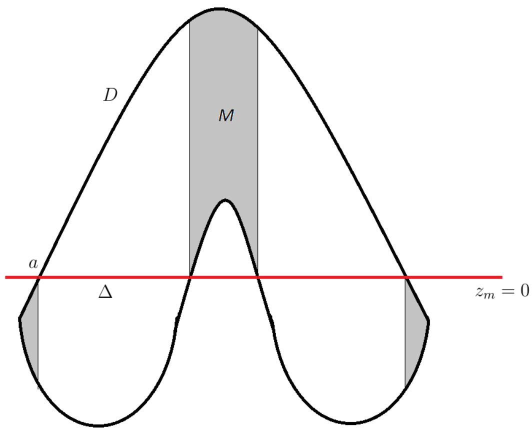
(2)⇒(3) Theorem 77 combined with the Sobolev embedding (Lemma 76.(3)).

(3)⇒(4) Trivial.

(4)⇒(1) We apply induction on m. The base of induction $m = 1$ says: Every open of $\mathbb{C}$ is a domain of holomorphy. This was shown in Example 45. Let D be an open of $\mathbb{C}^m$ satisfying Serre's condition. To prove that D is a (weak) domain of holomorphy it suffices to just check that for every open ball $B \subseteq D$ such that some $a \in \partial B$ is in ∂D there exits a holomorphic function on D which cannot be continued holomorphically across a. *Proof.* If D is not a domain of holomorphy then every holomorphic function in D can be extended holomorphically to a neighborhood U of some $a' \in \partial D$. Let the open ball $B(a', 2r)$ be contained in U. For every $a'' \in B(a', r) \cap D$, setting $r' := \mathrm{dist}(a', \partial D)$ we have $B(a'', r') \subseteq D$ and $\partial B(a'', r') \cap \partial D \neq \emptyset$. For every a in this latter nonempty set every holomorphic function on D can be extended holomorphically across a. Q.E.D. After a holomorphic change of coordinates one can assume $a = 0$ and $B_0 := B \cap \{z_m = 0\} \neq \emptyset$. Since B is convex it follows that $0 \in \partial B_0$, so $0 \in \partial \Delta$ where

$$\Delta := D \cap \{z_m = 0\},$$

is the open of $\mathbb{C}^{m-1}$ which we want to apply the induction hypothesis on.



Next we prove the following extension theorem:

For every $f \in C^\infty_{0,q}(\Delta)$ with $\overline{\partial} f = 0$ and $0 \le q \le m-2$ there exists $F \in C^\infty_{0,q}(D)$ such that $\overline{\partial} F = 0$ and $f = i^* F$, where $i : \Delta \hookrightarrow D$ is the inclusion.

Proof. As usual we first solve the problem in the smooth category and then do the required modifications by the d-bar problem. Let $\pi : \mathbb{C}^m \to \mathbb{C}^{m-1}$ be the canonical projection dropping the last coordinate. The most natural idea which comes to mind is to extend f to $\{z \in D : \pi(z) \in \Delta\}$ by $\pi^* f$ and set is zero elsewhere. Here is the way to implement this idea. Since Δ and $M := \{z \in D : \pi(z) \notin \Delta\}$ are disjoint and closed in D, one can find a bump function $\psi \in C^\infty(D)$ which equals 1 on some neighborhood on Δ and equals 0 on some neighborhood on M. The form $\psi\pi^* f$, defined to be zero wherever ψ vanishes, lives in $C^\infty_{0,q}(D)$ and $i^*\psi\pi^* f = f$ because $\pi \circ i = \mathrm{id}$. Set

$$F := \psi\pi^* f - z_m v,$$

where $v \in C^\infty_{0,q}(D)$ is chosen to make $\overline{\partial} F = 0$, namely $\overline{\partial} v = z_m^{-1}\overline{\partial}\psi \wedge \pi^* f$. This is possible by D satisfies the Serre condition. Clearly, $i^* F = f$. Q.E.D.

Using this extension theorem we can show that Δ satisfies the Serre's condition, namely for every $f \in C^\infty_{p,q+1}(\Delta)$ with $\overline{\partial} f = 0$ and $q \le m-3$ there exists $u \in C^\infty_{p,q}(\Delta)$ such that $\overline{\partial} u = f$. *Proof.* For such f find $F \in C^\infty_{0,q+1}(D)$ by the extension theorem above. Since D satisfies the Serre's condition and $q+1 \le m-2$, one can find $U \in C^\infty_{0,q}(D)$ such that $\overline{\partial} U = F$. Setting $u := i^* U$ we have $\overline{\partial} u = i^* \overline{\partial} U = f$. Q.E.D.

By the induction hypothesis Δ is a domain of holomorphy, so there exists $f \in \mathcal{O}(\Delta)$ which cannot be extended holomorphically across 0. The extension theorem above gives $F \in \mathcal{O}(D)$ which can not be extended holomorphically across 0. *Another argument.* [Ohs, 2.12]. ∎

Other proofs are given in [Nog, 4.4.20, 7.4.10][Kaup, 63.7] (in the language of sheaf cohomology) and [HL, page 85] [Ran, pages 196, 223] (by integral representations method).

