Infinitely Divisible Multivariate and Matrix Gamma Distributions

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Abstract

Classes of multivariate and cone valued infinitely divisible Gamma distributions are introduced. Particular emphasis is put on the cone-valued case, due to the relevance of infinitely divisible distributions on the positive semi-definite matrices in applications. The cone-valued class of generalised Gamma convolutions is studied. In particular, a characterisation in terms of an Itô-Wiener integral with respect to an infinitely divisible random measure associated to the jumps of a Lévy process is established.

A new example of an infinitely divisible positive definite Gamma random matrix is introduced. It has properties which make it appealing for modelling under an infinite divisibility framework. An interesting relation of the moments of the Lévy measure and the Wishart distribution is highlighted which we suppose to be important when considering the limiting distribution of the eigenvalues.

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1. Introduction

The classical examples of multivariate and matrix Gamma distributions in the probability and statistics literature are not necessarily infinitely divisible [14], [19], [40]. These examples are analogous to one-dimensional Gamma distributions and are obtained by a direct generalisation of the one-dimensional probability densities; see for example [15], [23], [24]. Working in the domain of Fourier transforms, some infinitely divisible matrix Gamma distributions have recently been considered in [5], [27]. Their Lévy measures are direct generalisations of the one-dimensional Gamma distribution. The work of [27] arose in the context of random matrix models relating classical and free infinitely divisible distributions.

The study of infinitely divisible random elements in cones has been considered in [4], [25], [26], [31] and references therein. They are important in the construction and modeling of cone increasing Lévy processes. In the particular case of infinitely divisible positive-definite random matrices, their importance in applications has been recently highlighted in [7], [8], [28] and [29]. This is due to the fact that infinite divisibility allows modelling by matrix Lévy and Ornstein-Uhlenbeck processes, which are in those papers used to model the time dynamics of a $d \times d$ covariance matrix to obtain a so-called stochastic volatility model (for observed series of financial data).

Generalized Gamma Convolutions (GGC) is a rich and interesting class of one-dimensional infinitely divisible distributions on the cone $\mathbb{R}_+ = [0, \infty)$. It is the smallest class of infinitely divisible distributions on $\mathbb{R}_+$ that contains all Gamma distributions and that is closed under classical convolution and weak convergence. This class was introduced by O. Thorin in a series of papers and further studied by L. Bondesson in his book [10]. The book of Steutel and Van Harn [39] contains also many results and examples about GGC. Several well known and important distributions on $\mathbb{R}_+$ are GGC. The recent survey paper by James, Roynette and Yor [16] contains a number of classical results and old and new examples of GGC. The multivariate case was considered in Barndorff-Nielsen, Maejima and Sato [3].

There are three main purposes in this paper. We formulate and study multivariate and cone valued Gamma distributions which are infinitely divisible. Second, we consider and characterise the corresponding class $GGC(K)$ of Generalised Gamma Convolutions on a finite dimensional cone K. Finally, we introduce a new example of a positive definite random matrix with infinitely divisible Gamma distribution and with explicit Lévy measure.

The main results and organisation of the paper are as follows. Section 2 briefly presents preliminaries on notation and results about one-dimensional GGC on $\mathbb{R}_+$ as well as some matrix notation. Section 3 introduces a class of infinitely divisible d-variate Gamma distributions $\Gamma_d(\alpha,\beta)$, whose Lévy measures are analogous to the Lévy measure of the one-dimensional Gamma distribution. The parameters α and β are measures and functions on $\mathbf{S}$ (the unit sphere with respect to a prescribed norm), respectively. It is shown that the distribution does not depend on the particular norm under consideration. The characteristic function is derived and it is shown that the Fourier-Laplace transform on $\mathbb{C}^d$ exists if β is bounded away from zero $\alpha-$ almost everywhere. Furthermore, the finiteness of moments of all orders is studied and some interesting examples exhibiting essential differences to univariate Gamma distributions are given.

Section 4 considers cone valued Gamma distributions and their corresponding class $GGC(K)$ of Generalised Gamma Convolutions on a cone K, defined as the smallest class of distributions on K which is closed under convolution and weak convergence and contains all the so-called elementary Gamma variables in K (and also all Gamma random variables in K in our new definition). This class is characterised as the stochastic integral of a non-random function with respect to the Poisson random measure of the jumps of a Gamma Lévy process on the cone. This is a new representation in the multivariate case extending the Wiener-Gamma integral characterization of one-dimensional GGC on $\mathbb{R}_+ = [0,\infty)$, as considered, for example, in [16].

Section 5 considers the special cone valued case of infinitely divisible positive-semidefinite $d \times d$ matrix Gamma distributions. New examples are introduced via an explicit form of their Lévy measure. They include as particular cases the examples considered in [5], [27]. A detailed study is done of the new two parameter positive definite matrix distribution $A\Gamma(\eta,\Sigma)$, where $\eta > (d-1)/2$ and Σ is a $d \times d$ positive definite matrix. This special infinitely divisible Gamma matrix distribution has several modeling features similar to the classical (but non-infinitely divisible) matrix Gamma distribution defined through a density, in particular the Wishart distribution. Namely, moments of all orders exist, the matrix mean is proportional to Σ and the matrix of covariances equals the second moment of the Wishart distribution. When Σ is the $d \times d$ identity matrix I_d, the distribution is invariant under orthogonal conjugations and the trace of a random matrix M with distribution $A\Gamma(\eta,\mathrm{I}_d)$ has a one-dimensional Gamma distribution. A relation of the moments of the Marchenko-Pastur distribution with the asymptotic moments of the Lévy measure is exhibited. Hence, this matrix Gamma distribution has a special role when dealing with a random covariance matrix and its time dynamics, e.g. by specifying it as a matrix Lévy or Ornstein-Uhlenbeck process. As an application, the matrix Normal-Gamma distribution is introduced, which is a matrix extension of the one-dimensional variance Gamma distribution of [22] which is popular in finance.

2. Preliminaries

For the general background in infinitely divisible distributions and Lévy processes we refer to the standard references, e.g. [36].

2.1. One-dimensional GGC

A positive random variable Y with law $\mu = \mathscr{L}(Y)$ belongs to the class of Generalised Gamma Convolutions (GGC) on $\mathbb{R}_+ = [0,\infty)$, denoted by $T(\mathbb{R}_+)$, if and only if there exists a positive Radon measure υ_μ on $(0,\infty)$ and $a > 0$ such that its Laplace transform is given by:

$$L_\mu(z) = \mathbb{E}e^{-zY} = \exp\left(-az - \int_0^\infty \ln\left(1+\frac{z}{s}\right)\ \upsilon_\mu(\mathrm{d}s)\right) \tag{2.1}$$

with

$$\int_0^1 |\log x| \upsilon_\mu(\mathrm{d}x) < \infty, \quad \int_1^\infty \frac{\upsilon_\mu(\mathrm{d}x)}{x} < \infty. \tag{2.2}$$

For convenience we shall work without the translation term, *i.e.* with $a = 0$. The measure υ_μ is called the Thorin measure of μ. Its Lévy measure is concentrated on $(0,\infty)$ and is such that:

$$\nu_\mu(\mathrm{d}x) = x^{-1} l_\mu(x)\mathrm{d}x, \tag{2.3}$$

where l_μ is a completely monotone function in $x > 0$ given by

$$l_\mu(\mathrm{d}x) = \int_0^\infty e^{-xs} \upsilon_\mu(\mathrm{d}s). \tag{2.4}$$

The class $T(\mathbb{R}_+)$ can be characterized by Wiener-Gamma representations. Specifically, a positive random variable Y belongs to $T(\mathbb{R}_+)$ if and only if there is a Borel function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\int_0^\infty \ln(1+h(t))\mathrm{d}t < \infty, \tag{2.5}$$

such that $Y \stackrel{\mathscr{L}}{=} Y^h$ has the Wiener-Gamma integral representation

$$Y^h \stackrel{\mathscr{L}}{=} \int_0^\infty h(u)\mathrm{d}\gamma_u, \tag{2.6}$$

where $(\gamma_t; t \geq 0)$ is the standard Gamma process with Lévy measure $\nu(\mathrm{d}x) = e^{-x}\frac{\mathrm{d}x}{x}$. The relation between the Thorin function h and the Thorin measure υ_μ is as follows: υ_μ is the image of the Lebesgue measure on $(0,\infty)$ under the application : $s \to 1/h(s)$. That is,

$$\int_0^\infty e^{-\frac{x}{h(s)}}\mathrm{d}s = \int_0^\infty e^{-xz}\upsilon_\mu(\mathrm{d}z), \quad x > 0. \tag{2.7}$$

On the other hand, if $F_{\upsilon_\mu}(x) = \int_0^x \upsilon_\mu(\mathrm{d}y)$ for $x \geq 0$ and $F_{\upsilon_\mu}^{-1}(s)$ is the the right continuous generalised inverse of $F_{\upsilon_\mu}(s)$, that is $F_{\upsilon_\mu}^{-1}(s) = \inf\{t > 0; F_{\upsilon_\mu}(t) \geq s\}$ for $s \geq 0$, then, $h(s) = 1/F_{\upsilon_\mu}^{-1}(s)$ for $s \geq 0$.

Many well known distributions belong to $T(\mathbb{R}_+)$. The positive α-stable distributions, $0 < \alpha < 1$, are GGC with $h(s) = \{s\theta\Gamma(\alpha+1)\}^{-\frac{1}{\alpha}}$ for a $\theta > 0$. In particular, for the $1/2-$stable distribution, $h(s) = 4\left(s^2\pi\right)^{-1}$. Beta distribution of the second kind, lognormal and Pareto are also GGC, see [16].

For more details on univariate GGCs we refer to [10, 16]

2.2. Notation

$\mathbb{M}_d(\mathbb{R})$ is the linear space of $d\times d$ matrices with real entries and $\mathbb{S}_d$ its subspace of symmetric matrices. By $\mathbb{S}_d^+$ and $\overline{\mathbb{S}}_d^+$ we denote the open (in $\mathbb{S}_d$) and closed cones of positive and nonnegative definite matrices in $\mathbb{M}_d(\mathbb{R})$. $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$ is the unit sphere on $\mathbb{R}^d$ with respect to the norm $\|\cdot\|$.

The Fourier transform $\widehat{\mu}$ of a measure μ on $\mathbb{M}=\mathbb{R}^d$ or $\mathbb{M}=\mathbb{M}_d(\mathbb{R})$ is given by

$$\hat{\mu}(z)=\int_{\mathbb{M}} e^{i\langle z,x\rangle}\mu(\mathrm{d}x) \quad z\in\mathbb{M}$$

where we use $\langle A,B\rangle=\mathrm{tr}(A^\top B)$ as the scalar product in the matrix case, where $A^\top$ denotes the transposed on $\mathbb{M}_d(\mathbb{R})$. By I_d we denote the $d\times d$ identity matrix and by $|A|$ the determinant of a square matrix A. For a matrix A in the linear group $\mathscr{GL}_d(\mathbb{R})$ we write $A^{-\top}=\left(A^\top\right)^{-1}$.

We say that the distribution of a symmetric random $d\times d$ matrix M is invariant under orthogonal conjugations if the distribution of $OMO^\top$ equals the distribution of M for any non-random matrix O in the orthogonal group $\mathscr{O}(\mathrm{d})$. Note that $M\to OMO^\top$ with $O\in\mathscr{O}(\mathrm{d})$ are all linear orthogonal maps on $\mathbb{S}_d^+$ (or $\mathbb{M}_d$) preserving $\mathbb{S}_d^+$.

3. Multivariate Gamma Distributions

3.1. Definition

Definition 3.1. *Let μ be an infinitely divisible probability distribution on $\mathbb{R}^d$. If there exists a finite measure α on the unit sphere $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$ with respect to the norm $\|\cdot\|$ equipped with the Borel σ-algebra and a Borel-measurable function $\beta:\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}\to\mathbb{R}_+$ such that*

$$\hat{\mu}(z)=\exp\left(\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\int_{\mathbb{R}_+}\left(e^{irv^\top z}-1\right)\frac{e^{-\beta(v)r}}{r}\mathrm{d}r\alpha(\mathrm{d}v)\right) \tag{3.1}$$

*for all $z\in\mathbb{R}^d$, then μ is called a d-*dimensional Gamma distribution *with parameters α and β, abbreviated $\Gamma_d(\alpha,\beta)$-distribution.*

*If β is constant, we call μ a $\|\cdot\|$-*homogeneous $\Gamma_d(\alpha,\beta)$-distribution.

Observe that the notation $\Gamma_d(\alpha,\beta)$ implicitly also specifies which norm we use, because α is a measure on the unit sphere with respect to the norm employed and β is a function on it. The parameters α and β play a comparable role as shape and scale parameters as in the usual positive univariate case.

