Infinitely divisible distributions of fractional transforms of Lévy measures and related Upsilon transformations

Thiele Seminar

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(based on joint works with O. Arizmendi, O. Barndorff-Nielsen, M. Maejima, K. Sato)

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Plan for the Seminar

I. **Motivation**

1. From free probability
2. From classical probability
3. Introduction: Gaussian representation

II. **Type G distributions again: a new look**

1. Lévy measure characterization
2. New Lévy measure characterization

III. **Distributions of class A**

1. Lévy measure characterization
2. Integral representation of type G distributions
3. Integral representation of distributions of class A

IV **Fractional transforms of Lévy measures**

1. Basic results
2. Relation to Upsilon transformations
3. Examples

Motivation

1. **Infinite divisibility with respect to non-classical convolutions**
    1. As Lévy measures (the arcsine case)
    2. As "Gaussian" distribution in different types of convolutions (Boolean, monotone, free)
2. *Goal of the talk:* **In classical infinite divisibility**
    1. They are not infinitely divisible, but
    2. As Lévy measures and their integral fractional transforms
3. *Introduction:* **Simple Gaussian representation**
    1. Consequences in infinite divisibility
    2. Motive to introduce distributions of Class A

I. Representation of the Gaussian distribution

- $\varphi(x;\tau)$ density of the Gaussian distribution *zero mean and variance* $\tau > 0$

$$\varphi(x;\tau) = (2\pi\tau)^{-1/2} \mathrm{e}^{-x^2/(2\tau)}, \; x \in \mathbb{R}. \tag{1}$$

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- $f_\tau(x)$ *exponential density* (Gamma $G(1, 2\tau)$):

$$f_\tau(x) = \frac{1}{2\tau} \exp(-\frac{1}{2\tau} x), \ x > 0. \qquad (2)$$

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- $\mathrm{a}(x,s)$ density of *arcsine distribution* $\mathrm{a}(x,s)\mathrm{d}x$

$$\mathrm{a}(x,s) = \begin{cases} \frac{1}{\pi}(s - x^2)^{-1/2}, & |x| < \sqrt{s} \\ 0 & |x| \geq \sqrt{s}. \end{cases} \tag{3}$$

A_s random variable with density $\mathrm{a}(x,s)$ on $(-\sqrt{s}, \sqrt{s})$. $(A = A_1)$.

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- **Arcsine distribution is not ID**.

I. A representation of the Gaussian distribution

Fact

$$\varphi(x;\tau) = \frac{1}{2\tau}\int_0^\infty e^{-s/(2\tau)} a(x;s)\,ds, \ \tau > 0, \ x \in \mathbb{R}. \tag{4}$$

Equivalently: If E_τ and A are independent random variables, then

$$Z_\tau \stackrel{L}{=} \sqrt{E_\tau}A.$$

Equivalently: Gaussian distribution is a exponential superposition of the arcsine distribution.

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- **Examples:** X^2 is infinitely divisible if X is stable symmetric, normal inverse Gaussian, normal variance gamma, $t-$student.

I. A characterization of Exponential Distribution

Theorem

Y_α, $\alpha > 0$, random variable with gamma distribution $G(\alpha, \beta)$ independent of A. Let

$$X = \sqrt{Y_\alpha} A.$$

Then X has an ID distribution if and only if $\alpha = 1$, in which case Y_1 has exponential distribution and X has Gaussian distribution.

- (Kingman (63)) $USP(\theta, \sigma)$: $\theta \geq -3/2$, $\sigma > 0$

$$f_\theta(x\,;\sigma) = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} \qquad -\sigma < x < \sigma \tag{4}$$

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I. Extension: Ultraspherical distributions

Similar representations of the Gaussian distribution

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- $\theta = -3/2$ is symmetric Bernoulli (role of Gaussian distribution in Boolean convolution)
- $\theta = 0$ is semicircle distribution (role of Gaussian distribution in free probability)
- $\theta = -1/2$ is uniform distribution,
- $\theta = \infty$ is Gaussian distribution: *Poincaré´s theorem*: $(\theta \to \infty)$

$$f_\theta(x\,;\sqrt{(\theta+2)/2}\sigma) \to \frac{1}{\sqrt{2\pi}\sigma}\exp(-x^2/(2\sigma^2)).$$