Remark 78. During the proof of (4)⇒(1) in Theorem 70 we in fact showed that for every open $D \subseteq \mathbb{C}^m$ and $\Delta := D \cap \{z_m = 0\} \subseteq \mathbb{C}^{m-1}$ with inclusion $i : \Delta \hookrightarrow D$:

$$H^{0,q}_{\mathrm{Db}}(D) = 0 \text{ for some } q \Rightarrow$$
$$\forall f \in C^\infty_{0,q}(\Delta) \text{ with } \overline{\partial} f = 0\ \exists F \in C^\infty_{0,q}(D) \text{ with } \overline{\partial} F = 0 \text{ and } f = i^* F \Rightarrow$$
$$H^{0,q-1}_{\mathrm{Db}}(\Delta) = 0.$$

Repeating this argument proves the following canonical extension theorem from linear complex submanifolds sometimes called **Serre's criterion**: *Let $D \subseteq \mathbb{C}^m$ be open and $\Delta := D \cap \{z_1 = \cdots = z_k = 0\}$ be a linear complex submanifold of D of codimension k. Let D has the property that for every $f \in C^\infty_{p,q+1}(D)$ with $\overline{\partial} f = 0$ and $0 \le q < k$ there exists $u \in C^\infty_{p,q}(D)$ such that $\overline{\partial} u = f$. (In cohomological terms: Dolbeault cohomologies $H^{0,q}_{\mathrm{Db}}(D)$ vanish for $1 \le q \le k$.) Then every holomorphic function on Δ can be holomorphically extended to D.* Example: If D is Hartogs H_2 (Theorem 1) then $f(z_1, z_2) := 1/(z_2 - 1/2)$ is holomorphic on the complex linear complex submanifold $M = D \cap \{z_1 = 0\}$, but can not be extended holomorphically to D because of the Hartogs extension phenomenon. ∎

Remark 79. The second statement in Theorem 77 is never true if $q > 0$, because of the following type of examples. On $\mathbb{C}^2$ consider

$$u := |x_2|\,(2H(x_1) - 1)\,d\bar{z}_1 + |x_1|\,(2H(x_2) - 1)\,d\bar{z}_2,$$

where $x_j = \mathrm{Re} z_j$ and $H : \mathbb{R} \to \mathbb{R}$ is the unit step function $1_{[0,\infty)}$. Since $\partial|x_j|/\partial x_j = 2H(x_j) - 1$ in the distributional sense it follows that

$$\bar{\partial}u = (2H(x_2) - 1)(2H(x_1) - 1)d\bar{z}_2 \wedge d\bar{z}_1 + (2H(x_1) - 1)(2H(x_2) - 1)d\bar{z}_1 \wedge d\bar{z}_2 = 0.$$

Note that $u \in L^2_{0,1,\mathrm{loc}}$ but $u \notin W^{2,1}_{0,1,\mathrm{loc}}$, because the Dirac unit mass distribution appears in the first distributional derivatives of u. Those differential operators P with smooth coefficients such that all distributional solutions of $Pu = f$ are smooth functions if f is so are called **hypoelliptic** [Fol-PDE, pages 63, 216]; most important examples are elliptic operators, for example $\partial/\partial\bar{z}$ acting on functions on $\mathbb{C}$, Hodge Laplacian $(dd^* + d^*d)$ and de Rham operator $(d+d^*)$ acting on differential forms on oriented Riemannian manifolds, Kohn Laplacian $(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$ and Dolbeault operator $(\bar{\partial} + \bar{\partial}^*)$ acting on differential forms on Hermitian manifolds, and Dirac type operators on spinors on spin manifolds [Tay, chapter 10]. Every standard PDE textbook proves the hypoellipticity of elliptic operators under the name of "interior (or local) regularity of linear elliptic equations": [Tay, volume I, page 442][Fol-PDE, 6.34][Jos-PDE, 9.3.2][Bre, 9.25][Eva, 6.3.1][Hör-PDE, 4.1.7], etc. The interior regularity of constant coefficient elliptic equations is even proved in some of the graduate level real analysis textbooks [Rud-FA, 8.12][Fol, 8.14][Jos, 23.7]. ■

Exercise: (1) Hörmander proved [Fol-PDE, 6.36]: *A constant coefficient differential operator $P \in \mathbb{C}[\partial/\partial x_1, \ldots, \partial/\partial x_m]$ is hypoelliptic if and only if $|\mathrm{Im}\xi| \to \infty$ as $|\xi| \to \infty$ inside $\{\xi \in \mathbb{C}^m : P(\sqrt{-1}\xi_1, \ldots, \sqrt{-1}\xi_m) = 0\}$.* Using this criterion show that $\partial/\partial\bar{z}_j$, the Laplacian $\Delta := \sum_{1\le j\le m} \partial^2/\partial x_j^2$ and the heat operator $\Delta - \partial/\partial x_{m+1}$ are hypoelliptic but not the wave operator $\Delta - \partial^2/\partial x_{m+1}^2$. (2) Find a nonsmooth function $f = f(x_1, x_2)$ which satisfies the wave equation $\partial^2 f/\partial x_1^2 = \partial^2 f/\partial x_2^2$ in the distributional sense. (*Hint.* Consider functions of the form $f = g(x_1 - x_2)$.)

4.2 First corollaries of Theorem 70

Theorem 80 (Behnke-Stein)**.** *The union of an increasing sequence of domains of holomorphy is a domain of holomorphy.*

Proof. The union of an increasing sequence of pseudoconvex opens is pseudoconvex (evident from several different characterizations of pseudoconvexity given Theorem 58, say Hartogs pseudoconvexity, or the plurisubharmonicity of $-\log \mathrm{dist}$), so we are done by Levi's problem. For a direct proof which does not use Levi's problem refer [Nog, 5.4.10][Bers, Theorem 38]. ■

Theorems 61 and 70 combined gives:

Theorem 81. *Being a domain of holomorphy is a local property of the boundary in the sense that an open $D \subseteq \mathbb{C}^m$ is a domain of holomorphy if and only if every point in ∂D has a neighborhood $U \subseteq \mathbb{C}^m$ such that $D \cap U$ is a domain of holomorphy.*

Another proof in the language of sheaf cohomology is given in [Kaup, 63.7].