Remark 3.2. *(i) Obviously the Lévy measure ν_μ of μ is given by*

$$\nu_\mu(E)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\int_{\mathbb{R}_+}1_E(rv)\frac{e^{-\beta(v)r}}{r}\mathrm{d}r\alpha(\mathrm{d}v) \tag{3.2}$$

for all $E\in\mathscr{B}(\mathbb{R}^d)$. This expression is equivalent to

$$\nu_\mu(\mathrm{d}x)=\frac{e^{-\beta(x/\|x\|)\|x\|}}{\|x\|}\widetilde{\alpha}(\mathrm{d}x), \quad x\in\mathbb{R}^d \tag{3.3}$$

where $\widetilde{\alpha}$ is a measure on $\mathbb{R}^d$ given by

$$\widetilde{\alpha}(E)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\int_0^\infty 1_E(rv)\mathrm{d}r\alpha(\mathrm{d}v), \quad E\in\mathscr{B}(\mathbb{R}^d). \tag{3.4}$$

(ii) Likewise we define $\mathbb{M}_d(\mathbb{R})$ and $\mathbb{S}_d$-valued Gamma distributions with parameters α and β (abbreviated $\Gamma_{\mathbb{M}_d}(\alpha,\beta)$ and $\Gamma_{\mathbb{S}_d}(\alpha,\beta)$, respectively) by replacing $\mathbb{R}^d$ with $\mathbb{M}_d(\mathbb{R})$ and $\mathbb{S}_d$, respectively, and the Euclidean scalar product with $\langle Z,X\rangle = \mathrm{tr}(X^\top Z)$. All upcoming results immediately generalise to this matrix-variate setting. We provide further details in Section 5.

If $d=1$ and $\alpha(\{-1\})=0$, then we have the usual one-dimensional $\Gamma(\alpha(\{1\}),\beta(1))$-distribution. In general it is elementary to see that for $d=1$ a random variable $X\sim\Gamma_1(\alpha,\beta)$ if and only if $X\overset{\mathscr{D}}{=}X_1-X_2$ with $X_1\sim\Gamma(\alpha(\{1\}),\beta(1))$ and $X_2\sim\Gamma(\alpha(\{-1\},\beta(-1))$ being two independent usual Gamma random variables, i.e. X has a bilateral Gamma distribution as analysed in [17, 18] and introduced in [11, 22] under the name variance Gamma distribution. If $\alpha(\{1\})=\alpha(\{-1\})$ and $\beta(1)=\beta(-1)$, it indeed can be represented as the variance mixture of a normal random variable with an independent positive Gamma one (a comprehensive summary of this case can be found in [39] where it is called sym-Gamma distribution).

Now we address the question which α,β we can take to obtain a Gamma distribution.

Proposition 3.3. *Let α be a finite measure on $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$ and $\beta:\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}\to\mathbb{R}_+$ a measurable function. Then* (3.2) *defines a Lévy measure ν_μ and thus there exists a $\Gamma_d(\alpha,\beta)$ probability distribution μ if and only if*

$$\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\ln\left(1+\frac{1}{\beta(v)}\right)\alpha(\mathrm{d}v)<\infty. \tag{3.5}$$

Moreover, $\int_{\mathbb{R}^d}(\|x\|\wedge 1)\nu_\mu(\mathrm{d}x)<\infty$ holds true.

The condition (3.5) is trivially satisfied, if β is bounded away from zero α-almost everywhere.

Proof:

$$\int_{\|x\|\le 1}\|x\|\nu_\mu(\mathrm{d}x)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\int_0^1 e^{-\beta(v)r}\mathrm{d}r\alpha(\mathrm{d}v)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\frac{1-e^{-\beta(v)}}{\beta(v)}\alpha(\mathrm{d}v)\le\alpha(\mathbf{S}_{\mathbb{R}^d,\|\cdot\|})<\infty$$

using the elementary inequality $1-e^{-x}\le x$, for each $x\in\mathbb{R}_+$. Denoting by E_1 the exponential integral function given by $E_1(z)=\int_z^\infty\frac{e^{-t}}{t}dt$ for $z\in\mathbb{R}_+$, we get

$$\int_{\|x\|>1}\nu_\mu(\mathrm{d}x)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\int_1^\infty\frac{e^{-\beta(v)r}}{r}\mathrm{d}r\alpha(\mathrm{d}v)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}E_1(\beta(v))\alpha(\mathrm{d}v) \tag{3.6}$$

$$=\int_0^\infty E_1(z)\tau(\mathrm{d}z), \tag{3.7}$$

where we made the substitution $z=\beta(v)$ and $\tau(E)=\alpha(\beta^{-1}(E))$ for all Borel sets E in $\mathbb{R}_+$. Since τ is a finite measure and $0\le E_1(z)\le e^{-z}\ln(1+1/z)\,\forall z\in\mathbb{R}_+$ (see [1, p. 229]),

$$\int_{1/2}^\infty E_1(z)\tau(\mathrm{d}z)<\infty.$$

The series representation $E_1(z)=-\gamma-\ln(z)-\sum_{i=1}^\infty\frac{(-1)^iz^i}{n\cdot n!}$ with γ being the Euler-Mascheroni constant ([1, p. 229]) implies that $\lim_{z\downarrow 0}E_1(z)/(-\ln(z))=1$. Hence,

$$\int_0^{1/2}E_1(z)\tau(\mathrm{d}z)<\infty\Leftrightarrow\int_0^{1/2}|\ln(z)|\tau(\mathrm{d}z)<\infty\Leftrightarrow\int_0^{1/2}\ln(1+1/z)\tau(\mathrm{d}z)<\infty$$

using $\ln(1+1/z) = \ln(1+z) - \ln(z)$ and the finiteness of τ in the second equivalence. Appealing to the finiteness of τ once more, the above conditions are equivalent to

$$\int_0^\infty \ln(1+1/z)\tau(\mathrm{d}z) = \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \ln(1+1/\beta(v))\alpha(\mathrm{d}v) < \infty.$$

■

The next proposition shows that the definition of a Gamma distribution does not depend on the norm, only the parametrisation changes when using different norms.

Proposition 3.4. *Let $\|\cdot\|_a$ be a norm on $\mathbb{R}^d$ and μ be a $\Gamma_d(\alpha,\beta)$ distribution with α being a finite measure on $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_a}$ and $\beta : \mathbf{S}_{\mathbb{R}^d,\|\cdot\|_a} \to \mathbb{R}_+$ measurable. If $\|\cdot\|_b$ is another norm on $\mathbb{R}^d$, then μ is a $\Gamma_d(\alpha_b,\beta_b)$ distribution with α_b being a finite measure on $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_b}$ and $\beta_b : \mathbf{S}_{\mathbb{R}^d,\|\cdot\|_b} \to \mathbb{R}_+$ measurable. Moreover, it holds that*

$$\alpha_b(E) = \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_a}} 1_E\left(\frac{v}{\|v\|_b}\right)\alpha(\mathrm{d}v) \qquad \forall E \in \mathscr{B}(\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_b}) \tag{3.8}$$

$$\beta_b(v_b) = \beta\left(\frac{v_b}{\|v_b\|_a}\right)\|v_b\|_a \qquad \forall v_b \in \mathbf{S}_{\mathbb{R}^d,\|\cdot\|_b}. \tag{3.9}$$

The above formulae show that the mass in the different directions, which is given by α, does not change, and β only needs to be adapted for the scale changes implied by the change of the norm.

Proof: Substituting first $v_b = v/\|v\|_b$ and then $s = r/\|v_b\|_a$ gives:

$$\begin{aligned}
&\exp\left(\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_a}}\int_{\mathbb{R}_+}\left(e^{irv^\top z}-1\right)\frac{e^{-\beta(v)r}}{r}\mathrm{d}r\alpha(\mathrm{d}v)\right)\\
&\quad = \exp\left(\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_b}}\int_{\mathbb{R}_+}\left(e^{i\frac{r}{\|v_b\|_a}v_b^\top z}-1\right)\frac{e^{-\beta\left(\frac{v_b}{\|v_b\|_a}\right)r}}{r}\mathrm{d}r\alpha_b(\mathrm{d}v_b)\right)\\
&\quad = \exp\left(\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_b}}\int_{\mathbb{R}_+}\left(e^{isv_b^\top z}-1\right)\frac{e^{-\beta\left(\frac{v_b}{\|v_b\|_a}\right)\|v_b\|_a s}}{s}\mathrm{d}s\alpha_b(\mathrm{d}v_b)\right).
\end{aligned}$$

■

3.2. Properties

In this section we study several fundamental properties of our Gamma distributions.

Proposition 3.5. *Any $\Gamma_d(\alpha,\beta)$-distribution is self-decomposable.*

Proof: This follows immediately from the definition and [36, Th. 15.10]. ■

Later on we will considerably improve this result by showing that we are in a very special subset of the self-decomposable distributions. This result has important implications for applications where one likes to work with distributions having densities, i.e. distributions which are absolutely continuous (with respect to the Lebesgue measure).

Proposition 3.6. *Assume that* $\operatorname{supp}\alpha$ *is of full dimension, i.e. that it contains d linearly independent vectors in $\mathbb{R}^d$. Then the $\Gamma_d(\alpha,\beta)$-distribution is absolutely continuous.*

Proof: It is immediate that the support of $\Gamma_d(\alpha,\beta)$ is the closed convex cone generated by supp α. Hence, the support of $\Gamma_d(\alpha,\beta)$ is of full dimension and so the distribution is non-degenerate. Thus [35] concludes. ■

It follows along the same lines that in the degenerate case the $\Gamma_d(\alpha,\beta)$-distribution is absolutely continuous with respect to the Lebesgue measure on the subspace generated by supp α. If supp α consists of exactly d linearly independent vectors, $\Gamma_d(\alpha,\beta)$ equals the distribution of a linear transformation of a vector of d independent univariate Gamma random variables with appropriate parameters and thus the density can be calculated easily using the density transformation theorem with an invertible linear map. If supp α is a finite set of full dimension, one can calculate the density from the density of independent univariate Gamma random variables by using the density transformation theorem with an invertible linear map and integrating out the non-relevant dimensions. In general the density can be determined via solving a partial integro-differential equation (see [37]). Moreover, criteria for qualitative properties of the density like continuity and continuous differentiability can be deduced from the results of [33, 34], but looking at the simple case of a vector of independent univariate Gamma distributions one immediately sees that the sufficient conditions given there are far from being sharp. Therefore we refrain from giving more details.

Next we show that our d-dimensional Gamma distribution has the same closedness properties regarding scaling and convolution as the usual univariate one.

Proposition 3.7. *(i) Let $X\sim\Gamma_d(\alpha,\beta)$ and $c>0$. Then $cX\sim\Gamma_d(\alpha,\beta/c)$.*

(ii) Let $X_1\sim\Gamma_d(\alpha_1,\beta)$ and $X_2\sim\Gamma_d(\alpha_2,\beta)$ be two independent d-dimensional Gamma variables. Then $X_1+X_2\sim\Gamma_d(\alpha_1+\alpha_2,\beta)$.

Proof: Follows immediately from considering the characteristic functions. ■

Likewise it is immediate to see the following distributional properties of the induced Lévy process.

Proposition 3.8. *Let L be a $\Gamma_d(\alpha,\beta)$ Lévy process, i.e. $L_1\sim\Gamma_d(\alpha,\beta)$. Then $L_t\sim\Gamma_d(t\alpha,\beta)$ for all $t\in\mathbb{R}_+$.*

Of high importance for applications is that the class of Γ_d distributions is invariant under invertible linear transformations.

Proposition 3.9. *Let $X\sim\Gamma_d(\alpha,\beta)$ (with respect to the norm $\|\cdot\|$) and A be an invertible $d\times d$ matrix. Then $AX\sim\Gamma_d(\alpha_A,\beta_A)$ with respect to the norm $\|\cdot\|_A=\|A^{-1}\cdot\|$ and*

$$\alpha_A(E)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}1_E(Av)\,\alpha(\mathrm{d}v)=\alpha(A^{-1}E)\qquad\forall E\in\mathscr{B}(\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_A})\tag{3.10}$$

$$\beta_A(v)=\beta\left(A^{-1}v\right)\qquad\forall v\in\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_A}.\tag{3.11}$$

Proof: We have for all $z\in\mathbb{R}^d$

$$\begin{aligned}E\left(e^{i<z,AX>}\right)&=\int_{\mathbb{R}^d}e^{i<z,Ax>}\mu(dx)=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}}\int_{\mathbb{R}_+}\left(e^{irv^\top A^\top z}-1\right)\frac{e^{-\beta(v)r}}{r}dr\alpha(dv)\\&=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_A}}\int_{\mathbb{R}_+}\left(e^{iru^\top z}-1\right)\frac{e^{-\beta(A^{-1}u)r}}{r}dr\alpha(A^{-1}du)\\&=\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_A}}\int_{\mathbb{R}_+}\left(e^{iru^\top z}-1\right)\frac{e^{-\beta_A(u)r}}{r}dr\alpha_A(du)\end{aligned}$$

where we substituted $u = Av$. ■

It is easy to see that the above proposition can be extended to $m \times d$ matrices of full rank with $m > d$. Obviously, such a result cannot hold in general for a linear transformation A with $\ker(A) \neq \{0\}$, since combinations of one dimensional Gamma distributions are in general not univariate Gamma distributions.

Next we present an alternative representation of the characteristic function.

