TYPE A DISTRIBUTIONS:
INFINITELY DIVISIBLE DISTRIBUTIONS
RELATED TO ARCSINE DENSITY

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ABSTRACT. Two transformations \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) of Lévy measures on \( \mathbb{R}^d \) based on the arcsine density are studied and their relation to general Upsilon transformations is considered. The domains of definition of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are determined and it is shown that they have the same range. Infinitely divisible distributions on \( \mathbb{R}^d \) with Lévy measures being in the common range are called type A distributions and expressed as the law of a stochastic integral

\[
\int_0^1 \cos(2^{-1} \pi t) \, dX_t
\]

with respect to Lévy process \( \{X_t\} \). This new class includes as a proper subclass the Jurek class of distributions. It is shown that generalized type G distributions are the image of type A distributions under a mapping defined by an appropriate stochastic integral. \( \mathcal{A}_2 \) is identified as an Upsilon transformation, while \( \mathcal{A}_1 \) is shown to be not.

1. INTRODUCTION

Let \( I(\mathbb{R}^d) \) denote the class of all infinitely divisible distributions on \( \mathbb{R}^d \). For \( \mu \in I(\mathbb{R}^d) \), we use the Lévy-Khintchine representation of its characteristic function \( \hat{\mu}(z) \) given by

\[
\hat{\mu}(z) = \exp \left\{-\frac{1}{2} \langle \Sigma z, z \rangle + i \langle \gamma, z \rangle \right\}
+ \int_{\mathbb{R}^d} \left( e^{i(x,z)} - 1 \right) \frac{i(x,z)}{1 + |x|^2} \nu(dx)
\]

where \( \Sigma \) is a symmetric nonnegative-definite \( d \times d \) matrix, \( \gamma \in \mathbb{R}^d \) and \( \nu \) is a measure on \( \mathbb{R}^d \) (called the Lévy measure) satisfying \( \nu(\{0\}) = 0 \) and \( \int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(dx) < \infty \). The triplet \( (\Sigma, \nu, \gamma) \) is called the Lévy-Khintchine triplet of \( \mu \in I(\mathbb{R}^d) \). Let \( \mathfrak{M}_L(\mathbb{R}^d) \)

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denote the class of Lévy measures of \( \mu \in I(\mathbb{R}^d) \). The class of Lévy measures \( \nu \) on \( \mathbb{R}^d \) satisfying \( \nu(\{0\}) = 0 \) and \( \int_{\mathbb{R}^d} (1 \wedge |x|) \nu(dx) < \infty \) is denoted by \( \mathcal{M}_1(\mathbb{R}^d) \). Sometimes we write \( \mathcal{M}_2(\mathbb{R}^d) = \mathcal{M}_L(\mathbb{R}^d) \). A measure \( \nu \) on \( (0, \infty) \) is also called a Lévy measure if it satisfies \( \int_{(0, \infty)} (1 \wedge x^2) \nu(dx) < \infty \), and denote by \( \mathcal{M}_L((0, \infty)) \) the class of all Lévy measures on \( (0, \infty) \).

Let
\[
a(x; s) = \begin{cases} 
\pi^{-1}(s - x^2)^{-1/2}, & |x| < s^{1/2}, \\
0, & \text{otherwise},
\end{cases}
\]

which is the density of the symmetric arcsine law with parameter \( s > 0 \). For a Lévy measure \( \rho \in \mathcal{M}_L((0, \infty)) \), define
\[
\ell(x) = \int_{\mathbb{R}^d_+} a(x; s) \rho(ds), \quad x \in \mathbb{R}.
\]

In \([1]\), a distribution such that its Lévy measure is either zero or has a density \( \ell \) of the form (1.2) is called a type A distribution on \( \mathbb{R} \). It is a one-dimensional symmetric distribution. Let \( Z \) be a standard normal random variable and \( V \) a positive infinitely divisible random variable independent of \( Z \). The distribution of the one-dimensional random variable \( V^{1/2}Z \) is infinitely divisible and is called of type \( G \). It is also shown in \([1]\) that an infinitely divisible distribution \( \tilde{\mu} \) on \( \mathbb{R} \) is of type \( G \) if and only if there exists a type \( A \) distribution \( \mu \) on \( \mathbb{R} \) with the following stochastic integral mapping representation
\[
\tilde{\mu} = \mathcal{L}\left( \int_0^{1/2} \left( \log \frac{1}{t} \right)^{1/2} dX_t^{(\mu)} \right).
\]

Here and in what follows, \( \mathcal{L} \) means “the law of” and \( \{X_t^{(\mu)}\} \) means a Lévy process on \( \mathbb{R}^d \) whose distribution at time 1 is \( \mu \in I(\mathbb{R}^d) \). (In (1.3), \( d = 1 \).)

In this paper, we study more about type \( A \) (not necessarily symmetric) distributions on \( \mathbb{R}^d \). The organization of this paper is the following.