Pseudoconvexity can not be characterized topologically (because: Hartogs H and the unit ball are homeomorphic) but here is a topological restriction on this notion:

Theorem 82 (Serre)**.** *The de Rham cohomology of a pseudoconvex open $D \subseteq \mathbb{C}^m$ can be computed by holomorphic forms in the sense that the natural coset enlargement map*

$$\frac{\{f \in C^\infty_{p,0}(D) : \bar\partial f = 0, \partial f = 0\}}{\{dg : g \in C^\infty_{p-1,0}(D), \bar\partial g = 0\}} \to \frac{\{f \in C^\infty_p(D) : df = 0\}}{\{dg : g \in C^\infty_{p-1}(D)\}} =: H^p_{\mathrm{dR}}(D; \mathbb{C}),$$

is an isomorphism of vector spaces for every $p \geq 0$. (Note that the de Rham cohomology $H^p_{\mathrm{dR}}(M; \mathbb{C})$ of a smooth manifold M is isomorphic to the singular cohomology of M [Lee, 18.14][War, 5.36], hence a topological invariant, and can be computed efficiently by simplicial [Hat, 2.27][Mun, page 103], cellunar [Hat, 2.35][Mil, page 36] or Morse theoretic [Mil, pages 20, 36][AD, 2.7.3] means. A good reference is [BT].) Specially, $H^p_{\mathrm{dR}}(D; \mathbb{C}) = 0$ for $p > m$.

Proof. The map is clearly a homomorphism of vector spaces. Its surjectivity and injectivity follows immediately from the following assertion: *For every $f \in C^\infty_r(D)$ with $df \in C^\infty_{r+1,0}(D)$ there exists $f' \in C^\infty_{r,0}(D)$ such that $f - f' \in dC^\infty_{r-1}(D)$.* We prove the assertion inductively by restricting to those $f = \sum_{0 \leq q \leq r} f_{r-q,q}$, $f_{r-q,q} \in C^\infty_{r-q,q}$, such that $f_{r-q,q} = 0$ for some $k \in \{0, 1, \dots, r\}$ and every $q > k$. The initial case $k = 0$ is true by setting $f' := f$. Since $df \in C^\infty_{r+1,0}$ it follows that $\bar\partial f_{r-k,k} = 0$, so by Theorem 70 we can write $f_{r-k,k} = \bar\partial g$ for some $g \in C^\infty_{r-k,k-1}$. The induction hypothesis applied to

$$f - dg = \sum_{0 \leq q \leq k-1} f_{r-q,q} - \partial g,$$

gives $f - dg - f' \in dC^\infty_{r-1}$ for some $f' \in C^\infty_{r,0}$. ■

Another proof in the language of sheaf cohomology is given in [Nog, 4.4.29].

Example 83. This example shows that in Theorem 82, $H^p_{\mathrm{dR}}(D; \mathbb{C})$ can be nonzero for $p \leq m$. Consider $D := \{z \in \mathbb{C}^m : 1/2 < |z_j| < 2, \forall j\}$. D being a product of opens of the complex plane is pseudoconvex. Consider the smooth form $f := (z_1 \cdots z_p)^{-1} dz_1 \wedge \ldots \wedge dz_p$ of degree $p \in \{1, \dots, m\}$. Clearly, $df = 0$ but the following argument shows that there is no smooth form g of degree $p - 1$ such that $f = dg$, hence $H^p_{\mathrm{dR}}(D; \mathbb{C}) \neq 0$. If such g exists then by Stokes' theorem the integral of f over the p-dimensional torus $\{|z_1| = \cdots = |z_p| = 1, z_{p+1} = \cdots = z_m = 1\}$ vanishes, however a direct computation shows that this integral equals $(2\pi\sqrt{-1})^p$. ■

4.3 d-bar problem with L^2 estimates

A careful analysis of the argument used to prove Theorem 74 gives the following strengthening:

Theorem 84 (d-bar problem; L^2 estimates). *Let $D \subseteq \mathbb{C}^m$ be a pseudoconvex open.*

(1) For every plurisubharmonic function φ on D and $f \in L^2_{p,q+1,\varphi}(D)$ satisfying $\overline{\partial} f = 0$ there exists $u \in L^2_{p,q,\mathrm{loc}}(D)$ such that $\overline{\partial} u = f$ and

$$\int_D |u|^2 \left(1 + |z|_2^2\right)^{-2} e^{-\varphi} d\mu \leq \int_D |f|^2 e^{-\varphi} d\mu.$$

(2) If D is bounded then there exists a positive real number C depending only on m and the diameter of D such that for every $f \in L^2_{p,q+1}(D)$ with $\overline{\partial} f = 0$ there exists $u \in L^2_{p,q}(D)$ such that $\overline{\partial} u = f$ and $\|u\|_\varphi \leq C \|f\|_\varphi$.

To prove this theorem we first show:

Lemma 85. *Suppose pseudoconvex open $D \subseteq \mathbb{C}^m$, C^2 function $\varphi : D \to \mathbb{R}$ satisfying*

$$\sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} t_j \overline{t_k} \geq c|t|_2^2 \quad \text{on } D, \quad \forall t \in \mathbb{C}^m, \tag{4.23}$$

for some positive-valued continuous function c on D. Then for every $f \in L^2_{p,q+1,\varphi}(D)$ satisfying $\overline{\partial} f = 0$ there exists $u \in L^2_{p,q,\varphi}(D)$ such that $\overline{\partial} u = f$ and

$$\int_D |u|^2 \left(1 + |z|_2^2\right)^{-2} e^{-\varphi} d\mu \leq 2 \int_D |f|^2 c^{-1} e^{-\varphi} d\mu.$$

Proof. ■

Proof of the Theorem 84. (1)

(2) Immediate from (1). ■

4.4 Interpolation problem

Theorem 86 (Interpolation problem). *An open $D \subseteq \mathbb{C}^m$ is a domain of holomorphy if and only if for every sequence of complex numbers q_j and every sequence p_j of points of D which does not accumulate anywhere in D there exists a holomorphic function f on D such that $f(p_j) = q_j$ for all j.*

Proof. If the interpolation problem is solvable in D with $q_j = j$ then condition (4) in Theorem 44 (Cartan-Thullen) is satisfied, so D is a domain of holomorphy. For the other direction refer [Ohs, 5.3]. ■

Another proof in the language of sheaf cohomology is given in [Nog, 4.4.21]. A generalization is Theorem 106.