Proposition 3.10. *Let μ be $\Gamma_d(\alpha,\beta)$ distributed. Then the characteristic function is given by*

$$\hat{\mu}(z) = \exp\left(\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \ln\left(\frac{\beta(v)}{\beta(v) - iv^\top z}\right)\alpha(\mathrm{d}v)\right) \text{ for all } z \in \mathbb{R}^d \tag{3.12}$$

where ln *is the main branch of the complex logarithm.*

Proof: Follows from the definition and the well known fact

$$\int_0^\infty \left(e^{-r(-iv^\top)z} - 1\right)\frac{e^{-\beta(v)r}}{r}\mathrm{d}r = \ln\left(\frac{\beta(v)}{\beta(v) - iv^\top z}\right).$$

■

Note that if α has countable support $\{v_j\}_{j\in\mathbb{N}}$, then

$$\hat{\mu}(z) = \prod_{j\in\mathbb{N}}\left(\frac{\beta(v_j)}{\beta(v_j) - iv_j^\top z}\right)^{\alpha(\{v_j\})}.$$

We now show that the Fourier-Laplace transform of a Gamma distribution exists if and to a certain extent only if β is bounded away from zero α almost everywhere.

Theorem 3.11. *(i) The Fourier-Laplace transform $\hat{\mu}$ of a $\Gamma_d(\alpha,\beta)$ distribution μ exists for all z in a neighborhood $U \subseteq \mathbb{C}^d$ of zero, if $\beta(v) \geq \kappa$ for $v \in \mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$ α-a.e. with $\kappa > 0$. $\hat{\mu}$ is analytic there and given by formula* (3.12)*.*

(ii) If there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$ with $\lim_{n\to\infty}\beta(v_n) = 0$ and $\alpha(\{v_n\}) > 0$ for all $n \in \mathbb{N}$, then the Fourier-Laplace transform $\hat{\mu}$ exists in no neighborhood $U \subseteq \mathbb{C}^d$ of zero.

Proof: Using Proposition 3.4 we can assume w.l.o.g. that the Euclidean norm $\|\cdot\|_2$ is used for the definition of the $\Gamma_d(\alpha,\beta)$ distribution.

(i) We will now show (i) for $U = B_\kappa(0) \subseteq \mathbb{C}^d$, where $B_\kappa(0) := \{x \in \mathbb{C}^d : \|x\|_2 < \kappa\}$. From Proposition 3.10 it is clear that $\hat{\mu}(z)$ exists for all $z \in B_\kappa(0) \subseteq \mathbb{C}^d$, if and only if

$$\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot|_2}} \ln\left(\frac{\beta(v)}{\beta(v) - iv^\top z}\right)\alpha(\mathrm{d}v) = -\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \ln\left(1 - \frac{iv^\top z}{\beta(v)}\right)\alpha(\mathrm{d}v)$$

exists for all $z \in B_\kappa(0)$. Consider now an arbitrary $\delta \in (0,1)$ and $z \in \overline{B}_{\delta\kappa}(0)$. Then the Cauchy-Schwarz inequality implies $|iv^\top z| \leq \|z\|_2 \leq \delta\kappa$ and hence $|(iv^\top z)/\beta(v)| \leq \delta$. Therefore $\ln\left(1 - \frac{iv^\top z}{\beta(v)}\right)$ exists and is bounded on $\overline{B}_{\delta\kappa}(0)$ α-a.e. This implies that

$$-\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \ln\left(1 - \frac{iv^\top z}{\beta(v)}\right)\alpha(\mathrm{d}v)$$

exists on $\overline{B}_{\delta\kappa}(0)$. Since $\delta \in (0,1)$ was arbitrary, this concludes the proof of (i), since the analyticity follows immediately from the appendix of [12].

(ii) W.l.o.g. assume $\beta(v_n) < 1/n$. For $n \in \mathbb{N}$ set $z_n = -i\beta(v_n)v_n$. Then $\|z_n\|_2 = \beta(v_n) < 1/n$ and $1-(iv_n^\top z_n)/\beta(v_n) = 0$. Hence,

$$\int_{\{v_n\}} \ln\left(1-\frac{iv^\top z_n}{\beta(v)}\right)\alpha(\mathrm{d}v) \quad \text{and thereby} \quad \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \ln\left(1-\frac{iv^\top z_n}{\beta(v)}\right)\alpha(\mathrm{d}v)$$

do not exist. This implies that $\hat{\mu}$ is not defined on $B_{1/n}(0)$. Since $n \in \mathbb{N}$ was arbitrary, this shows (ii). ∎

Proposition 3.12. *A $\Gamma_d(\alpha,\beta)$ distribution μ has a finite moment of order $k>0$, i.e. $\int_{\mathbb{R}^d}\|x\|^k\mu(\mathrm{d}x) < \infty$, if and only if*

$$\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \beta(v)^{-k}\alpha(\mathrm{d}v) < \infty. \tag{3.13}$$

Moreover, if m is the mean vector and $\Sigma = (\sigma_{ij})_{i,j=1,\dots,d}$ is the covariance matrix of $\Gamma_d(\alpha,\beta)$

$$m = \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \beta(v)^{-1}v\alpha(\mathrm{d}v). \tag{3.14}$$

and

$$\Sigma = \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \beta(v)^{-2}vv^\top\alpha(\mathrm{d}v) \tag{3.15}$$

Proof: If β is bounded away from zero, (3.13) holds trivially and Theorem 3.11 implies that μ has finite moments of all orders $k>0$. So w.l.o.g. assume that β is not bounded away from zero in the following. By [36, p. 162] μ has a finite moment of order k, if and only if

$$\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \int_1^\infty r^k\frac{e^{-\beta(v)r}}{r}\mathrm{d}r\alpha(\mathrm{d}v) < \infty.$$

Substituting $s = r\beta(v)$ this is equivalent to

$$\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \beta(v)^{-k}\int_{\beta(v)}^\infty s^{k-1}e^{-s}\mathrm{d}s\alpha(\mathrm{d}v) < \infty. \tag{3.16}$$

Assuming without loss of generality that $\beta(v) \le 1$ for all $v \in \mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$, we have that

$$0 < C(k) := \int_1^\infty s^{k-1}e^{-s}\mathrm{d}s \le \int_{\beta(v)}^\infty s^{k-1}e^{-s}\mathrm{d}s \le \Gamma(k).$$

Hence, (3.16) is equivalent to (3.13). Finally, (3.14) and (3.15) follow from Example 25.12 in [36] and observing that that the infinitely divisible distribution $\Gamma_d(\alpha,\beta)$ with Fourier transform (3.1) has Lévy triplet $(\zeta, 0, \nu_\mu)$, where $\zeta = \int_{\|x\|\le 1} x\nu_\mu(\mathrm{d}x)$. ∎

Corollary 3.13. *A $\|\cdot\|$-homogeneous $\Gamma_d(\alpha,\beta)$ distribution has an analytic Fourier-Laplace transform in $B_\beta(0)$ and finite moments of all orders.*

Hence, any homogeneous Gamma distribution behaves like one would expect it from the univariate case. However, the behaviour in the non-homogeneous case may be drastically different, as the following examples illustrate.

Example 3.14. *Consider $d = 2$. Let α be concentrated on $\{v_n\}_{n\in\mathbb{N}}$ with*

$$v_n = (\sin(n^{-1}), \cos(n^{-1}))$$

and set $\alpha(\{v_n\}) = e^{-n}$ and $\beta(v_n) = 1/n$ for all $n \in \mathbb{N}$. Then by Theorem 3.11 (ii) the Fourier-Laplace transform exists in no neighbourhood of zero.

$\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \beta(v)^{-k}\alpha(\mathrm{d}v) = \sum_{n\in\mathbb{N}} n^k e^{-n}$ *is finite for all $k > 0$ using the quotient criterion, because*

$$\lim_{n\to\infty} \frac{(n+1)^k e^{-(n+1)}}{n^k e^{-n}} = e^{-1} < 1.$$

Thus, we have moments of all orders, but the Fourier-Laplace transform exists in no complex neighbourhood of zero.

Example 3.15. *Consider the set-up of Example 3.14, but set now $\alpha(\{v_n\}) = 1/n^{1+m}$ for some real $m > 0$. $\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \beta(v)^{-k}\alpha(\mathrm{d}v) = \sum_{n\in\mathbb{N}} \frac{n^k}{n^{1+m}}$ is finite if and only if $k < m$.*

It is easy to see that condition (3.5) *is satisfied if condition* (3.13) *holds for some $k > 0$. Hence, the $\Gamma_2(\alpha,\beta)$ distribution exists indeed, but only moments of orders smaller than m are finite.*

Example 3.16. *Consider again the set-up of Example 3.14. Set now $\alpha(\{v_n\}) = (\ln(1+n)^3(n+1))^{-1}$.*

Then $\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \ln\left(1+\frac{1}{\beta(v)}\right)\alpha(\mathrm{d}v) = \sum_{n\in\mathbb{N}} \frac{1}{\ln(1+n)^2(1+n)} < \infty$ (see [32, Theorem 3.29] and thus the $\Gamma_2(\alpha,\beta)$ distribution is well-defined.

Yet, $\int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|_2}} \beta(v)^{-k}\alpha(\mathrm{d}v) = \sum_{n\in\mathbb{N}} \frac{n^k}{\ln(1+n)^3(1+n)} = \infty$ for all real $k > 0$ and so the $\Gamma_2(\alpha,\beta)$ distribution has no finite moments of positive orders at all.

4. Gamma and Generalised Gamma Convolutions on Cones

4.1. Cone-valued infinitely divisible random elements

We first review several facts about infinitely divisible elements with values in a cone of a finite dimensional Euclidean space B with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. A nonempty convex set K of B is said to be a *cone* if $\lambda \geq 0$ and $x \in K$ imply $\lambda x \in K$. A cone is *proper* if $x = 0$ whenever x and $-x$ are in K. The *dual cone* K' of K is defined as $K' = \{y \in B' : \langle y, s\rangle \geq 0 \text{ for every } s \in K\}$. A proper cone K induces a partial order on B by defining $x_1 \leq_K x_2$ whenever $x_2 - x_1 \in K$ for $x_1 \in B$ and $x_2 \in B$. Examples of proper cones are $\mathbb{R}_+$, $\mathbb{R}^d_+ = [0,\infty)^d$, $\mathbb{S}^+_d$ and $\overline{\mathbb{S}}^+_d$.

A random element X in K is *infinitely divisible (ID)* if and only if for each integer $p \geq 1$ there exist p independent identically distributed random elements $X_1,...,X_p$ in K such that $X \overset{law}{=} X_1 + ... + X_p$. A probability measure μ on K is ID if it is the distribution of an ID element in K. It is known (see [38]) that such a distribution μ is concentrated on a cone K if and only if its Laplace transform $L_\mu(\Theta) = \int_K \exp(-\langle\Theta, x\rangle)\mu(\mathrm{d}x)$ is given by the *regular Lévy-Khintchine representation*

$$L_\mu(\Theta) = \exp\left\{-\langle\Theta,\Psi_0\rangle - \int_K \left(1-e^{-\langle\Theta,x\rangle}\right)\nu_\mu(\mathrm{d}x)\right\} \quad \text{for all } \Theta \in K', \tag{4.1}$$

where $\Psi_0 \in K$ and the Lévy measure is such that $\nu_\mu(K^c) = 0$ and

$$\int_K (\|x\| \wedge 1)\,\nu_\mu(\mathrm{d}x) < \infty. \tag{4.2}$$

If $X = \{X(t); t \geq 0\}$ is the K-increasing Lévy process (K-valued subordinator) associated to μ, its Lévy-Itô decomposition is of the form

$$\begin{aligned} X(t) &= t\Psi_0 + \int_0^t \int_K x N(\mathrm{d}t, \mathrm{d}x) \\ &= t\Psi_0 + \sum_{s \leq t} \Delta X(s) \quad a.s., \end{aligned} \tag{4.3}$$

where $\Delta X(s) \in K$ for all $s \geq 0$ a.s. and $N(dt, \mathrm{d}x)$ is a Poisson random measure on $\mathbb{R}_+ \times K$ with

$$\mathbb{E}\{N(\mathrm{d}t, \mathrm{d}x)\} = \nu_\mu(\mathrm{d}x)\mathrm{d}t. \tag{4.4}$$

4.2. Cone-valued Gamma distributions

Let $\mathbf{S}_{\|\cdot\|}$ be the unit sphere of B with respect to the norm $\|\cdot\|$ and let K be a proper cone of B. We write $\mathbf{S}_{\|\cdot\|}^K = \mathbf{S}_{\|\cdot\|} \cap K$ and denote by $\mathscr{B}(\mathbf{S}_{\|\cdot\|}^K)$ the Borel sets of $\mathbf{S}_{\|\cdot\|}^K$,

Similar to Definition 3.1 we have Gamma distributions in the cone K.