I. Other Gaussian representations

- $USP(\theta, \sigma)$: $\theta \geq -3/2$, $\sigma > 0$

$$f_\theta(x\,;\sigma) = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} \quad -\sigma < x < \sigma \tag{5}$$

- $USP(\theta,\sigma)$: $\theta \geq -3/2$, $\sigma > 0$

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Theorem (Kingman (63))

Let Y_α, $\alpha > 0$, r.v. with gamma distribution $G(\alpha,\beta)$ independent of r.v. S_θ with distribution $USP(\theta,1)$. Let

$$X \overset{L}{=} \sqrt{Y_\alpha}S_\theta \tag{6}$$

When $\alpha = \theta + 2$, X has a Gaussian distribution.

Moreover, *the distribution of X is infinitely divisible iff $\alpha = \theta + 2$ in which case X has a classical Gaussian distribution.*

- S_θ is r.v. with distribution $USP(\theta,1)$. For $\theta > -1/2$ it holds that

$$S_\theta \stackrel{L}{=} U^{1/(2(\theta+1))} S_{\theta-1}$$

where U is r.v. with uniform distribution $U(0,1)$ independent of r.v. $S_{\theta-1}$ with distribution $USP(\theta-1,1)$.

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- In particular, the *semicircle distribution is a mixture of the arcsine*

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I. Recursive representations

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- In particular, the *semicircle distribution is a mixture of the arcsine*

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- *This fact and the Gaussian representation suggest that the arcsine distribution is a "nice small" distribution to mixture with.*

- *Definition*: A mixture of Gaussians $X = \sqrt{V}Z$ has a *type G* distribution if $V > 0$ has an ID divisible distribution.

II. Type G distributions

Definition and relevance

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- Several well-known ID distributions are type G.
- $X_t^2 = (B_{V_t})^2$ is always infinitely divisible.
- **Open problem: ID of $(B_{V_t})^2$ as a process.**

II. Type G distributions: Lévy measure characterization

- If $V > 0$ is ID with Lévy measure ρ, then $\mu \stackrel{L}{=} \sqrt{V}Z$ is ID with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$

$$l(x) = \int_{\mathbb{R}_+} \varphi(x; s)\rho(\mathrm{d}s), \quad x \in \mathbb{R}. \tag{7}$$

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Theorem (Rosinski (91))

A symmetric distribution μ on $\mathbb{R}$ is type G iff is infinitely divisible and its Lévy measure is zero or $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where $l(x)$ is representable as

$$l(r) = g(r^2), \tag{8}$$

g is completely monotone on $(0, \infty)$ and $\int_0^\infty \min(1, r^2)g(r^2)\mathrm{d}r < \infty$.

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- In general $G(\mathbb{R})$ is the class of *generalized type G* distributions with Lévy measure (8).

II. Type G distributions: new characterization

- Using Gaussian representation in $I(x) = \int_{\mathbb{R}_+} \varphi(x;s)\rho(\mathrm{d}s)$:

$$I(x) = \int_0^\infty \mathrm{a}(x;s)\eta(s)\mathrm{d}s. \tag{9}$$

where $\eta(s) := \eta(s;\rho)$ is the completely monotone function

$$\eta(s;\rho) = \int_{\mathbb{R}_+} (2r)^{-1} e^{-s(2r)^{-1}} \rho(\mathrm{d}r). \tag{10}$$

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$$\int_0^\infty \min(1, s)\eta(s)\mathrm{d}s < \infty.$$

Lemma

Let g be a real function. The following statements are equivalent:
(a) g is completely monotone on $(0,\infty)$ *with*

$$\int_0^\infty (1\wedge r^2)g(r^2)\mathrm{d}r < \infty. \tag{11}$$

(b) There is a function $h(s)$ *completely monotone on* $(0,\infty)$*, with* $\int_0^\infty (1\wedge s)h(s)\mathrm{d}s < \infty$ *and* $g(r^2)$ *has the arcsine transform*

$$g(r^2) = \int_0^\infty \mathrm{a}^+(r;s)h(s)\mathrm{d}s, \quad r>0, \tag{12}$$

where

$$\mathrm{a}^+(r;s) = \begin{cases} 2\pi^{-1}(s-r^2)^{-1/2}, & 0<r<s^{1/2}, \\ 0, & \textit{otherwise}. \end{cases} \tag{13}$$