4.5 Division problem

We aim to prove the following theorem:

Theorem 87 (Division problem; Oka). *For an open $D \subseteq \mathbb{C}^m$ the followings are equivalent:*

(1) D is a domain of holomorphy.

(2) If $f_1, \ldots, f_k$ is a finite collection of holomorphic functions on D with no common zeros then there exists holomorphic functions $g_1, \ldots, g_k$ on D such that $1 = \sum f_j g_j$.

(2′) If f_j is a sequence of holomorphic functions on D with no common zeros then there exists holomorphic functions g_j on D such that $1 = \sum f_j g_j$, the uniform convergence on compacts.

(3) The (closed) maximal ideals of $\mathcal{O}(D)$ are given by $\mathfrak{m}_a := \{f \in \mathcal{O}(D) : f(a) = 0\}$, $a \in D$. (Here closedness is with respect to the topology of uniform convergence on compacts.)

Proof. (1)⇒(2) [Ohs, page 90–92].

(2′)⇒(3) Every $\mathfrak{m}_a$ is maximal because for every $g \in \mathcal{O}(D) \setminus \mathfrak{m}_a$ the identity

$$f = \frac{f(a)}{g(a)} g + \left(f - \frac{f(a)}{g(a)} g \right), \quad f \in \mathcal{O}(D),$$

shows that $\mathcal{O}(D)$ is the only ideal containing $\mathfrak{m}_a \cup \{g\}$. Let $\mathfrak{m}$ be a maximal ideal not of this form. Then assuming a dense sequence p_j of points of D, one can find $f_j \in \mathfrak{m}$ which is nonzero on p_j hence nonzero on some neighborhood of p_j. This sequence f_j has no common zero so $1 = \sum f_j g_j$ for $g_j \in \mathcal{O}(D)$. We have the contradiction $1 \in \mathfrak{m}$.

(3)⇒(1) Assuming $p \in \partial D$ we will find a holomorphic function f_p on D which can not extended holomorphically across p. The ideal generated by $z_j - p_j$, $j = 1, \ldots, m$, is not contained in any maximal ideal, so by Zorn's lemma it should be the whole algebra $\mathcal{O}(D)$. This means $1 = \sum_{1 \le j \le m} (z_j - b_j) g_j$ for some $g_j \in \mathcal{O}(D)$. Set $f_p := 1 / \sum (z_j - p_j) g_j$. ■

Another proof in the language of sheaf cohomology is given in [Kaup, 63.7].

4.6 Cousin problems

4.7 Other applications of the d-bar technique

Theorem 88. *An open $D \subseteq \mathbb{C}^m$ with C^1 boundary is a domain of holomorphy if and only if it carries a complete smooth Kähler metric.*

Proof. Only if part was proved in Theorem 54. For the other direction refer [Ohs2]. ■

Chapter 5

d-bar problem on strongly pseudoconvex domains (via integral representations' methods) with applications

References: [Ran, HL, LM, CS, Ada].

With integral representation we can prove L^p, $p \neq 2$, estimates for the d-bar problem which can not be done by the Hilbert space methods of Chapter 4.

Theorem 89 (d-bar problem; L^p and Lipschitz estimates. Kerzman-Øvrelid)**.** *For every bounded strongly pseudoconvex open $D \subseteq \mathbb{C}^m$ with C^3 boundary there exist bounded operators $S_q : L^1_{0,q}(D) \to L^1_{0,q}(D)$, $q = 1, \dots, m$, such that if $f \in L^1$ satisfies $\overline{\partial} f = 0$ (in distributional sense) then $f = \overline{\partial} S_q f$. Furthermore:*

$$\|S_q\|_{L^p \to L^p} < \infty, \quad p \in [1, \infty],$$

$$\|S_q\|_{L^\infty \to \Lambda^\alpha} < \infty, \quad \alpha \in (0, 1/2], \quad \Lambda^\alpha = \left\{ f : \sup \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \right\},$$

$$f \in L^1 \cap C^k \Rightarrow S_q f \in L^1 \cap C^k, \quad k \in [1, \infty].$$

Since $\Lambda^\alpha(\Omega) \subseteq C(\overline{\Omega})$ [Ada, 3.6]) it follows that

$$f \in L^\infty \cap C^\infty \Rightarrow S_q f \in C^\infty(\Omega) \cap C(\overline{\Omega}).$$

Theorem 90 (Henkin)**.** *Let $D \subseteq \mathbb{C}^m$ be a bounded strongly pseudoconvex open with C^2 boundary. Then every element of $\mathcal{O}(D) \cap C(\overline{D})$ (respectively the weighted Bergman space $\mathcal{O}(D) \cap L^p_\alpha(D)$, $\alpha \geq -1$, $p \in [1, \infty)$) can be approximated uniformly (respectively in L^p norm) by elements of $\mathcal{O}(\overline{D})$.*

Chapter 6

Boundary values of holomorphic and harmonic functions (Hardy spaces)

References: [Ste][Kra, chapter 1, 8][Rud-SCV].

6.1 Green and Poisson kernels

6.2 Bergman kernels

6.3 Szegö kernels

Chapter 7

Coherent cohomology of complex spaces

References: [Hör, Nog].

Oka was the first who proved that the obstruction to Cousin problems are purely topological in the sense that if the problem admits a continuous solution then it admits a holomorphic solution.

7.1 Weierstrass preparation theorem

References: [GH, chapter 0].

A holomorphic germ $f \in \mathcal{O}_{\mathbb{C}^m,0}$ is said to be **regular in z_m of order k** (k a positive integer) if $f(0,\dots,0,z_m)$ has 0 as a zero of order k. It is straightforward to show that every holomorphic germ which vanishes at the origin becomes regular in z_m after a complex linear change of coordinates. A holomorphic germ $W \in \mathcal{O}_{\mathbb{C}^m,0}$ is said to be a **Weierstrass polynomial in z_m of degree k** if it has the form $W = z_m^k + z_m^{k-1}w_{k-1} + \cdots + w_0$ where w_j are holomorphic germs in variables $z_1,\dots,z_{m-1}$ which vanish at the origin.