Definition 4.1. *Let μ be an infinitely divisible distribution on the cone K. If there exist a finite measure α on $\mathbf{S}_{\|\cdot\|}^K$, and a measurable function $\beta : \mathbf{S}_{\|\cdot\|}^K \to \mathbb{R}_+$ such that*

$$\mathrm{L}_\mu(\Theta) = \exp\left\{ -\int_{\mathbf{S}_{\|\cdot\|}^K} \int_0^\infty \left(1 - e^{-r\langle \Theta, U\rangle}\right) \frac{e^{-\beta(U)r}}{r} \mathrm{d}r \alpha(\mathrm{d}U) \right\} \tag{4.5}$$

for all $\Theta \in K'$, then μ is called a K-Gamma distribution with parameters α and β, and we write $\mu \sim \Gamma_K(\alpha, \beta)$. The Lévy measure ν_μ of μ is

$$\nu_\mu(E) = \int_{\mathbf{S}_{\|\cdot\|}^K} \int_0^\infty 1_E(rU) \frac{e^{-\beta(U)r}}{r} \mathrm{d}r \alpha(\mathrm{d}U), \quad E \in \mathscr{B}(K), \tag{4.6}$$

and satisfies

$$\int_K \min(1, \|x\|) \nu_\mu(\mathrm{d}x) < \infty. \tag{4.7}$$

The expression (4.6) is equivalent to

$$\nu_\mu(\mathrm{d}X) = \frac{e^{-\beta(X/\|X\|)\|X\|}}{\|X\|} 1_K(X) \widetilde{\alpha}(\mathrm{d}X), \tag{4.8}$$

where $\widetilde{\alpha}$ is a measure on K given by

$$\widetilde{\alpha}(E) = \int_{\mathbf{S}_{\|\cdot\|}^K} \int_0^\infty 1_E(rU) \mathrm{d}r \alpha(\mathrm{d}U), \quad E \in \mathscr{B}(K). \tag{4.9}$$

All properties of the multivariate Gamma distribution in Section 3 are also true for the cone-valued Gamma distribution. As in Proposition 3.3 we can in particular show that there exists a $\Gamma_K(\alpha, \beta)$ probability measure μ if and only if

$$\int_{\mathbf{S}_{\|\cdot\|}^K} \ln\left(1 + \frac{1}{\beta(U)}\right) \alpha(\mathrm{d}U) < \infty, \tag{4.10}$$

in which case we have (4.7). Also, as for Proposition 3.10 the Laplace transform of a $\Gamma_K(\alpha,\beta)$ probability measure μ is also given by

$$L_\mu(\Theta)=\exp\left\{-\int_{\mathbf{S}^K_{\|\cdot\|}}\ln\left(1+\frac{\langle\Theta,U\rangle)}{\beta(U)}\right)\alpha(\mathrm{d}U)\right\},\quad \Theta\in K'. \tag{4.11}$$

If $\|\cdot\|_b$ is another norm on K and if μ has distribution $\Gamma_K(\alpha,\beta)$, then μ has distribution $\Gamma_K(\alpha_b,\beta_b)$ where α_b,β_b are given as in (3.8) and (3.9) respectively. Also, $\Gamma_K(\alpha,\beta)$, has a finite moment of order $k>0$, if and only if

$$\int_{\mathbf{S}^K_{\|\cdot\|}}\beta(U)^{-k}\alpha(\mathrm{d}U)<\infty. \tag{4.12}$$

In the homogeneous case, i.e. $\beta(U)=\beta_0>0$ for any $U\in\mathbf{S}^K_{\|\cdot\|}$, we have $\mathbb{E}\|M\|^k<\infty$ for any $k>0$.

If M is a random element in K with distribution $\Gamma_K(\alpha,\beta)$ and (4.12) is satisfied with $k=1$,

$$\mathbb{E}(M)=\int_{\mathbf{S}^K_{\|\cdot\|}}\beta(U)^{-1}U\alpha(\mathrm{d}U). \tag{4.13}$$

4.3. Itô-Wiener-Gamma integrals

In this section we formulate an Itô-Wiener-Gamma integral for K-valued Gamma process, similar to the Itô-Wiener-Gamma integral (2.6) with respect to the one-dimensional Gamma process.

Let $\gamma=\gamma(\alpha,\beta)=(\gamma_t;t\geq 0)$ be a K-valued Gamma process. That is, γ is the K-increasing Lévy process such that $\mu_{\alpha,\beta}=\Gamma_K(\alpha,\beta)$ is the distribution of γ_1. Let $N_\gamma(\mathrm{d}s,\mathrm{d}x)$ be the random measure on $\mathbb{R}_+\times K$ associated to the K-valued jumps of γ and $\nu_{\mu_{\alpha,\beta}}$ be the Lévy measure of γ_1. Hence $\mathbb{E}\{N(\mathrm{d}t,\mathrm{d}x)\}=\nu_{\mu_{\alpha,\beta}}(\mathrm{d}x)\mathrm{d}t$, where

$$\nu_{\mu_{\alpha,\beta}}(E)=\int_{\mathbf{S}^K_{\|\cdot\|}}\int_0^\infty 1_E(rU)\frac{e^{-\beta(U)r}}{r}\mathrm{d}r\alpha(\mathrm{d}U),\quad E\in\mathscr{B}(K).$$

Let $h:\mathbb{R}_+\times\mathbf{S}^K_{\|\cdot\|}\to\mathbb{R}_+$ be a measurable function such that

$$\int_{\mathbf{S}^K_{\|\cdot\|}}\int_0^\infty\ln\left(1+\frac{h(s,U)}{\beta(U)}\right)\alpha(\mathrm{d}U)\mathrm{d}s<\infty, \tag{4.14}$$

in which case we say that h belongs to $L(\Gamma_K(\alpha,\beta))$. The last condition is the cone analogon of the one-dimensional condition (2.5).

We prove in the next proposition that the following Itô-Wiener-Gamma integral type is well defined

$$Y^h=\int_0^\infty\int_K h\left(s,\frac{x}{\|x\|}\right)xN(\mathrm{d}s,\mathrm{d}x), \tag{4.15}$$

in the framework of integration with respect to infinitely divisible independently scattered random measures (i.d.i.s.r.m.) in Rajput and Rosinski [30] (see also [9] for the special case of random matrices).

Proposition 4.2. *The integral* (4.15) *is well defined if and only if the function* $h:\mathbb{R}_+\times\mathbf{S}^K_{\|\cdot\|}\to\mathbb{R}_+$ *belongs to* $L(\Gamma_K(\alpha,\beta))$.

Moreover, $h\in L(\Gamma_K(\alpha,\beta))$ *if and only if*

$$\int_0^\infty\int_K\min(1,\|h(s,x/\|x\|)x\|)\nu_{\mu_{\alpha,\beta}}(\mathrm{d}x)\mathrm{d}s<\infty, \tag{4.16}$$

and (4.16) is equivalent to the following two conditions

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^{1/2} |\ln(z)|\, G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty \tag{4.17}$$

and

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_{1/2}^{\infty} \frac{1}{z} G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty \tag{4.18}$$

where $G_U(\mathrm{d}z)$ is the measure on $\mathbb{R}_+$ which is the image of the Lebesgue measure on $\mathbb{R}_+$ under the change of variable $s \to \beta(U)/h(s,U)$.

Proof: By [9, 30] the existence of the integral is equivalent to (4.16).

Since $\beta(U) < 0$ *a.e.* U, Fubini's theorem and elementary computations give

$$\begin{aligned}
I &= \int_0^\infty \int_K \min(1, \|h(s, x/\|x\|)x\|)\nu_{\mu_{\alpha,\beta}}(\mathrm{d}x)\mathrm{d}s \qquad (4.19)\\
&= \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \int_0^\infty \min(1, \|h(s,U)rU\|) \frac{e^{-r\beta(U)}}{r} \mathrm{d}r\alpha(\mathrm{d}U)\mathrm{d}s\\
&= \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \int_0^{1/h(s,U)} h(s,U)e^{-r\beta(U)}\mathrm{d}r\alpha(\mathrm{d}U)\\
&+ \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \int_{1/h(s,U)}^\infty \frac{e^{-r\beta(U)}}{r}\mathrm{d}r\alpha(\mathrm{d}U)\\
&\overset{z=r\beta(U)}{=} \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \frac{h(s,U)}{\beta(U)} \left(1 - e^{-r\beta(U)/h(s,U)}\right) \mathrm{d}s\alpha(\mathrm{d}U)\\
&+ \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty E_1\left(\frac{h(s,U)}{\beta(U)}\right) \mathrm{d}s\alpha(\mathrm{d}U) \qquad (4.20)
\end{aligned}$$

where E_1 is the exponential integral function as in the proof of Proposition 3.3.

Using the change of variable $z = \beta(U)/h(s,U)$ we have

$$\begin{aligned}
I &= \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \frac{1-e^{-z}}{z} G_U(\mathrm{d}z)\alpha(\mathrm{d}U)\\
&+ \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty E_1(z)\, G_U(\mathrm{d}z)\mathrm{d}s\alpha(\mathrm{d}U) = I_1 + I_2 \qquad (4.21)
\end{aligned}$$

We shall in the following show that $I < \infty$ if and only if

$$I_3 = \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \ln\left(1 + \frac{1}{z}\right) G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty \tag{4.22}$$

if and only if (4.17) and (4.18) are satisfied. This concludes, as obviously (4.22) and $h \in L(\Gamma_K(\alpha,\beta))$ are equivalent.

First,

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_{1/2}^\infty \frac{1-e^{-z}}{z} G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty \tag{4.23}$$

if and only if (4.18), if and only if

$$I_4 = \int_{\mathbf{S}^K_{\|\cdot\|}} \int_{1/2}^{\infty} \ln\left(1+\frac{1}{z}\right) G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty$$

since $(1/z)/\ln(1+z^{-1}) \to 1$ as $z \to \infty$.

On the other hand,

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_{0}^{1/2} E_1(z)\, G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty$$

if and only if (4.17) holds (since $E_1(z)/(-\ln(z)) \to 1$ as $z \to 0$) if and only if

$$I_5 = \int_{\mathbf{S}^K_{\|\cdot\|}} \int_{0}^{1/2} \ln\left(1+\frac{1}{z}\right) G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty$$

because $\ln(1+z^{-1})/(-\ln(z)) \to 1$ as $z \to 0$.

$I_5 < \infty$ and (4.17) both imply

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_{0}^{1/2} G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty.$$

Thus

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_{0}^{1/2} \frac{1-e^{-z}}{z} G_U(\mathrm{d}z)\alpha(\mathrm{d}U) \le \int_{\mathbf{S}^K_{\|\cdot\|}} \int_{0}^{1/2} G_U(\mathrm{d}z)\alpha(\mathrm{d}U) < \infty,$$

provided $I_5 < \infty$ or (4.17) hold.

Since $0 \le E_1(z) \le e^{-z}\ln(1+1/z)\,\forall z \in \mathbb{R}_+$ (see [1, p. 229]), using (4.23) implies

$$\begin{aligned}
&\int_{\mathbf{S}^K_{\|\cdot\|}} \int_{1/2}^{\infty} E_1(z)\, G_U(\mathrm{d}z)\mathrm{d}s\alpha(\mathrm{d}U) \\
&\le \int_{\mathbf{S}^K_{\|\cdot\|}} \int_{1/2}^{\infty} e^{-z}\ln\left(1+\frac{1}{z}\right) G_U(\mathrm{d}z)\alpha(\mathrm{d}U) \le e^{-1/2} I_4.
\end{aligned}$$

■

Proposition 4.3. *Let $h \in L(\Gamma_K(\alpha,\beta))$. Then the distribution of the K-valued random variable Y^h is infinitely divisible and has Laplace transform*

$$\mathrm{L}_{Y^h}(\Theta) = \exp\left(-\int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^{\infty} \int_0^{\infty} \left(1-e^{-rh(t,U)\langle\Theta,U\rangle}\right) \frac{e^{-\beta(U)r}}{r} \mathrm{d}t\mathrm{d}r\alpha(\mathrm{d}U)\right) \tag{4.24}$$

$$= \exp(-\int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^{\infty} \ln\left(1+\frac{\langle\Theta,U\rangle}{z}\right) G_U(\mathrm{d}z)\alpha(\mathrm{d}U). \tag{4.25}$$

where for α-a.e. U, G_U is a Thorin measure measure on $\mathbb{R}_+$ which is the image of Lebesgue measure on $\mathbb{R}_+$ under the change of variable $s \to \beta(U)/h(s,U)$. Moreover, the Lévy measure of Y^h is

$$\nu_{Y^h}(E) = \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^{\infty} 1_E(rU)\frac{k_U(r)}{r}\mathrm{d}r\alpha(\mathrm{d}U), \quad E \in \mathscr{B}(K) \tag{4.26}$$

where

$$k_U(r) = \int_0^{\infty} e^{-rz} G_U(\mathrm{d}z). \tag{4.27}$$

Proof: Using the obvious analogue for the Laplace transform of the formulae for the characteristic functions of the integrals with respect to i.d.i.s.r.m.s in [9, 30] we obtain

$$L_{Y^h}(\Theta) = \exp\left(-\int_K \int_0^\infty \left(1 - e^{-\left\langle \Theta, h(s, \frac{x}{\|x\|})x \right\rangle}\right) \nu_{\mu_{\alpha,\beta}}(\mathrm{d}x)\mathrm{d}s\right)$$
$$= \exp\left(\int_0^\infty \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \left(1 - e^{-rh(s,U)\langle \Theta, U\rangle}\right) \frac{e^{-r\beta(U)}}{r} \mathrm{d}r\alpha(\mathrm{d}U)\mathrm{d}s\right).$$

As in the last proposition, let $G_U(\mathrm{d}z)$ be the *measure* on $\mathbb{R}_+$ which is the image of the Lebesgue measure on $\mathbb{R}_+$ under the change of variable $s \to \beta(U)/h(s,U) = z(U)$. Then

$$L_{Y^h}(\Theta) = \exp\left(\int_0^\infty \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \left(1 - e^{-r\langle \Theta, U\rangle}\right) \frac{e^{-r\beta(U)/h(s,U)}}{r} \mathrm{d}r\alpha(\mathrm{d}U)\mathrm{d}s\right)$$
$$= \exp\left(\int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \int_0^\infty \left(1 - e^{-r\langle \Theta, U\rangle}\right) \frac{e^{-rz}}{r} G_U(dz)\mathrm{d}r\alpha(\mathrm{d}U)\right)$$
$$= \exp\left(\int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \left(1 - e^{-r\langle \Theta, U\rangle}\right) \frac{k_U(r)}{r} \mathrm{d}r\alpha(\mathrm{d}U)\right).$$

Hence, combining (2.2) with the existence conditions for the integral, $G_U(\mathrm{d}t)$ is a Thorin measure on $\mathbb{R}_+$ for α-*a.e.* U. ■

4.4. Characterisation of Cone Valued GGC

In this section we define Generalized Gamma Convolutions *GGC*(K) in the cone K and characterize this class as the distributions of the K-valued random elements represented by the stochastic integral (4.15). The result is an extension to the cone valued case of the Wiener-Gamma integral representation of one-dimensional generalised Gamma convolutions, see Section 1.2 in [16].