II. Type G distributions: Summary new representation

- Lévy measure is a (special) mixture of arcsine measure: There is a completely monotone function $\eta(s)$ on $(0, \infty)$ such that

$$l(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s)\mathrm{d}s. \tag{14}$$

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- **Next problem**: Characterization of ID distributions when Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$ **is the arcsine transform**

$$l(x) = \int_0^\infty \mathrm{a}(x;s)\lambda(\mathrm{d}s). \tag{15}$$

III. Distributions of Class A

Definition

$A(\mathbb{R})$ is the class of A of distributions on $\mathbb{R}$: ID distributions with Lévy measure $\nu(\mathrm{d}x) = l(x)\mathrm{d}x$, where

$$l(x) = \int_{\mathbb{R}_+} \mathrm{a}(x; s)\lambda(\mathrm{d}s) \tag{16}$$

and λ is a Lévy measure on $\mathbb{R}_+ = (0, \infty)$.

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- Maejima, PA, Sato (11): $A(\mathbb{R}^d)$ including non-symmetric case, stochastic integral representation, relation to Upsilon transformations, comparison to other known ID classes.
- $G(\mathbb{R}) \subset A(\mathbb{R})$.
- **How large is the class** $A(\mathbb{R})$?

- $ID(\mathbb{R}^d)$ **class of infinitely divisible distributions on** $\mathbb{R}^d$.

III. Recall some known classes of ID distributions

Characterization via Lévy measure

- $ID(\mathbb{R}^d)$ **class of infinitely divisible distributions on** $\mathbb{R}^d$.
- Polar decomposition of Lévy measure ν (univariate case $\xi = -1, 1$)

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- Observation: Arcsine density $a(x; s)$ is increasing in $r \in (0, \sqrt{s})$

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Let $\Psi : ID(\mathbb{R}^d) \to ID(\mathbb{R}^d)$ be the mapping given by

$$\Psi(\mu) = \mathcal{L}\left(\int_0^{1/2}\left(\log\frac{1}{s}\right)^{1/2} dX_s^{(\mu)}\right). \tag{18}$$

An ID distribution $\widetilde{\mu}$ belongs to $G(\mathbb{R}^d)$ iff there exists a type A distribution μ such that $\widetilde{\mu} = \Psi(\mu)$. That is

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- **Next problem: integral representation for type A distributions?**

- Jurek (85): $U(\mathbb{R}^d) = \mathcal{U}(ID(\mathbb{R}^d))$,

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III. Stochastic integral representations for some ID classes

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- Barndorff-Nielsen, Maejima, Sato (06): $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$ and $T(\mathbb{R}^d) = \Upsilon(L(\mathbb{R}^d))$

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{s}\mathrm{d}X_s^{(\mu)}\right).$$

III. Class A of distributions

Stochastic integral representation

Theorem (Maejima, PA, Sato (11))

Let $\Phi_{\cos} : ID(\mathbb{R}^d) \to ID(\mathbb{R}^d)$ *be the mapping*

$$\Phi_{\cos}(\mu) = \mathcal{L}\left(\int_0^1 \cos(\frac{\pi}{2}s)\mathrm{d}X_s^{(\mu)}\right), \quad \mu \in ID(\mathbb{R}^d). \tag{20}$$

Then

$$A(\mathbb{R}^d) = \Phi_{\cos}(ID(\mathbb{R}^d)). \tag{21}$$

.