Theorem 91 (Weierstrass preparation and division theorems)**.** *Let $f \in \mathcal{O}_{\mathbb{C}^m,0}$ be a holomorphic germ regular in z_m of order k. Then: (1) f can be uniquely written as $f = Wg$ where W is a Weierstrass polynomial in z_m of degree k and g is a unit germ in $\mathcal{O}_{\mathbb{C}^m,0}$ namely $g(0) \neq 0$. (2) Every holomorphic germ $h \in \mathcal{O}_{\mathbb{C}^m,0}$ can be written uniquely as $h = fq + r$ where $r \in \mathcal{O}_{\mathbb{C}^{m-1},0}[z_m]$ is polynomial in z_m of degree less than k.*

Proof. Use $z = (z', z_m)$ to coordinate $\mathbb{C}^m = \mathbb{C}^{m-1} \times \mathbb{C}$. Find positive numbers R and r such that

$$0 < |z_m| \leq R \Rightarrow f(0, z_m) \neq 0, \qquad |z_m| = R,\ |z'|_2 \leq r \Rightarrow f(z', z_m) \neq 0.$$

(1) For every $|z'|_2 < r$ the function $f(z)$ has exactly k zeros $z_m = \alpha_j(z')$, $j = 1,\dots,k$, on $|z_m| < R$ because the expression

$$\frac{1}{2\pi\sqrt{-1}} \int_{|z_m|=R} \frac{\partial f/\partial z_m(z)}{f(z)} dz_m,$$

is a continuous (even holomorphic) integer-valued function hence equals its value at the origin which is k. We assert that

$$W(z) := \prod_{j=1}^{k}(z_m - \alpha_j(z')) = z_m^k + w_{k-1}(z')z_m^{k-1} + \cdots + w_0(z')$$

works. All functions $w_j(z')$ are holomorphic vanishing at the origin because

$$\omega_l(z') := \sum_{j=1}^{k}(\alpha_j(z'))^l = \frac{1}{2\pi\sqrt{-1}}\int_{|z_m|=R}\frac{z_m^l \partial f/\partial z_m(z)}{f(z)}dz_m, \quad l = 1, 2, \ldots,$$

are holomorphic and each a_j is a polynomial in ω_l's (say by Waring's second formula). Consider the partially defined function $g := f/W$. For every $|z'|_2 < r$ remove the singularities of g and denote the resulting function again by g. The formula

$$g(z) = \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta|<R}\frac{g(z', \zeta)}{z_m - \zeta}d\zeta, \quad |z'|_2 < r, \quad |z_m| < R$$

show that g is holomorphic in z'. This proves the existence. For uniqueness, assuming $f = W_1 g_1 = W_2 g_2$, for every fixed $|z'|_2 < r$ the roots of the monic polynomials W_1 and W_2 are the same, so they are identical. Since $(g_1 - g_2)W_1 = 0$ it follows that the Taylor coefficients of the holomorphic function $g_1 - g_2$ are all zero.

(2) By (1) we can assume f is a Weierstrass polynomial in z_m of degree k, say $f = z_m^k + f_{k-1}(z')z_m^{k-1} + \cdots + f_0(z')$. We assert that

$$q(z) := \frac{1}{2\pi\sqrt{-1}}\int_{|\zeta_m|=R}\frac{h(z', \zeta_m)}{f(z', \zeta_m)}\frac{d\zeta_m}{\zeta_m - z_m}, \quad |z_m| < R, \quad |z'| < r,$$

works. Indeed

$$\begin{aligned} h - fq &= \frac{1}{2\pi\sqrt{-1}}\int \frac{h(z', \zeta_m)}{f(z', \zeta_m)}\frac{f(z', \zeta_m) - f(z', z_m)}{\zeta_m - z_m}d\zeta_m \\ &= \frac{1}{2\pi\sqrt{-1}}\int \frac{h(z', \zeta_m)}{f(z', \zeta_m)}\frac{\zeta_m^k - z_m^k + f_{m-1}(z')\left(\zeta_m^{k-1} - z_m^{k-1}\right) + \cdots}{\zeta_m - z_m}d\zeta_m \\ &= \frac{1}{2\pi\sqrt{-1}}\int \frac{h(z', \zeta_m)}{f(z', \zeta_m)}\left(z_m^{k-1} + F_{m-2}(z', \zeta_m)z_m^{k-2} + \cdots\right)d\zeta_m \end{aligned}$$

is a polynomial in z_m of degree less than k. For uniqueness, assuming two representations $h = fq_1 + r_1 = fq_2 + r_2$, comparing the number of zeros in two sides of $f(q_1 - q_2) = r_2 - r_1$ for every fixed $|z'|_2 < r$ shows that $q_1 - q_2 = r_2 - r_1 = 0$. ■

For other proofs refer [Bers, page 40][KP, 6.1.3][Hör, 6.1.1]

7.2 Stalk study of the sheaf of holomorphic functions

Theorem 92. *The ring of germs of holomorphic functions $\mathcal{O}_{\mathbb{C}^m,0}$ is a Noetherian normal unique factorization domain.*

7.3 Local study of the sheaf of holomorphic functions: coherency

Proposition 93. *The class of coherent sheaves on an open $D \subseteq \mathbb{C}^m$ is closed under*

Theorem 94 (Coherence theorems of Oka and Cartan)**.** *Let $D \subseteq \mathbb{C}^m$ be open. (1) The structure sheaf $\mathcal{O}_D$ of holomorphic functions on D is coherent analytic. (2) The sheaf of ideals $\mathcal{I}_Y$ of every analytic subset Y of D is coherent analytic.*

7.4 Sheaf cohomology and its universal role among cohomology theories

[War, Chapter 5]