Similar to the multivariate case (see [3]), we define *GGC*(K) as follows

Definition 4.4. *The class $GGC(K)$ is the collection of all infinitely divisible distributions on K with Lévy measure ν_μ having a polar decomposition*

$$\nu_\mu(E) = \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty 1_E(rU) \frac{k_U(r)}{r} \mathrm{d}r\alpha(\mathrm{d}U), \quad E \in \mathscr{B}(K), \tag{4.28}$$

where $k_U(r)$ is a measurable function in U and completely monotone in r for α-a.e. U.

A probabilistic interpretation of the class $GGC(K)$ is provided by Theorem 4.6 below.

Proposition 4.3 says that the class of distributions of the Wiener-Gamma integrals Y_h with $h \in L(\Gamma_K(\alpha,\beta))$ are *GGC*(K).

We now prove that all distributions in *GGC*(K) have a Wiener-Gamma integral representation. For simplicity we consider the case without drift, that is $\Psi_0 = 0$ in (4.1) and (4.3). Otherwise

$$Y^h = \Psi_0 + \int_0^\infty \int_K h\left(s, \frac{x}{\|x\|}\right) xN(\mathrm{d}s, \mathrm{d}x).$$

Theorem 4.5 (Wiener-Gamma characterization of $GGC(K)$**).** *For any fixed Borel-measurable function* $\beta : \mathbf{S}^K_{\|\cdot\|} \to \mathbb{R}_+$ *bounded away from zero it holds that*

$$\left\{ Y^h = \int_0^\infty \int_K h(s, x/\|x\|) x N(\mathrm{d}s, \mathrm{d}x) : \ \alpha \text{ a finite measure on } \mathscr{B}(\mathbf{S}^K_{\|\cdot\|}), h \in L(\Gamma_K(\alpha, \beta)) \right\}$$
$$= GGC_0(K)$$

with $GGC_0(K)$ *denoting all generalized Gamma convolutions on K without drift.*

The condition on β above is needed to ensure the existence of the Gamma random variables for all finite measures α. The result implies that starting with any fixed homogeneous (or non-homogeneous with β bounded away from 0) Gamma distribution one can obtain all generalized Gamma convolutions as the sum of a constant and a Wiener-Itô integral with respect to the jump measure obtained from this fixed distribution.

Proof: Let $\mu \in GGC_0(K)$ with Lévy measure given by (4.28). Since $k_U(r)$ is completely monotone in r for α-a.e. U, there exists a Radon measure G_U such that $k_U(r) = \int_0^\infty e^{-rz} G_U(\mathrm{d}z)$. Moreover

$$\int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \min(1, r) \frac{k_U(r)}{r} \mathrm{d}r \alpha(\mathrm{d}U) < \infty. \tag{4.29}$$

Let $F_{G_U}(x) = \int_0^x G_U(\mathrm{d}z)$ for $x \geq 0$ and $F^{-1}_{G_U}(s)$ be the right continuous generalised inverse of $F_{G_U}(s)$. Let $\tilde{h}(s, U) = 1/F^{-1}_{G_U}(s)$ and $h(s, U) = \beta(U)\tilde{h}(s, U)$ for $s \geq 0$. It follows as in the one dimensional case that $G_U\mu$, α-a.e. U, is a Thorin measure which is the image of Lebesgue measure on $(0, \infty)$ under the change of variable $s \to 1/h(s, U)$. That is,

$$\int_0^\infty e^{-\frac{\beta(U)x}{h(s,U)}} \mathrm{d}s = \int_0^\infty e^{-\frac{x}{\tilde{h}(s,U)}} \mathrm{d}s = \int_0^\infty e^{-xz} G_U(\mathrm{d}z), \quad x > 0$$

and

$$\begin{aligned} I &= \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \int_0^\infty \frac{\min(1, r)}{r} e^{-rz} G_U(\mathrm{d}z) \mathrm{d}r \alpha(\mathrm{d}U) \\ &= \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty \int_0^\infty \frac{h(s, U)}{\beta(U)} \left(1 - e^{-\beta(U)r/h(s,U)}\right) dr \mathrm{d}s \alpha(\mathrm{d}U) \\ &+ \int_{\mathbf{S}^K_{\|\cdot\|}} \int_0^\infty E_1 \left(\frac{h(s, U)}{\beta(U)}\right) \mathrm{d}s \alpha(\mathrm{d}U). \end{aligned}$$

Thus (4.29), Proposition 4.2 and (4.20) imply $h \in L(\Gamma_K(\alpha, \beta))$. Let $N(\mathrm{d}s, \mathrm{d}x)$ be the Poisson random measure associated to $\Gamma_K(\alpha, \beta)$ and $Y^h = \int_0^\infty \int_K h(s, x/\|x\|) x N(\mathrm{d}s, \mathrm{d}x)$. Then Proposition 4.3 shows that μ is the distribution of Y^h which concludes the proof. ■

We also have another characterization of $GGC(K)$, similar to a characterization of multivariate GGC proved in [3, Theorem F]. This gives another probabilistic interpretation of $GGC(K)$.

We call XV an elementary Gamma variable in K if X is a non-random non-zero vector in K and V is a non-negative real random variable with one-dimensional Gamma distribution $\Gamma(\alpha, \beta)$.

Theorem 4.6. *GGC(K) is the smallest class of distributions on K closed under convolution and weak convergence and containing the distributions of all elementary Gamma variables in K.*

Proof: The proof is along the same lines to that of Theorem F in [3, p. 27]. $\blacksquare$

This implies also that GGC(K) is the smallest class of distributions closed under convolution and weak convergence containing all K-valued Γ distributions in the sense of Definition 4.1. It is trivial to see that GGC(K) includes all K-valued stable distribution using the spectral representation of stable Lévy measures and maps h of the form $h(s,U) = \{s\theta(U)\Gamma(\alpha+1)\}^{-\frac{1}{\alpha}}$ with $0 < \alpha < 1$.

5. Positive Definite Matrix Gamma Distributions

In this section we consider the important case of non-negative definite Gamma random matrices. This corresponds to the closed cone $K = \overline{\mathbb{S}}_d^+$ of symmetric non-negative definite $d \times d$ matrices with inner product $\langle X, Y \rangle = \mathrm{tr}(X^\top Y)$, $X, Y \in \overline{\mathbb{S}}_d^+$. When X is in the open cone $\mathbb{S}_d^+$, we write $X > 0$. When dealing with random matrices, a useful matrix norm is the *trace norm* defined for $X \in \mathbb{M}_d(\mathbb{R})$ as $\|X\| = \mathrm{tr}(\left(X^\top X\right)^{1/2})$. We write $\mathbf{S}^+_{\|\cdot\|} = \mathbf{S}^{\overline{\mathbb{S}}_d^+}_{\|\cdot\|} = \mathbf{S}_{\|\cdot\|} \cap \overline{\mathbb{S}}_d^+$. For $X \in \overline{\mathbb{S}}_d^+$, $\|X\| = \mathrm{tr}(X)$ and, in particular, if $U \in \mathbf{S}^+_{\|\cdot\|}$, $\mathrm{tr}(U) = \|U\| = 1$. By Proposition 3.4 it is not important which norm we use. So we choose the one most convenient to work with.

5.1. General Case

The matrix Gamma distribution $\mu \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\alpha, \beta)$ on $\overline{\mathbb{S}}_d^+$ has the Laplace transform

$$L_\mu(\Theta) = \exp\left\{ -\int_{\mathbf{S}^+_{\|\cdot\|}} \int_0^\infty \left(1 - e^{-r\mathrm{tr}(\Theta U)}\right) \frac{e^{-\beta(U)r}}{r} \mathrm{d}r \alpha(\mathrm{d}U) \right\}, \forall \Theta \in \overline{\mathbb{S}}_d^+ \tag{5.1}$$

with alternative representation

$$L_\mu(\Theta) = \exp\left\{ -\int_{\mathbf{S}^+_{\|\cdot\|}} \ln\left(1 + \frac{\mathrm{tr}(U\Theta)}{\beta(U)}\right) \alpha(\mathrm{d}U) \right\}, \forall \Theta \in \overline{\mathbb{S}}_d^+. \tag{5.2}$$

Additional properties of Gamma random matrices to those for the general cone valued case in Section 4.2 are the following.

If M is a symmetric random matrix with Gamma distribution $\Gamma_{\overline{\mathbb{S}}_d^+}(\alpha, \beta)$, $\mathrm{tr}(M)$ follows a one-dimensional Gamma convolution law. However, in the homogeneous case $\beta(U) = \beta_0 > 0$, $\mathrm{tr}(M)$ has a one-dimensional Gamma distribution $\Gamma(\alpha(\mathbf{S}^+_{\|\cdot\|}), \beta_0)$.

Proposition 5.1. *a) If $M \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\alpha, \beta)$, $\mathrm{tr}(M)$ has a one-dimensional GGC law with Laplace transform*

$$\mathbb{E}e^{-\theta \mathrm{tr}(M)} = \exp\left\{ -\int_0^\infty \ln\left(1 + \frac{\theta}{s}\right) \upsilon_{\alpha,\beta}(\mathrm{d}s) \right\} \tag{5.3}$$

where $\upsilon_{\alpha,\beta}$ is the Thorin measure on $(0, \infty)$ induced by $\alpha(\mathrm{d}U)$ on $\mathbf{S}^+_{\|\cdot\|}$ under the transformation $U \to \beta(U)$.

b) If $M \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\alpha, \beta)$ with $\beta(U) = \beta_0$, then $\mathrm{tr}(M)$ has the one-dimensional Gamma distribution $\Gamma(\alpha(\mathbf{S}^+_{\|\cdot\|}), \beta_0)$.

Proof: For $\theta > 0$, let $\Theta = \theta \mathrm{I}_d$. Since $\mathbb{E}e^{-\theta \mathrm{tr}(M)} = L_\mu(\Theta)$, from (4.11)

$$\mathbb{E}e^{-\theta \mathrm{tr}(M)} = \exp\left\{ -\int_{\mathbf{S}^+_{\|\cdot\|}} \ln\left(1 + \frac{\theta \mathrm{tr}(U)}{\beta(U)}\right) \alpha(\mathrm{d}U) \right\}$$

$$= \exp\left\{ -\int_0^\infty \ln\left(1+\frac{\theta}{s}\right) \upsilon_{\alpha,\beta}(\mathrm{d}s) \right\}$$

where $\upsilon_{\alpha,\beta}$ is the measure on $(0,\infty)$ induced by $\alpha(\mathrm{d}U)$ on $\mathbf{S}^+_{\|\cdot\|}$ under the transformation $U \to \beta(U)$. Then, using (2.1) we obtain (a). For (b) we observe that from the first equality in the last expression with $\beta(U) = \beta_0$, we obtain $\mathbb{E}e^{-\theta \mathrm{tr}(M)} = (1+\theta/\beta_0)^{-\alpha(\mathbf{S}^+_{\|\cdot\|})}$. ∎

Any matrix Gamma distribution $\Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$ is self-decomposable and if $\mathrm{supp}(\alpha)$ is of full dimension, it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}_d$ (which can be identified with $\mathbb{R}^{d(d+1)/2}$) and so there is a density. The proof follows from the multivariate case, identifying the cone $\mathbb{S}_d^+$ with $\mathbb{R}^{d(d+1)/2}$. Moreover, since the Lebesgue measure of $\overline{\mathbb{S}}_d^+ \backslash \mathbb{S}_d^+$ is zero, the distribution $\Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$ is supported in the the open cone $X > 0$. In other words

Corollary 5.2. *Let M be a random matrix with Gamma distribution $\Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$ with* $\mathrm{supp}(\alpha)$ *of full dimension. Then $\mathbb{P}(M>0) = 1$.*

The following result is an adaptation of Proposition 3.8 to special linear operators preserving the cone $\mathbb{S}_d^+$. Observe that all invertible surjective linear operators preserving $\mathbb{S}_d^+$ are of the form $X \mapsto CXC^T$ with some $C \in GL_d(\mathbb{R})$ (see [20, 21]).