Section 2 considers two arcsine transformations \( A_1 \) and \( A_2 \) of Lévy measures on \( \mathbb{R}^d \) based on (1.1) and a reparametrization of this density of the arcsine law. It is shown that the domains of the transformations \( A_1 \) and \( A_2 \) are \( \mathcal{M}_1(\mathbb{R}^d) \) and \( \mathcal{M}_2(\mathbb{R}^d) \), respectively, but they are identical modulo some \( (p) \)-transformation. We see that both transformations are one-to-one and that they have the same range \( \mathcal{R}(A_k) \). It is shown that this range contains as a proper subclass the Jurek class \( U(\mathbb{R}^d) \) of distributions on \( \mathbb{R}^d \) studied in \([6], [9]\). \( U(\mathbb{R}^d) \) includes several known classes of multivariate distributions characterized by the radial part of their Lévy measures, such as
the Goldie-Steutel-Bondesson class $B(\mathbb{R}^d)$, the class of selfdecomposable distributions $L(\mathbb{R}^d)$ and the Thorin class $T(\mathbb{R}^d)$, see [2]. Recently, other bigger classes than the Jurek class have been discussed in the study of extension of selfdecomposability, see [7] and [14].

Section 3 deals with the class $A(\mathbb{R}^d)$ of type A distributions on $\mathbb{R}^d$ defined as those infinitely divisible distributions on $\mathbb{R}^d$ which Lévy measure $\nu$ belongs to $\mathcal{R}(A_k)$. Some probabilistic interpretations are considered and the relation to the class $G(\mathbb{R}^d)$ of generalized type G distributions on $\mathbb{R}^d$ introduced in [9] is studied. It is shown that $A(\mathbb{R}^d) = \Phi_{\cos}(I(\mathbb{R}^d))$, where $\Phi_{\cos}$ is the stochastic integral mapping

$$\Phi_{\cos}(\mu) = \mathcal{L} \left( \int_0^1 \cos(2^{-1}\pi t) dX_t(\mu) \right), \quad \mu \in I(\mathbb{R}^d).$$

It is also shown that the class of Lévy measure in $G(\mathbb{R}^d)$ is the image of the class of Lévy measures in $B(\mathbb{R}^d) \cap \mathcal{M}_1(\mathbb{R}^d)$ under $A_1$. In order to prove this, and as a result of independent interest, a new arcsine representation of completely monotone functions is first obtained. In addition, the class $G(\mathbb{R}^d)$ is described as the image of $A(\mathbb{R}^d)$ under the stochastic integral mapping (1.3), $d \geq 1$, including the multivariate and non-symmetric cases. For doing this, we first have to prove that $A_2$ is an Upsilon transformation in the sense of [4]. However, we see that, remarkably, $A_1$ is not an Upsilon transformation and it is not commuting with a specific Upsilon transformation, which is different from other cases considered so far. Finally, Section 4 contains examples of $A_1$ and $A_2$ transformations of Lévy measures where the modified Bessel function $K_0$ plays an important role.

2. Two arcsine transformations $A_1$ and $A_2$ on $\mathbb{R}^d$

2.1. Definitions and domains. Besides the arcsine density (1.1), we consider two one-sided arcsine densities with different parameters $s > 0$ and $s^2$ as follows:

$$a_1(r; s) = \begin{cases} 2\pi^{-1}(s - r^2)^{-1/2}, & 0 < r < s^{1/2}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$a_2(r; s) = \begin{cases} 2\pi^{-1}(s^2 - r^2)^{-1/2}, & 0 < r < s, \\ 0, & \text{otherwise}. \end{cases}$$

Then we consider two arcsine transformations $A_1$ and $A_2$ of measures on $\mathbb{R}^d$ based on these two one-sided arcsine densities.
\textbf{Definition 2.1.} Let \( \nu \) be a measure on \( \mathbb{R}^d \) satisfying \( \nu(\{0\}) = 0 \). Then, for \( k = 1, 2 \), define the \textit{arcsine transformation} \( A_k \) of \( \nu \) by

\[
(2.1) \quad A_k(\nu)(B) = \int_{\mathbb{R}^d \setminus \{0\}} \nu(dx) \int_0^\infty a_k(r; |x|)1_B \left( \frac{x}{|x|} \right) \, dr, \quad B \in \mathcal{B}(\mathbb{R}^d).
\]

The domain \( \mathcal{D}(A_k) \) is the class of measures \( \nu \) on \( \mathbb{R}^d \) such that \( \nu(\{0\}) = 0 \) and the right-hand side of (2.1) is a Lévy measure in \( \mathcal{M}_L(\mathbb{R}^d) \). The range is

\[
\mathcal{R}(A_k) = \{ A_k(\nu) : \nu \in \mathcal{D}(A_k) \}.
\]

We have the following result about domains of \( A_1 \) and \( A_2 \).