Theorem 95 (Ubiquity of the sheaf cohomology)**.** *(1) If X is a topological manifold then the singular cohomology $H^k_{\mathrm{Sing}}(X, \mathbb{C})$ is given by the sheaf cohomology $H^q(X, \mathbb{C})$ with respect to the sheaf of locally constant functions. (2) If X is a smooth manifold then the de Rham cohomology $H^k_{\mathrm{dR}}(X, \mathbb{C})$ is given by the sheaf cohomology $H^k(X, \mathbb{C})$ with respect to the sheaf of locally constant functions. (3) If X is a complex manifold then the Dolbeault cohomology $H^{p,q}_{\mathrm{Db}}(X, \mathbb{C})$ is given by the sheaf cohomology $H^q(X, \mathcal{O}^{(p)})$ with respect to the sheaf of holomorphic forms.*

Proof. (2) Let Ω^k be the sheaf of germs of smooth differentiable forms of degree k on X. Then the complex of sheaves

$$0 \to \mathbb{C} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim_{\mathbb{R}} X} \to 0,$$

is fine (by partition of unity) and exact (by the Poincaré lemma). Apply de Rham-Weil.

(3) Let $\Omega^{(p,q)}$ be the sheaf of germs of smooth differentiable forms of type (p, q) on X. Then the complex of sheaves

$$0 \to \mathcal{O}^{(p)} \hookrightarrow \Omega^{(p,0)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{(p,\dim_{\mathbb{C}} X)} \to 0,$$

is fine (by partition of unity) and exact (by Dolbeault-Grothendieck lemma, Theorem 25). Apply de Rham-Weil. ■

7.5 Global study of the sheaf of holomorphic functions: Cartan B

Theorem 96 (Cartan B)**.** *An open $D \subseteq \mathbb{C}^m$ is a domain of holomorphy if and only if all the sheaf cohomologies $H^q(D, \mathcal{F})$, $q > 0$, of D with respect to coherent analytic sheaves $\mathcal{F}$ vanish.*

Chapter 8

Holomorphic approximations of functions

References: [Hör, Ran].

For two subsets $A \subseteq B$ of $\mathbb{C}^m$, (B, A) is called a **Runge pair** if $\mathcal{O}(B)$ is dense in $\mathcal{O}(A)$ in the sense that every holomorphic function on A can be uniformly approximated on compacts by holomorphic functions on B. A is called **Runge** if $(\mathbb{C}^m, A)$ is a Runge pair. Theorem 12 gave a topological characterization of Runge pairs (D, K), open $D \subseteq \mathbb{C}$, compact K.

Example: The power series representation of holomorphic functions on polydiscs shows that polydiscs are Runge. Therefore, by Hartogs extension theorem, Hartogs H is also Runge.

Theorem 97. *For two opens $D \subseteq D'$ in the complex plane, (D', D) is a Runge pair if and only if any of the following equivalent conditions hold:*

(1) If $D' \setminus D$ can be written as the union of a closed $F \subseteq D'$ and a compact L such that $F \cap L = \emptyset$ then $L = \emptyset$.

(2) For every compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D')} = \hat{K}_{\mathcal{O}(D)}$.

(3) For every compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D')} \cap D = \hat{K}_{\mathcal{O}(D)}$.

(4) For every compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D')} \cap D$ is compact. [Hör, 1.3.4]

Example: Open $D \subseteq \mathbb{C}$ is Runge if and only if $\mathbb{C} \setminus \overline{D}$ is connected.

Exercise: Show that Runge subsets of $\mathbb{C}^m$ remain Runge under biholomorphic maps $\mathbb{C}^m \to \mathbb{C}^m$.

Contrary to such topological characterization in the plane, in higher dimensions there is not even a biholomorphic intrinsic characterization of Runge opens, as shown by the following example.

Example 98 (Wermer [Wer2, Wer3, Kaup])**.** Consider the map

$$F : \mathbb{C}^3 \to \mathbb{C}^3, \quad z \mapsto \left(z_1, z_1 z_2 + z_3, z_1 z_2^2 - z_2 + 2 z_2 z_3\right).$$

The determinant of the Jacobian of F equals $1 - 2z_3$, so F is locally biholomorphic on every point of the polydisc $P_\epsilon := \{|z_1| < 1 + \epsilon, |z_2| < 1 + \epsilon, |z_3| < \epsilon\}$, $0 < \epsilon < 1/2$. We assert that $F : P_\epsilon \to F(P_\epsilon)$ is biholomorphic for sufficiently small ϵ. If F is not injective on any P_ϵ then there exists two sequences of points $p^j = (p_1^j, p_2^j, p_3^j)$ and $q^j = (q_1^j, q_2^j, q_3^j)$ in $P_{1/j}$ such that $p^j \neq q^j$, $F(p^j) = F(q^j)$, $p_3^j \to 0$, $q_3^j \to 0$, and after passing to a subsequence we can assume that both sequences converge. It is straightforward to deduce that $|p_2^j - q_2^j|$ and $|p_3^j - q_3^j|$ both tend to zero, namely p^j and q^j converge to a common point, but then F can not be locally injective on this common point of convergence. From now on fix some ϵ such that $F : P_\epsilon \to F(P_\epsilon)$ is biholomorphic. The power series representation of holomorphic functions on polydiscs shows that P_ϵ is Runge, but assuming $D := F(P_\epsilon)$ Runge leads to a contradiction as follows. Consider

$$K = \{(w_1, 1, 0) \in \mathbb{C}^3 : |w_1| = 1\}, \quad D' = \{(w_1, 1, 0) \in D\}, \quad f = \mathrm{proj}_2 \circ F^{-1}.$$