Proposition 5.3. *Let $M \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$ with respect to the trace norm $\|\cdot\|$ and let $C \in \mathcal{GL}_d(\mathbb{R})$. Then $Y = CMC^\top \sim \Gamma_{\mathbf{S}^+_{\|\cdot\|_C}}(\alpha_c,\beta_c)$, where $\mathbf{S}^+_{\|\cdot\|_C} = \mathbf{S}_{\|\cdot\|_c} \cap \overline{\mathbb{S}}_d^+$ for $\|B\|_C = \left\|C^{-1}BC^{-\top}\right\|$ and*

$$\alpha_C(E) = \alpha(C^{-1}EC^{-\top}), \quad \forall E \in \mathscr{B}(\mathbf{S}^+_{\|\cdot\|_C}), \tag{5.4}$$

and

$$\beta_C(U) = \beta(C^{-1}EC^{-\top}), \quad \forall U \in \mathbf{S}^+_{\|\cdot\|_C}. \tag{5.5}$$

Moreover, $Y \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\widehat{\alpha}_C,\widehat{\beta}_C)$ with respect to the trace norm $\|\cdot\|$ where

$$\widehat{\alpha}_C(E) = \int_{\mathbf{S}_d^+(C)} 1_E\left(\frac{U}{\|U\|}\right)\alpha_C(\mathrm{d}U), \quad \forall E \in \mathscr{B}(\mathbf{S}^+_{\|\cdot\|}) \tag{5.6}$$

and

$$\widehat{\beta}_C(U) = \beta_C\left(\frac{U}{\|U\|_C}\right)\|U\|_C, \quad \forall U \in \mathbf{S}^+_{\|\cdot\|}. \tag{5.7}$$

Example 5.4. *(Diagonal matrix with independent entries). As pointed out in [5], an infinitely divisible non-negative definite random matrix M has independent components if and only if it is diagonal and therefore its Lévy measure is concentrated in the diagonal matrix axes $E^{ii} \in \overline{\mathbb{S}}_d^+$, $i = 1,\ldots d$. Thus, a Gamma random matrix $M \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$ has independent components, if and only if there exist non-negative numbers $\beta_1,\ldots,\beta_d$ such that the Lévy measure ν_X is given by*

$$\nu_M(E) = \sum_{i=1}^d \alpha(E_{ii}) \int_0^\infty 1_E(rU)\frac{e^{-r\beta_i}}{r}\mathrm{d}r \quad E \in \mathscr{B}(\overline{\mathbb{S}}_d^+).$$

Further examples are considered in the next section.

5.2. The $A\Gamma$-distribution

We now introduce a special matrix distribution $A\Gamma_d(\eta,\Sigma)$ in the open cone $\mathbb{S}_d^+$ with parameters, $\eta > (d-1)/2$ and $\Sigma \in \mathbb{S}_d^+$. We study several properties including a relation between cumulants of $A\Gamma(\eta,\Sigma)$ and the moments of a Wishart distribution.

The multivariate Gamma function, denoted by $\Gamma_d(\eta)$, is defined for $\mathrm{Re}(\eta) > (d-1)/2$ as

$$\Gamma_d(\eta) = \int_{X>0} e^{-\mathrm{tr(X)}} |X|^{\eta-(d+1)/2} \mathrm{d}X, \tag{5.8}$$

where $\mathrm{d}X$ is the Lebesgue measure on $\mathbb{S}_d^+$ (identified with $\mathbb{R}^{d(d+1)/2}$); see for example [24, p. 61]. An alternative expression for $\Gamma_d(\eta)$ is ([24, Theorem 2.1.12])

$$\Gamma_d(\eta) = \pi^{d(d-1)/4}\prod_{i=1}^{d}\Gamma\left(\eta-\frac{1}{2}(i-1)\right), \quad \mathrm{Re}(\eta) > (d-1)/2. \tag{5.9}$$

The special infinitely divisible matrix Gamma distribution $A\Gamma_d(\eta,\Sigma)$ is defined as follows. For $\eta > (d-1)/2$, consider the measure $\rho_\eta(\mathrm{d}X) = g_\eta(X)\mathrm{d}X$ on the open cone $\mathbb{S}_d^+$ where

$$g_\eta(X) = c_{d,\eta}\frac{e^{-\mathrm{tr}(X)}}{(\mathrm{tr}(X))^{\eta d}} |X|^{\eta-(d+1)/2}, \quad X>0, \tag{5.10}$$

and

$$c_{d,\eta} = \omega_{d,\eta}\frac{\Gamma(\eta d)}{\Gamma_d(\eta)} \tag{5.11}$$

and $\omega_{d,\eta} > 0$ is given.

Proposition 5.5. *Let $\eta > (d-1)/2$. Then there exists a homogeneous Gamma matrix distribution $\Gamma_{\mathbb{S}_d^+}(\alpha_\eta,\beta)$ with respect to the trace norm where $\beta(U) = 1$ for each $U \in \mathbf{S}_{\|\cdot\|}^+$ and α_η is the measure on $\mathbf{S}_{\|\cdot\|}^+$ given by*

$$\alpha_\eta(\mathrm{d}U) = c_{d,\eta}|U|^\eta \frac{\mathrm{d}U}{|U|^{(d+1)/2}} \tag{5.12}$$

with $\alpha_\eta(\mathbf{S}_{\|\cdot\|}^+) = \omega_{d,\eta}$. Moreover, the Lévy measure of $\Gamma_{\mathbb{S}_d^+}(\alpha_\eta,\beta)$ is ρ_η and has a polar decomposition

$$\rho_\eta(E) = \int_E g_\eta(X)\mathrm{d}X = c_{d,\eta}\int_{\mathbf{S}_{\|\cdot\|}^+}\int_0^\infty 1_E(rU)\frac{e^{-r}}{r}\mathrm{d}r\frac{|U|^\eta\,\mathrm{d}U}{|U|^{(d+1)/2}}, \quad E \in \mathscr{B}(\overline{\mathbb{S}}_d^+). \tag{5.13}$$

Proof: To show existence of the matrix distribution $\Gamma_{\mathbb{S}_d^+}(\alpha_\eta,\beta)$, by Proposition 3.3 it suffices to prove that α_η is a finite measure, since trivially β satisfies (4.12). The fact that $\Gamma_{\mathbb{S}_d^+}(\alpha_\eta,\beta)$ is concentrated in the open cone $\mathbb{S}_d^+$ will follow by Corollary 5.2 since from (5.12) $\mathrm{supp}(\alpha_\eta)$ has full dimension.

For $X>0$ make the change of variable

$$X = rU, r = \mathrm{tr}(X),\ \mathrm{tr}(U) = 1, \mathrm{d}X = r^{d(\mathrm{d}+1)/2-1}\mathrm{d}r\mathrm{d}U \tag{5.14}$$

([23, p. 111]). Using this in (5.8) and the fact that $|rU| = r^d|U|$

$$\begin{aligned}\Gamma_d(\eta) &= \int_0^\infty\int_{\mathbf{S}_{\|\cdot\|}^+} r^{d\eta-1}e^{-r}\mathrm{d}r|U|^\eta\frac{\mathrm{d}U}{|U|^{(d+1)/2}} \\ &= \Gamma(\eta d)\int_{\mathbf{S}_{\|\cdot\|}^+}|U|^\eta\frac{\mathrm{d}U}{|U|^{(d+1)/2}}\end{aligned} \tag{5.15}$$

and hence $\alpha_\eta(\mathbf{S}_{\|\cdot\|}^+) = \omega_{d,\eta}$. Using again the change of variable (5.14) we have (5.13). ■

Definition 5.6. *Let $\eta > (d-1)/2$ and $\Sigma \in \mathbb{S}_d^+$. An infinitely divisible $p \times p$ positive definite random matrix M is said to follow the distribution $A\Gamma_d(\eta,\Sigma)$ if it has Gamma distribution $\Gamma_{\mathbb{S}_d^+}(\alpha_{\eta,\Sigma},\beta_\Sigma)$ with respect to the trace norm where $\beta_\Sigma(U) = \mathrm{tr}(\Sigma^{-1}U)$ and*

$$\alpha_{\eta,\Sigma}(\mathrm{d}U) = \frac{1}{|\Sigma|^\eta \,\mathrm{tr}(\Sigma^{-1}U)^{\eta d}}\alpha_\eta(\mathrm{d}U) \tag{5.16}$$

and α_η is given by (5.12).

Remark 5.7. *a) The distribution $A\Gamma_d(\eta,\Sigma)$ has also as a parameter the total mass $\alpha_\eta(\mathbf{S}^+_{\|\cdot\|}) = \omega_{d,\eta}$. This parameter is conjectured to be of particular importance when considering the limiting spectral (eigenvalue) distribution as the dimension goes to infinity, since it may then depend on η or d. Particularly, interesting choices of $\omega_{d,\eta}$ in this connection should be $d\eta$, d or a constant.*

b) The case $\eta = (d+1)/2$ was considered in Barndorff-Nielsen and Pérez-Abreu [5].

c) Note that for $\eta \in ((d-1)/2,(d+1)/2)$ the Lévy density becomes infinity at the non-invertible elements of $\mathbb{S}_d^+$ (i.e. the matrices which are positive semi-definite, but not strictly), whereas for $\eta > (d+1)/2$ the Lévy density becomes zero at the non-invertible elements of $\mathbb{S}_d^+$. For $\eta = (d+1)/2$ we have that α_{η,I_d} is the uniform measure on the unit sphere.

d) The Fourier-Laplace of the distribution $A\Gamma_d(\eta,\mathrm{I}_d)$ is

$$L_\mu(\Theta) = \exp\left\{-c_{d,\eta}\int_{\mathbf{S}^+_{\|\cdot\|}} \ln\left(1+\frac{\mathrm{tr}(U\Theta)}{\beta(U)}\right)|U|^\eta \frac{\mathrm{d}U}{|U|^{(d+1)/2}}\right\}, \forall\Theta \in \mathbb{S}_d^+.$$

e) This is an absolutely continuous Lévy measure. For similar alternatives with continuous singular Lévy measures, see the $B\Gamma_d$ distributions with $q < d$ in the next section.

f) $\eta > (d-1)/2$ is a necessary and sufficient condition to define a proper Lévy measure. The sufficiency is shown above. For the necessity it is enough to note that the poles of the Lévy density different from zero need to be of an order lower than one in order to give a valid Lévy measure.

Note that if $M \sim A\Gamma(\eta,\mathrm{I}_d)$, then $\Sigma^{1/2}M\Sigma^{1/2} \sim A\Gamma_d(\eta,\Sigma)$. This follows from Proposition 5.3 which also gives together with Proposition 5.5, that $A\Gamma_d(\eta,\Sigma)$ has Lévy measure $\rho_{\eta,\Sigma}(\mathrm{d}X) = g_{\eta,\Sigma}(X)\mathrm{d}X$ where

$$g_{\eta,\Sigma}(X) = \frac{c_{d,\eta}}{|\Sigma|^\eta}\frac{e^{-\mathrm{tr}(\Sigma^{-1}X)}}{[\mathrm{tr}(\Sigma^{-1}X)]^{\eta d}}|X|^{\eta-(d+1)/2}, \quad X > 0. \tag{5.17}$$

The existence of moments of all orders of $A\Gamma_d(\eta,\mathrm{I}_d)$ follows since (4.12) is trivially satisfied. The same is true for $A\Gamma_d(\eta,\Sigma)$ since $\Sigma^{1/2}M\Sigma^{1/2} \sim A\Gamma_d(\eta,\Sigma)$.

In the homogeneous case the distribution $A\Gamma_d(\eta,\sigma\mathrm{I}_d)$ with $\sigma \in \mathbb{R}^+$ is invariant under orthogonal conjugations and the trace follows a one-dimensional Gamma distribution.

Lemma 5.8. *Let $\eta > (d-1)/2$ and $\sigma > 0$.*

a) The distribution $A\Gamma_d(\eta,\sigma\mathrm{I}_d)$ is invariant under orthogonal conjugations.

b) If $M \sim A\Gamma_d(\eta,\sigma\mathrm{I}_d)$, then $\mathrm{tr}(M)$ follows a one-dimensional Gamma distribution $\Gamma(\omega_{d,\eta},\sigma)$.