IV. General framework

- **Upsilon transformations of Lévy measures:**

$$\Upsilon_\sigma(\rho)(B) = \int_0^\infty \rho(u^{-1}B)\sigma(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}^d). \qquad (22)$$

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- **Fractional transformations of Lévy measures:**

$$(\mathcal{A}_{q,p}^{\alpha,\beta}\nu)(C) = \frac{1}{\Gamma(p)}\int_0^\infty r^{-q-1}dr\int_{\mathbb{R}^d} 1_C(r\frac{x}{|x|})(|x|^\beta - r^\alpha)_+^{p-1}\nu(\mathrm{d}x),$$

$p, \alpha, \beta \in \mathbb{R}_+$, $q \in \mathbb{R}$ [Maejima, PA, Sato (2011b), Sato (10)].

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- **How large is the class $A_{q,p}^{\alpha}(\mathbb{R})$?**

IV. Fractional transformations of measures

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- **We do not know if there are stochastic integrals representations for** $q \geq 1$.

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- Flexibility in choice of parameters

Teorema

$$U(\mathbb{R}) \subset A^{\alpha}_{q,p}(\mathbb{R}) \qquad \textit{if } 0 < p \leq 1,\ q \leq -1, \tag{23}$$

$$A^{\alpha}_{q,p}(\mathbb{R}) \subset U(\mathbb{R}) \qquad \textit{if } p \geq 1,\ -1 \leq q < 2. \tag{24}$$

- Examples:
  - Arcsine distribution $(q = -1, p = 1/2, \alpha = 2, \beta = 1)$ then (23).
  - Semicircle distribution $(q = -1, p = 3/2,\ \alpha = 2,\ \beta = 2)$ then (24)
  - Uniform $(q = -1, p = 1)$ then (23) and (24).
- There are stochastic integrals representations when $q < 1$.
- **We do not know if there are stochastic integrals representations for** $q \geq 1$.
- **Open problem: Do all infinitely divisible distributions admit a stochastic integral representation?**:

- There are stochastic representations when $q < 1$.

IV. Examples of integral representations

- There are stochastic representations when $q < 1$.
- **Special case:** $\alpha > 0, p > 0$, $q = -\alpha$. Let $\Phi_{\alpha,p} : ID(\mathbb{R}) \rightarrow ID(\mathbb{R})$

$$\Phi_{\alpha,p}(\mu) = \mathcal{L}\left(c_{p+1}^{-1/(\alpha p)} \int_0^{c_{p+1}} \left(c_{p+1}^{1/p} - s^{1/p} \right)^{1/\alpha} dX_s^{(\mu)} \right). \qquad (25)$$

with $c_p = 1/\Gamma(p)$. Then $A^{\alpha}_{-\alpha,p}(\mathbb{R}) = \Phi_{\alpha,p}(ID(\mathbb{R}))$.

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Example

If $p = 1/2$, $\alpha = 1$, $(q = -1)$

$$A^1_{-1,1/2}(\mathbb{R}) = \Phi_{1,1/2}(ID(\mathbb{R})),$$

$$\Phi_{1,1/2}(\mu) = \frac{\pi}{4} \int_0^{2/\sqrt{\pi}} \left(\frac{4}{\pi} - s^2 \right) dX_s^{(\mu)}, \quad \mu \in ID(\mathbb{R}).$$

IV. Relation to Upsilon transformations

Non-commutative relations

$$(\Upsilon_{\beta,\theta}\,\rho)(B) = \int_0^\infty \rho(t^{-1}B)\theta t^{-\beta-1}e^{-t^\theta}dt, \quad B \in \mathcal{B}(\mathbb{R}_0^d).$$

$$(\mathcal{A}_{\alpha,p}^{q,r}\rho)(B) = c_p \int_0^\infty u^{-\alpha-1}du \int_{\mathbb{R}_0^d} 1_B(ux/|x|)(|x|^r - u^q)_+^{p-1}\rho(dx).$$

Teorema

a) Non-commutative relation holds

$$\Upsilon_{\beta,\theta}\mathcal{A}_{\alpha,p}^{q,r} = \mathcal{A}_{\alpha,p}^{q,r}\Upsilon_{r(p-1+(\beta-\alpha)/q),\theta r/q},$$

b) $A_{\alpha,p}^{q,r}$ *and* $\Upsilon_{\beta,\theta}$ *are commutative iff*

$$q = r \quad and \quad q(p-1) - \alpha = 0.$$

IV. Relation to Upsilon transformations

- General upsilon transformation

$$\Upsilon_\sigma(\rho)(B) = \int_0^\infty \rho(t^{-1}B)\sigma(\mathrm{d}t). \tag{26}$$