Note that $K \subseteq D' \subseteq D$ because

$$F(z_1, 1/z_1, 0) = (z_1, 1, 0) \quad \text{for} \quad |z_1| = 1.$$

Also, $\|f\|_K = 1$ because f restricted to D' is given by $f(w_1, 1, 0) = \mathrm{proj}_2(w_1, 1/w_1, 0) = 1/w_1$. Every polynomial p in three complex variables according to the maximum principle satisfies $p(w_1, 1, 0) \leq \|p\|_K$ for $|w_1| \leq 1$, and since f can be uniformly approximated on compact $K \subseteq D$ by polynomials we have $f(w_1, 1, 0) \leq \|f\|_K$ for $|w_1| \leq 1$. Since $D \subseteq \mathbb{C}^3$ is an open containing K so there exists $w \in D'$ with $|w_1| < 1$, but then for any such point according to the estimation above we get the contradiction $1 < |w_1|^{-1} = |f(w)| \leq \|f\|_K = 1$. ■

Theorem 99 (Oka-Weil)**.** *A compact $K \subseteq \mathbb{C}^m$ is Runge if it is polynomially convex [Hör, 2.7.7]. The converse is also true if K is a Stein compactum [Ran, page 219].*

Here is a generalization.

Theorem 100. *For domain of holomorphy $D \subseteq \mathbb{C}^m$ and compact $K \subseteq D$, (D, K) is a Runge pair if K is plurisubharmonically convex in D, namely $K = \hat{K}_{PS(D)}$. [Hör, 4.3.2]*

Theorem 101 (Behnke)**.** *For two domains of holomorphy $D \subseteq D' \subseteq \mathbb{C}^m$, (D', D) is a Runge pair if and only if any of the following equivalent properties hold:*

(1) For every compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D')} = \hat{K}_{\mathcal{O}(D)}$.

(2) For every compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D')} \cap D = \hat{K}_{\mathcal{O}(D)}$.

(3) For every compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D')} \cap D$ is compactly supported in D'. [Hör, 4.3.3]

Theorem 102. *For domain of holomorphy $D \subseteq \mathbb{C}^m$ and compact $K \subseteq D$ we have $\hat{K}_{\mathcal{O}(D)} = \hat{K}_{PS(D)}$. [Hör, 4.3.4]*

Theorem 103. *Every domain of holomorphy can be exhausted by domains of holomorphy D_j such that all (D_{j+1}, D_j) and (D, D_j) are Runge pairs.*

Chapter 9

Holomorphic extensions of functions

9.1 From complements of thin sets

References: [Kaup, Gun, Ohs]

Theorem 104 (Riemann's removable singularities). *Consider an open $D \subseteq \mathbb{C}^m$ and a subset $A \subseteq D$.*

(1) If A is a closed subset of a proper analytic subset of D then every holomorphic function on $D \setminus A$ which is locally bounded on A (namely every $a \in A$ has a neighborhood $U \subseteq D$ such that the function is bounded on $U \setminus \{a\}$) can be extended holomorphically to D.

(1) If A is thin then every holomorphic function on $D \setminus A$ which is locally bounded on A can be extended holomorphically to D.

(2) If A is an analytic subset then every L^p function on $D \setminus A$, $p \in [2, \infty]$, can be extended holomorphically to D.

(3) If A is an analytic subset of codimension > 1 then any holomorphic function on $D \setminus A$ can be extended holomorphically to D.

Theorem 105 (Rado). *If a continuous function f on open $D \subseteq \mathbb{C}^m$ is holomorphic on D minus a smooth real hypersurface $S \subseteq \mathbb{C}^m$ then f is holomorphic on D.[CS, page 37]*

9.2 From complex submanifolds

References: [Ohs, EE]

Theorem 106 (Extension from complex submanifolds). *An open $D \subseteq \mathbb{C}^m$ is a domain of holomorphy if and only if every function holomorphic on a complex submanifold of D can be holomorphically extended to D.*

Proof. [Ohs, 5.6]. ∎

Another proof in the language of sheaf cohomology is given in [Nog, 4.5.11]. The theorem remains true if "submanifold" is replace by "analytic subsets" [Hör, 7.4.8][Nog, 6.12.7]. Note that submanifolds of dimension zero are just discrete subsets, so this Theorem solves the interpolation problem 86. Normed controlled versions are given in [Ohs, section 5.3].

9.3 From real hypersurfaces.

References: [KR, CS, EE].

Theorem 107 (Bochner)**.** *If $D \subseteq \mathbb{C}^m$, $m > 1$, is a bounded domain with connected C^1 boundary then every C^1 function on the boundary which satisfies tangential Cauchy-Riemann equations can be extended to a function holomorphic on D and continuous up to the boundary. Moreover, if the boundary is C^k, and $f \in C^l(D)$, $l \leq k$, then the extension in is $\mathcal{O}(D) \cap C^l(\overline{D})$. [Ran, page 165][CS, page 38][Hör, page 31]*

Lemma 108 (Jump formula for Bochner-Martinelli integrals)**.** *Assume bounded open $D \subseteq \mathbb{C}^m$, $m > 1$, with connected C^1 boundary and $f \in C^1(\partial D)$. Define $F(z) = \int_{\partial D} f(\zeta)K_0(\zeta, z)$ on $\mathbb{C}^m \backslash \partial D$ where K_0 is the Bochner-Martinelli kernel for functions, and set $F_-(z) := F(z)$ for $z \in D$ and $F_+(z) := F(z)$ for $z \in \mathbb{C}^m \setminus \overline{D}$. Then F_- and F_+ have C^α extensions to the boundary for every $0 < \alpha < 1$, and we have $f(z) = F_-(z) - F_+(z)$.*

Chapter 10

Special topics

10.1 Zero sets of holomorphic functions

References: [Rud-SCV, chapter 17].

10.1.1 The local geometry of analytic varieties

Theorem 109 (Henkin-Skoda)**.**

10.2 Hermitian symmetric spaces, Kähler manifolds, Einstein manifolds

References: [Helg, chapter 8].

10.2.1 Biholomorphic-invariant metrics

Bergman metric as the first Kahler metric and a generalization of the Poincar metric on the unit disk), KobayashiRoyden metric, Sibony metric, Caratheodory metric

Fefferman used the Bergman metric to prove his extension theorem. An accessible proof sketch is given in [Ohs]. More details are in [Ran, 7.8].