Proof: It is well known that the measure $\mathrm{d}X/|X|^{(d+1)/2}$ is invariant under the conjugation $X \to CXC^\top$, for $X > 0$ and any non-singular matrix C (see [13, Example 6.19]). The determinant and the trace norm functions are invariant under the conjugation $X \to OXO^\top$, for $X > 0$ and any $O \in \mathscr{O}(\mathrm{d})$. Thus the Lévy measure $\rho_{\eta,\sigma\mathrm{I}_d}$ with density (5.17), $\Sigma = \sigma\mathrm{I}_d$, is invariant under orthogonal conjugations and so the matrix distribution $A\Gamma(\eta,\sigma\mathrm{I}_d)$ is. Proposition 5.1(b) gives (b). ■

The cumulants of the distribution $A\Gamma_d(\eta,\sigma I_d)$ are related to the moments of the Wishart distribution, as we prove below. Recall that a $d\times d$ positive definite random matrix W is said to have a Wishart distribution $W_d(n,d)$ with parameters $n>d-1$ and $\Sigma\in\mathbb{S}_d^+$, if its density function is given by

$$f_W(A)=\frac{1}{2^{\frac{dn}{2}}\Gamma_d(\frac{n}{2})\,|\Sigma|^{\frac{n}{2}}}e^{-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}A)}\,|A|^{(n-d-1)/2},\ A>0. \tag{5.18}$$

As usual we denote by $A\otimes B$ the tensor product of the matrices A and B. We recall that if A and B are in $\mathbb{M}_n$, then $\mathrm{tr}(A\otimes B)=\mathrm{tr}(A)\mathrm{tr}(B)$ and $|A\otimes B|=|A|^n\,|B|^n$.

We use the notation $B(n,m)=\Gamma(n)\Gamma(m)/\Gamma(n+m)$ for the Beta function, with $\mathrm{Re}(n)>0,\mathrm{Re}(m)>0$, and for the multivariate Beta function, denoted by $B_d(x,y)$,

$$\mathrm{B}_d(n,m)=\frac{\Gamma_d(n)\Gamma_d(m)}{\Gamma_d(n+m)},\quad \mathrm{Re}(n)>0,\mathrm{Re}(m)>0. \tag{5.19}$$

Proposition 5.9. *Let $\eta>(d-1)/2$ and $g_{\eta,\Sigma}(X)$ be the Lévy density of $A\Gamma_d(\eta,\Sigma)$. Let W be a random matrix with Wishart distribution $W_d(2\eta,\Sigma)$. For any integer $p>0$ the following three identities hold:*

a)

$$\int_{X>0}X^p g_{\eta,\Sigma}(X)\mathrm{d}X=\frac{\omega_{d,\eta}}{2^p}\mathrm{B}(\eta d,p)\mathbb{E}W^p. \tag{5.20}$$

b)

$$\int_{X>0}X^{\otimes p}g_{\eta,\Sigma}(X)\mathrm{d}X=\frac{\omega_{d,\eta}}{2^p}\mathrm{B}(\eta d,p)\mathbb{E}(W^{\otimes p}). \tag{5.21}$$

c)

$$\int_{X>0}|X|^p\,g_{\eta,\Sigma}(X)\mathrm{d}X=\frac{\omega_{d,\eta}}{2^{pd}}\mathrm{B}(\eta d,pd)\mathbb{E}(|W|^p) \tag{5.22}$$

$$=\omega_{d,\eta}\Gamma_d(p)\frac{\mathrm{B}(\eta d,pd)}{\mathrm{B}_d(\eta,p)}\,|\Sigma|^p. \tag{5.23}$$

Proof: The existence of the integrals (5.20)-(5.22) is seen as follows. The finiteness of the p-th moment of $A\Gamma_d(\eta,\Sigma)$ is equivalent to the existence of the p-th moment of its Lévy measure (away from the origin). Then $\int_{X>0}\|X\|^p\,g_{\eta,\Sigma}(X)\mathrm{d}X<\infty$ for any $p>0$ gives the existence of the integral in (5.20). Since for $X>0$ $\mathrm{tr}(X^{\otimes p})=(\mathrm{tr}(X))^p=\|X\|^p$ and $|X|\le\|X\|^d$, one also obtains the existence of the integrals in (5.21) and (5.22), respectively.

The identities in (a)-(c) are consequences of a more general result for q-homogeneous functions which we now prove: Let $h:\mathbb{S}_d^+\to H$ be a function such that $h(rX)=r^q h(X)$ for any $r>0$ and $X\in\mathbb{S}_d^+$ and some fixed $q>0$, where H is $\mathbb{S}_d^+$, $\left(\mathbb{S}_d^+\right)^{\otimes p}$ and $(0,\infty)$ for (a),(b) and (c) respectively. In the cases (a) and (b) we have $q=p$ while for (c) $q=dp$.

The change of variable $V=\Sigma^{-1/2}X\Sigma^{-1/2}$, the invariance of the measure $|X|^{-(d+1)/2}\mathrm{d}X$ under non-singular linear transformations and writing $h_\Sigma(Y)=h(\Sigma^{1/2}Y\Sigma^{1/2})$ give:

$$J=\int_{X>0}h(X)g_{\eta,\Sigma}(X)\mathrm{d}X=c_{d,\eta}\int_{X>0}h(X)\frac{c_{d,\eta}}{|\Sigma|^\eta}\frac{e^{-\mathrm{tr}(\Sigma^{-1}X)}}{(\mathrm{tr}(\Sigma^{-1}X))^{\eta d}}\,|X|^{\eta-(d+1)/2}\mathrm{d}X$$

$$=c_{d,\eta}\int_{V>0}h_\Sigma(V)\frac{e^{-\mathrm{tr}(V)}}{(\mathrm{tr}(V))^{\eta d}}\,|V|^{\eta-(d+1)/2}\mathrm{d}V.$$

Using (5.14), the definition of the Gamma function, first for $\Gamma(q)$ and then for $\Gamma(\eta d+q)$, and (5.11) give

$$
\begin{aligned}
J &= c_{d,\eta}\int_0^\infty \int_{\mathbf{S}^+_{\|\cdot\|}} r^{q-1} h_\Sigma(U) e^{-r} \mathrm{d}r\, |U|^\eta \frac{\mathrm{d}U}{|U|^{(d+1)/2}} \\
&= c_{d,\eta}\Gamma(q) \int_{\mathbf{S}^+_{\|\cdot\|}} h_\Sigma(U)\, |U|^\eta \frac{\mathrm{d}U}{|U|^{(d+1)/2}} \\
&= \frac{c_{d,\eta}\Gamma(q)}{\Gamma(\eta d+q)} \int_{\mathbf{S}^+_{\|\cdot\|}} \int_0^\infty r^{\eta d+q-1} h_\Sigma(U) e^{-r} \mathrm{d}r\, |U|^\eta \frac{\mathrm{d}U}{|U|^{(d+1)/2}} \\
&= \frac{\omega_{d,\eta}\mathrm{B}(\eta d,q)}{\Gamma_d(\eta)} \int_{V>0} h_\Sigma(V) e^{-\mathrm{tr}(V)}\, |V|^{\eta-(d+1)/2}\, \mathrm{d}V && (5.24)\\
&= \frac{\omega_{d,\eta}\mathrm{B}(\eta d,q)}{2^{\eta d}\Gamma_d(\eta)} \int_A h_\Sigma\left(\frac{1}{2}A\right) e^{-\frac{1}{2}\mathrm{tr}(A)}\, |A|^\eta \frac{\mathrm{d}A}{|A|^{(d+1)/2}} && (5.25)\\
&= \omega_{d,\eta}\mathrm{B}(\eta d,q) \int_A h\left(\frac{1}{2}A\right) f_W(A)\mathrm{d}A = \omega_{d,\eta}\mathrm{B}(\eta d,q)\left(\frac{1}{2}\right)^q \mathbb{E}(h(W)), && (5.26)
\end{aligned}
$$

where in (5.24) we used again (5.14), in (5.25) the change of variable $A = V/2$ (with $\mathrm{d}V = (1/2)^{(d+1)/2}\mathrm{d}A$), and for (5.26) the fact that f_W is the density (5.18) of the random matrix W with Wishart distribution $W_d(2\eta,\Sigma)$.

Then (a), (b) and the first equality in (5.22) are proved. The second equality in (5.22) follows using the fact that $\mathbb{E}\,|W^p| = |\Sigma|^p\, 2^{dp}\Gamma_d(\eta+p)/\Gamma_d(\eta)$ (see Muirhead [24, p. 101]). ■

In particular, the mean $\mathbb{E}(M)$ and covariance matrix $\mathrm{Cov}(M) = \mathbb{E}(M\otimes M) - \mathbb{E}(M)\otimes\mathbb{E}(M)$ of $M \sim A\Gamma_d(\eta,\Sigma)$ are expressed in terms of the mean $\mathbb{E}(W)$ and the second tensor moment $\mathbb{E}(W)\otimes\mathbb{E}(W)$ of the Wishart distribution $W_d(2\eta,\Sigma)$. Recall that the commutation $d^2\times d^2$ matrix K is defined as

$$K = \sum_{i,j=1}^{d} H_{ij}\otimes H_{ij}^\top$$

where H_{ij} denotes the $d\times d$ matrix with $h_{ij}=1$ and all other elements zero. The m-th moments and cumulants of a $d\times d$ random matrix are d^{2m}-dimensional objects which need to be represented in a concise and at the same time easy to handle way. As usual for random matrices we define the moments and cumulants using the tensor product, e.g. the m-th moment of a random matrix X is understood to be $E(X^{\otimes m})$. An alternative would be to use the vec-operator to transfer the matrix into an element of $\mathbb{R}^{d^2}$ first, but typically this leads to formulae that are more cumbersome to handle. Now we have:

Corollary 5.10. *The cumulants of the random matrix $M \sim A\Gamma_d(\eta,\Sigma)$ are proportional to the tensor moments of the Wishart distribution. In particular*

$$\mathbb{E}(M) = \frac{\omega_{d,\eta}}{d}\Sigma \tag{5.27}$$

and the matrix of covariances between elements of M is given by

$$\mathrm{Cov}(M) = \omega_{d,\eta}\frac{\eta}{d(nd+1)}\left(\left(1+\frac{1}{2\eta}\right)\mathrm{I}_{d^2}+K\right)(\Sigma\otimes\Sigma). \tag{5.28}$$

Proof: The first assertion follows from (b) in Proposition 5.9. Since the first moment of M equals its first cumulant and its matrix of covariances equals its second cumulant, then (5.27) follows from (a) in Proposition 5.9 with $p = 1$ and since $\mathbb{E}(W) = 2\eta\Sigma$ for $W \sim W_d(2\eta, \Sigma)$. From [24, p. 90] we have

$$\mathrm{Cov}(W) = 2\eta(\mathrm{I}_{d^2} + K)(\Sigma \otimes \Sigma).$$

Using (b) in Proposition 5.9 with $p = 2$ we have

$$\begin{aligned}\mathrm{Cov}(M) &= \int_{X>0} X^{\otimes 2} g_{\eta,\Sigma}(X)\mathrm{d}X = \frac{\omega_{d,\eta}}{4}\frac{1}{(nd+1)\eta d}\mathbb{E}(W^{\otimes 2}) \\ &= \frac{\omega_{d,\eta}}{4}\frac{1}{(nd+1)\eta d}\left[\mathrm{Cov}(W) + [\mathbb{E}(W)]^{\otimes 2}\right].\end{aligned}$$

Hence (5.28) follows. ■

In particular, when $\omega_{d,\eta} = d\eta$, $\mathbb{E}(M) = \eta\Sigma$, as in the Wishart case. On the other hand, when $\omega_{d,\eta} = d$, $\mathbb{E}(M) = \Sigma$.