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- Four parameter transformation

$$(\mathcal{A}_{\alpha,p}^{q,r}\rho)(B) = c_p \int_0^\infty u^{-\alpha-1}du \int_{\mathbb{R}_0^d} 1_B(ux/|x|)(|x|^r - u^q)_+^{p-1}\rho(dx). \tag{28}$$

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- Four parameter transformation

$$(\mathcal{A}_{\alpha,p}^{q,r}\rho)(B) = c_p \int_0^{\infty} u^{-\alpha-1} du \int_{\mathbb{R}_0^d} 1_B(ux/|x|)(|x|^r - u^q)_+^{p-1}\rho(dx). \tag{28}$$

- Goals:
  - when is $\mathcal{A}_{\alpha,p}^{q,r}\rho$ an Upsilon transformation?
  - What is the role of the Upsilon transformation $\Upsilon_{\beta,\theta}$?

$$(\Upsilon_{\beta,\theta}\,\rho)(B) = \int_0^\infty \rho(t^{-1}B)\theta t^{-\beta-1}e^{-t^\theta}dt, \quad B \in \mathcal{B}(\mathbb{R}_0^d).$$

$$(\mathcal{A}_{\alpha,p}^{q,r}\rho)(B) = c_p \int_0^\infty u^{-\alpha-1}du \int_{\mathbb{R}_0^d} 1_B(ux/|x|)(|x|^r - u^q)_+^{p-1}\rho(dx).$$

Teorema

a) Non-commutative relation holds

$$\Upsilon_{\beta,\theta}\mathcal{A}_{\alpha,p}^{q,r} = \mathcal{A}_{\alpha,p}^{q,r}\Upsilon_{r(p-1+(\beta-\alpha)/q),\theta r/q},$$

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$$(\mathcal{A}_{\alpha,p}^{q,r}\rho)(B) = c_p \int_0^\infty u^{-\alpha-1}du \int_{\mathbb{R}_0^d} 1_B(ux/|x|)(|x|^r - u^q)_+^{p-1}\rho(dx).$$

Teorema

a) $A_{\alpha,p}^{q,r}$ is an Upsilon transformation iff $q = r$, in which case

$$\sigma(du) = c_p 1_{(0,1)}(u)(1-u^q)^{p-1}u^{-\alpha-1}du$$

b) $A_{\alpha,p}^{q,r}$ is Upsilon transformation iff $A_{\alpha,p}^{q,r}$ commutes with some $\Upsilon_{\beta,\theta}$.

IV. A class of examples

- $$\mathcal{A}^q_{\alpha,p}(\rho)(B) = c_p \int_0^\infty 1_B(u) u^{-\alpha-1} du \int_{(u^q,\infty)} (x-u^q)^{p-1} \rho(dx)$$
$$\widetilde{\rho}(dx) = \rho_{\beta,c,\theta}(dx) = x^{-\beta-1} e^{-cx^\theta} 1_{(0,\infty)}(x) dx$$
- $\widetilde{\rho}(dx) = \left(\mathcal{A}^q_{\alpha,p}\rho_{\beta,c,\theta}\right)(du)$

$$\widetilde{\rho}(dx) = c_p u^{-\alpha-1+q(p-\beta-1)} \left(\int_1^\infty (y-1)^{p-1} y^{-\beta-1} e^{-c(yu^q)^\theta} dy \right) 1_{(0,\infty)}(u) du.$$

Example (James, Roynette & Yor (08))

$$\rho(dx) = 2^{\alpha-1/2}\sqrt{\pi} x^{\alpha-1} e^{-2x} 1_{(0,\infty)}(x) dx. \quad (29)$$

$$\widetilde{\rho}(dx) = x^{-1} e^{-x} K_{\alpha-1/2}(x) 1_{(0,\infty)}(x) dx \quad (30)$$

$\widetilde{\rho}$ is GGC subordinator [$K_{\alpha-1/2}$ is Bessel function of order $\alpha - 1/2$]

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