10.3 Classification of domains up to biholomorphism

References: [BFG].

10.4 Neumann d-bar problem on strongly pseudo-convex domains with applications

References: [FK, CS, Tay, BS].

1. More general problem than $\overline{\partial}u = f$ is $\Box u = f$, which is a BVP, because to make formal adjoint $\overline{\partial}^*$ a true adjoint (or make $\Box$ true SA), the right hand side of integration by parts formula $(\partial u, v) - (p, \overline{\partial} v)$, which is $\int_{\partial\Omega}$ symbols should vanish, and it gives Neumann BCs. Spencer, based on works of Hodge (closed Riemannian), Weil (closed Kahler), de Rham and Kodairo (complete Riemannian), asked whether the linear algebra fact: *For SA matrix A, the equation $Au = f$ is solvable exactly when $f \perp \ker(A)$; if so there is a unique solution being orthogonal to $\ker(A)$; in other words, for any f, if $u = Nf$ denotes the unique solution of $Au = f - H(f)$ being orthogonal to $\ker(A)$ (H is the orthogonal projection onto $\ker(A)$), then we have Hodge decomposition $f = AN(f) + H(f)$.* The main issue for the Hodge orthogonal direct sum decomposition $\mathcal{H} = \ker(A) \oplus \operatorname{im}(A)$, for a formally SA diff operator is to first find a truely SA extension of A (also denoted by A), and then ask whether A is closed range or not, because if A is closed range then $\ker(A)^\perp = \overline{\operatorname{im}(A)} = \operatorname{im}(A)$. By the famous closed-range theorem of FA, this reduces to proving the fundamental estimate that A is bounded below $\|Au\| \geq C\|u\|$ for $u \perp \ker(A)$, because then if $Au_i \to f$, then Au_u is Cauchy, so u_i is Cauchy, so $u_i \to u$, so $f = Au$.

2. In case of manifolds for solvability of $\overline{\partial}u = f$, f need to satisfy $\overline{\partial} f = 0$ and $f \perp \ker(\Box)$, or equivalently just $f \perp \ker(\overline{\partial}^*)$.

3. On a smoothly bounded strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^m$, $\Box$ (or better say its Friedrichs extension denoted by F in Folland-Kohn or $\mathcal{L}$ by Taylor) is invertible when $q > 0$, with compact inverse $N = \Box^{-1}$ called Neumann solution operator. $u = \Box N u = \overline{\partial}\overline{\partial}^* u + \overline{\partial}^* \overline{\partial} u$ is Hodge decomposition.

   For $q = 0$, $\Box$ has large kernel.

   For $p = q = 0$, we have orthogonal decomposition $u = (u - \overline{\partial}^* N \overline{\partial} u) + \overline{\partial}^* N \overline{\partial} u$, so the formula $Bu = u - \overline{\partial}^* N \overline{\partial} u$ for Bergman projection on functions.

4. In case of manifolds for solvability of $\Box u = f$, f need to satisfy $f \perp \ker(\Box)$. If so, one gets $f = \Box N f + H f$.

Theorem 110 (d-bar problem; smoothness up to the boundary)**.** *Let $D \subseteq \mathbb{C}^m$ be a bounded strongly pseudoconvex open with smooth boundary. Then for any $f \in C^\infty_{p,q+1}(\overline{D})$ with $\overline{\partial} f = 0$ there exists $u \in C^\infty_{p,q}(\overline{D})$ such that $\overline{\partial} u = f$.*

10.5 Worms

References: [CS, 6.4].

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function with the following properties:

1. η is nonnegative, even and convex.
2. $\eta^{-1}(0) = [-\beta + \pi/2, \beta - \pi/2]$ for some fixed $\beta > \pi/2$.
3. there exists $a > 0$ such that $\eta > 1$ outside $[-a, a]$.
4. $\eta'(x) \neq 0$ if $\eta(x) = 1$.

For each

$$W_\beta := \left\{ (z_1, z_2) \in \mathbb{C}^2 : \left| z_1 + e^{\sqrt{-1}\log|z_2|^2} \right|^2 < 1 - \eta\left(\log|z_2|^2\right) \right\}$$

Proposition 111. *W_β is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$.*

Theorem 112. *There is no C^2 global defining function for W_β which is plurisubharmonic on the boundary.*

Theorem 113. *For $\beta \geq 3\pi/2$ there does not exist a sequence D_j of pseudoconvex domains in $\mathbb{C}^2$ such that the closure of W_β equals $\bigcap D_j$.*

Theorem 114. *For $\beta > \pi/2$ the Bergman projection of W_β does not map $W^k \to W^k$ when $k \geq \pi/(2\beta - \pi)$.*

10.6 Biholomorphisms $\mathbb{C}^m \to \mathbb{C}^m$

References: [Fors, chapter 4].

10.7 Corona problem

References: [Gar, chapter 8][DKSTW]

Theorem 115 (Corona problem; Carleson)**.** *Let D be the open unit disc of the complex place, and let $H^\infty(D)$ be the (commutative Banach) algebra of bounded holomorphic functions on D. Then:*

(1) D is dense in the maximal ideal space of $H^\infty(D)$;

(2) For every $f_1, \ldots, f_n \in H^\infty(D)$ satisfying $\max|f_j(z)| \geq \delta$ for some $\delta > 0$ and every $z \in D$, there exists $g_1, \ldots, g_n \in H^\infty(D)$ such that $1 = \sum f_j g_j$;

(3) For every positive integer n and positive real δ there exists positive real C such that for every $f_1, \ldots, f_n \in H^\infty(D)$ satisfying $\max|f_j(z)| \geq \delta$ for every $z \in D$, there exists $g_1, \ldots, g_n \in H^\infty(D)$ such that $1 = \sum f_j g_j$ and $\|g_j\| < C$.

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