This result is of particular importance in applications, since it implies that the second order moment structure is explicitly known which may allow method of moments based estimation of models using $A\Gamma_d(\eta, \Sigma)$ matrix subordinators as the stochastic input (e.g. [29])

The following result states an interesting relation with the so-called Marchenko-Pastur distribution of parameter $\lambda > 0$. Recall that the moments of this distribution are given by (see [2])

$$\mu_p(\lambda) = \sum_{j=0}^{p-1} \frac{1}{j+1}\binom{p}{j}\binom{p-1}{j}(\lambda)^j. \tag{5.29}$$

Lemma 5.11. *Let $\varepsilon \in \mathbb{R}$. For any integer $p > 0$, as $d \to \infty$ and $d/\eta \to \lambda > 0$*

$$\frac{1}{d}\int_{X>0} \mathrm{tr}\left(\frac{X}{d^\varepsilon}\right)^p g_{\eta,\mathrm{I}_d}(X)\mathrm{d}X \sim K_p(\lambda)\omega_{d,\eta} d^{-p(1+\varepsilon)} \tag{5.30}$$

where $K_p(\lambda) = \Gamma(p)\mu_p(2\lambda)$. In particular, for $\varepsilon = 1$

$$\lim_{d\to\infty} \frac{1}{d}\int_{X>0} \left[\mathrm{tr}\left(\frac{X}{d}\right)^p\right] g_{\eta,\mathrm{I}_d}(X)\mathrm{d}X = \begin{cases}\lambda, \text{ if } p = 1 \\ 0, \text{ if } p \geq 2\end{cases}. \tag{5.31}$$

Proof: If $W \sim W_d(2\eta, \mathrm{I}_d)$, then

$$\lim_{d\to\infty} \frac{1}{d}\mathbb{E}\mathrm{tr}\left(\frac{W}{2\eta}\right)^p = \mu_p(2\lambda). \tag{5.32}$$

as $2\eta/d \to 2\lambda$ due to the well known Marchenko-Pastur Theorem [2], for any $p > 0$,

By the Stirling approximation $\Gamma(z+1) \sim \sqrt{2\pi z}(z/e)^z$ for $z \to \infty$, for η and d large

$$\frac{\Gamma(\eta d)}{\Gamma(\eta d + p)} \sim (nd)^{-p}. \tag{5.33}$$

Using $\eta/d \to \lambda$ and (5.32) in (5.20) gives:

$$\frac{1}{d}\int_{X>0} \mathrm{tr}\left(\frac{X}{d^\varepsilon}\right)^p g_{\eta,\mathrm{I}_d}(X)\mathrm{d}X = \frac{\omega_{d,\eta}}{d^{\varepsilon p+1}2^p} B(\eta d, p)\mathbb{E}\mathrm{tr}(W^p)$$

$$
\begin{aligned}
&= \frac{\omega_{d,\eta}}{d^{\varepsilon p}} \frac{\Gamma(\eta d)\Gamma(p)}{\Gamma(\eta d + p)} \frac{1}{d} \mathbb{E}\mathrm{tr}(\frac{W^p}{2}) \\
&\sim \Gamma(p)\omega_{d,\eta}(\eta d)^{-p} d^{-p\varepsilon} \left[\frac{1}{d} \mathbb{E}\mathrm{tr}\left(\frac{W}{2}\right)^p \right] \\
&\sim \Gamma(p)\omega_{d,\eta} d^{-p-p\varepsilon} \left[\frac{1}{d} \mathbb{E}\mathrm{tr}\left(\frac{W}{2}\right)^p \right] && (5.34) \\
&\sim K_p(\lambda)\omega_{d,\eta} d^{-p(1+\varepsilon)}, \text{ for } \eta, d \text{ large}, && (5.35)
\end{aligned}
$$

which proves the lemma. ■

Conjecture 5.12. *We conjecture that the above Lemma is a first step to study the asymptotic spectral distribution of the random matrix $M \sim A\Gamma_d(\eta,\Sigma)$. More specifically, the right hand side of (5.31) must be related to the pth-cumulant of the pth-moment of the mean spectral distribution of M, which in turn should allow the identification of the limiting spectral distribution.*

5.3. Further examples

5.3.1. $B\Gamma$-distributions

Let $d \geq 1$ and $q = 1, ..., d$ be fixed. Consider the Lévy measure on $\overline{\mathbb{S}}_d^+$ given by

$$
\nu_q(\mathrm{d}X) = \frac{e^{-\beta_0\|X\|}}{\|X\|} \widetilde{\alpha}_{d,q}(\mathrm{d}X), \quad X \in \overline{\mathbb{S}}_d^+ \backslash \{0\} \tag{5.36}
$$

where $\beta_0 > 0$ and

$$
\widetilde{\alpha}_{d,q}(E) = \int_{\mathbf{S}^+_{\|\cdot\|}} \int_0^\infty 1_E(rU) \mathrm{d}r \alpha_{d,q}(\mathrm{d}U). \tag{5.37}
$$

Here $\alpha_{d,q}/d$ is the probability measure on the sphere $\mathbf{S}^+_{\|\cdot\|}$ induced by the transformation $V \to U = VV^\top$, where the $d \times q$ matrix V is uniformly distributed on the unit sphere of the linear space $\mathbb{M}_{d\times q}(\mathbb{R})$ of $d \times q$ matrices with real entries, with the Frobenius norm $\|Y\|_2^2 = \mathrm{tr}(Y^\top Y)$.

An infinitely divisible $d \times d$ symmetric random matrix M with Lévy measure ν_q has the Gamma distribution $\Gamma_{\mathbb{S}_d^+}(\alpha_{d,q}, \beta_0)$, since ν_q has a polar decomposition

$$
\nu_q(E) = \int_{\mathbf{S}^+_{\|\cdot\|}} \int_0^\infty 1_E(rU) \frac{e^{-r}}{r} \mathrm{d}r \alpha_{d,q}(\mathrm{d}U), \quad E \in \mathscr{B}(\overline{\mathbb{S}}_d^+).
$$

We call this distribution the $B\Gamma_d(q, \beta_0)$ distribution.

Remark 5.13. *a) We observe that the support of ν_q is concentrated in matrices of rank q in $\overline{\mathbb{S}}_d^+$. Hence this support is of full dimension. Then, by Corollary 5.2, $\Gamma_{\mathbb{S}_d^+}(\alpha_{d,q}, \beta_0)$ has support in the open cone $\mathbb{S}_d^+$.*

b) The case $q = 1$ was considered in Pérez-Abreu and Sakuma [27] in the context of random matrix models for free generalised Gamma convolutions. They considered the Hermitian case for which working in the setup of $\mathbb{M}_{d\times q}(\mathbb{C})$ is needed, but otherwise the above steps can be carried out in a straightforward way.

c) For $\Sigma \in \mathbb{S}_d^+$ one can consider invertible linear transformations of $\Gamma_{\mathbb{S}_d^+}(\alpha_{d,q}, \beta_0)$ to obtain infinitely divisible positive definite matrix Gamma distributions $\Gamma_{\mathbb{S}_d^+}(\alpha_{\eta,\Sigma}, \beta_\Sigma)$ with Lévy measures of the form

$$
\nu_q(\mathrm{d}X) = \frac{e^{-\|\Sigma^{-1}X\|}}{\|\Sigma^{-1}X\|} \widetilde{\alpha}_{d,q}(\Sigma^{-1/2}\mathrm{d}X\Sigma^{-1/2})
$$

similar to the family Γ *of matrix Gamma distributions considered in the last section.*

The following properties are easily proved.

Proposition 5.14. *Let* $M \sim \Gamma_{\overline{\mathbb{S}}_d^+}(\alpha_{d,q},\beta_0)$ *and* $q = 1,...,d$ *be fixed. Then,*
a) M *has an invariant distribution under orthogonal conjugations.*
b) $\mathbb{E}\,\|M\|^k < \infty$ *for any* $k > 0$.
d) $\mathrm{tr}(M)$ *has a one-dimensional Gamma distribution* $\Gamma(d,\beta_0)$.

5.3.2. Matrix Gamma-Normal distribution

In the one-dimensional case, the so called variance gamma distribution is popular in applications in finance, see [22]. This distribution is a mixture of Gaussians having a random variance following the one-dimensional Gamma distribution. As an application of the matrix Gamma distribution, we now present a matrix extension of the one-dimensional variance Gamma distribution.

Let Z be a $d\times q$ random matrix with independent standard Gaussian distributed entries, i.e.

$$\mathbb{E}\exp(i\mathrm{tr}(\Theta^\top Z)) = \exp\left(-\frac{1}{2}\mathrm{tr}(\Theta^\top\Theta)\right), \quad \forall\Theta\in\mathbb{M}_{d\times q}(\mathbb{R}).$$

Let X be a random matrix with the Gamma distribution $\Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$ and independent of Z. Consider the *random linear transformation* $Y = X^{1/2}Z$. Using a standard conditional argument we compute the characteristic function of the $d\times q$ matrix as follows:

$$\begin{aligned}\mathbb{E}\exp(i\mathrm{tr}(\Theta^\top Y)) &= \mathbb{E}_X\mathbb{E}_Z\left[\exp(i\mathrm{tr}(\Theta^\top X^{1/2}Z))\Big| X\right]\\ &= \mathbb{E}_X\left\{\exp(-\frac{1}{2}\mathrm{tr}(X^{1/2}\Theta\Theta^\top X^{1/2}))\right\}\\ &= \mathbb{E}_X\left\{\exp(-\frac{1}{2}\mathrm{tr}(\Theta\Theta^\top X))\right\}.\end{aligned}$$

Then, using (4.11)

$$\mathbb{E}\exp(i\mathrm{tr}(\Theta^\top Y)) = \exp\left\{-\int_{\mathbf{S}^+_{\|\cdot\|}}\ln\left(1+\frac{1}{2}\frac{\mathrm{tr}(U\Theta\Theta^\top)}{\beta(U)}\right)\alpha(\mathrm{d}U)\right\}, \tag{5.38}$$

for each $\Theta\in\mathbb{M}_{d\times q}(\mathbb{R})$. Using the terminology in [6], we can say that Y has a Mat$G$ distribution, which is infinitely divisible in $\mathbb{M}_{d\times q}(\mathbb{R})$.

Similar to the one-dimensional case, we call this distribution the matrix Gamma-Normal distribution with parameters α and β or more specifically the $d\times q$-dimensional matrix $\Gamma_{\overline{\mathbb{S}}_d^+}(\alpha,\beta)$-Normal distribution. We observe that Y has a symmetric distribution in the sense that $-Y \overset{law}{=} Y$ and also that Y has a distribution invariant under orthogonal conjugations if $\beta(U)=\beta_0$ and $\alpha(\mathrm{d}U)$ is invariant under orthogonal conjugations.

Remark 5.15. *If* $\beta(U)=\beta_0>0$ *and* $q=d$, $\mathrm{tr}(Y)$ *has a one-dimensional variance Gamma distribution with the following characteristic function: for* $\theta\in\mathbb{R}$, $\Theta=\theta\mathrm{I}_d$,

$$\mathbb{E}\exp(i\theta\mathrm{tr}(Y)) = \exp\left\{-\int_{\mathbf{S}^+_{\|\cdot\|}}\ln\left(1+\frac{1}{2\beta_0}\theta^2\right)\alpha(\mathrm{d}U)\right\}$$

$$= \left(1 + \frac{1}{2\beta_0}\theta^2\right)^{-\alpha(\mathbf{S}^+_{\|\cdot\|})}$$
$$= \left(1 - i\frac{1}{\sqrt{2\beta_0}}\theta\right)^{-\alpha(\mathbf{S}^+_{\|\cdot\|})} \left(1 + i\frac{1}{\sqrt{2\beta_0}}\theta\right)^{-\alpha(\mathbf{S}^+_{\|\cdot\|})}..$$

Thus, $\mathrm{tr}(Y)$ *has the same distribution as* $V - V'$*, where* V *has a one-dimensional Gamma distribution* $\Gamma(\alpha(\mathbf{S}^+_{\|\cdot\|}), \sqrt{2\beta_0})$ *and* V' *is an independent copy of* V.

Remark 5.16. *Let* $q = 1$*, i.e. the resulting matrix Gamma-normal distribution is* $\mathbb{R}^d$*-valued. Thus for* $\Theta \in \mathbb{R}^d$*,* $\Theta\Theta^\top$ *has rank one.*

Assume additionally that the measure α *is concentrated on the rank one matrices, that is* $U = \mathrm{u}\mathrm{u}^\top$ *with* $\mathrm{u} \in \mathbb{R}^d$ *(and the first non-zero component of* u *being positive, to make* u *unique given* U*). Let* $\widetilde{\alpha}$ *be the measure on the unit sphere* $\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}$ *of* $\mathbb{R}^d$ *induced by* α *under this transformation. Using this we write the integral in the right hand side of (5.38) as follows*

$$\begin{aligned}
&\int_{\mathbf{S}^+_{\|\cdot\|}} \ln\left(1 + \frac{1}{2}\frac{\mathrm{tr}(U\Theta\Theta^\top)}{\beta(U)}\right)\alpha(\mathrm{d}U) \\
&= \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \ln\left(1 + \frac{1}{2}\frac{(\Theta^\top \mathrm{u})^2)}{\beta(\mathrm{u}\mathrm{u}^\top)}\right)\widetilde{\alpha}(\mathrm{d}\mathrm{u}) \\
&= \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \ln\left(1 - i\frac{\Theta^\top \mathrm{u}}{\sqrt{2\beta(\mathrm{u}\mathrm{u}^\top)}}\right)\widetilde{\alpha}(\mathrm{d}\mathrm{u}) \\
&\quad + \int_{\mathbf{S}_{\mathbb{R}^d,\|\cdot\|}} \ln\left(1 + i\frac{\Theta^\top \mathrm{u}}{\sqrt{2\beta(\mathrm{u}\mathrm{u}^\top)}}\right)\widetilde{\alpha}(\mathrm{d}\mathrm{u}).
\end{aligned} \tag{5.39}$$

Interestingly, (5.39) implies that the matrix Gamma-normal random variable can be represented (in this special case) as $X_1 - X_2$ *with* $X_1, X_2 \sim \Gamma_d(\tilde{\alpha}, \tilde{\beta})$ *being independent where* $\tilde{\beta} = \sqrt{2\beta(\mathrm{u}\mathrm{u}^\top)}$.

Hence, the matrix Gamma-normal distribution with $q = 1$*, which can indeed be regarded as a d-dimensional generalisation of the univariate variance Gamma distribution, inherits interesting properties well-known in the univariate case.*

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