\textbf{Theorem 2.2.} The domains of \( A_k \) are as follows:

\[
\mathcal{D}(A_k) = \mathcal{M}^k(\mathbb{R}^d), \quad k = 1, 2.
\]

\textit{Proof.} We write \( c = 2\pi^{-1} \). First, let us show that \( \mathcal{D}(A_k) \subset \mathcal{M}^k(\mathbb{R}^d) \). Suppose that \( \nu \in \mathcal{D}(A_k) \). Write \( \tilde{\nu}_k = A_k(\nu) \). Then

\[
(2.2) \quad \tilde{\nu}_k(B) = \int_{\mathbb{R}^d \setminus \{0\}} \nu(dx) \int_0^{||x||/2} c(||x||^k - r^2)^{-1/2}1_B \left( \frac{x}{|x|} \right) \, dr.
\]

Hence, for all nonnegative measurable functions \( f \) on \( \mathbb{R}^d \setminus \{0\} \),

\[
\int_{\mathbb{R}^d \setminus \{0\}} f(x)\tilde{\nu}_k(dx) = \int_{\mathbb{R}^d \setminus \{0\}} \nu(dx) \int_0^{||x||/2} c(||x||^k - r^2)^{-1/2} f \left( \frac{x}{|x|} \right) \, dr.
\]

In particular,

\[
(2.3) \quad \int_{\mathbb{R}^d} (1 \wedge ||x||^2)\tilde{\nu}_k(dx) = \int_{\mathbb{R}^d} \nu(dx) \int_0^{||x||^2/2} (||x||^k - r^2)^{-1/2}(1 \wedge r^2) \, dr.
\]

Since \( \int (1 \wedge ||x||^2)\tilde{\nu}_k(dx) < \infty \), we see that

\[
\infty > c \int_{\mathbb{R}^d} \nu(dx) \int_0^1 (1 - u^2)^{-1/2}(1 \wedge (||x||^2u^2)) \, du
\]

\[
\geq c \int_{\mathbb{R}^d} (1 \wedge ||x||^k)\nu(dx) \int_0^1 (1 - u^2)^{-1/2} u \, du.
\]

Hence \( \nu \in \mathcal{M}^{k}(\mathbb{R}^d) \).

Next let us show that \( \mathcal{M}^{k}(\mathbb{R}^d) \subset \mathcal{D}(A_k) \). Suppose that \( \nu \in \mathcal{M}^{k}(\mathbb{R}^d) \). Let \( \tilde{\nu}_k(B) \) denote the right-hand side of (2.1). Then \( \tilde{\nu}_k \) is a measure on \( \mathbb{R}^d \) with \( \tilde{\nu}_k(\{0\}) = 0 \) and (2.2) and (2.3) hold. Hence

\[
\int_{\mathbb{R}^d} (1 \wedge ||x||^2)\tilde{\nu}_k(dx) = c \int_{\mathbb{R}^d} \nu(dx) \int_0^1 (1 - u^2)^{-1/2}(1 \wedge (||x||^2u^2)) \, du
\]
\[
\leq c \int_{\mathbb{R}^d} (1 \wedge |x|^k) \nu(dx) \int_0^1 (1 - u^2)^{-1/2} du < \infty.
\]

This shows that \( \nu \in \mathcal{D}(\mathcal{A}_k) \).

In order to study the relation between \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), we use the following transformation of measures.

**Definition 2.3.** Let \( p > 0 \). For any measure \( \rho \) on \((0, \infty)\), define a measure \( \rho^{(p)} \) on \((0, \infty)\) by

\[
\rho^{(p)}(E) = \int_{(0, \infty)} 1_E(s^p) \rho(ds), \quad E \in \mathcal{B}((0, \infty)).
\]

More generally, for any measure \( \nu \) on \(\mathbb{R}^d\) with \( \nu(\{0\}) = 0 \), define a measure \( \nu^{(p)} \) on \(\mathbb{R}^d\) by

\[
\nu^{(p)}(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B \left( \frac{|x|^p}{|x|} \right) \nu(dx), \quad B \in \mathcal{B}(\mathbb{R}^d).
\]

We call the mapping from \( \nu \) to \( \nu^{(p)} \) \( (p) \)-transformation.

The following result is the polar decomposition of a Lévy measure in \( \mathfrak{M}_L(\mathbb{R}^d) \), (see [2], [10]). Here we include the case \( \nu = 0 \). It is a basic tool to study multivariate infinitely divisible distributions.

**Proposition 2.4.** Let \( \nu \in \mathfrak{M}_L(\mathbb{R}^d) \). Then there exists a measure \( \lambda \) on the unit sphere \( \mathbb{S} = \{ \xi \in \mathbb{R}^d : |\xi| = 1 \} \) with \( 0 \leq \lambda(\mathbb{S}) \leq \infty \) and a family \( \{ \nu_\xi : \xi \in \mathbb{S} \} \) of measures on \((0, \infty)\) such that \( \nu_\xi(E) \) is measurable in \( \xi \) for each \( E \in \mathcal{B}((0, \infty)) \), \( 0 < \nu_{\xi(0, \infty)} \leq \infty \) for each \( \xi \in \mathbb{S} \), and

\[
\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d).
\]

\( \nu_\xi \) is called the radial component of \( \nu \). Here \( \lambda \) and \( \{ \nu_\xi \} \) are uniquely determined by \( \nu \) in the following sense: if \( (\lambda, \nu_\xi) \) and \( (\lambda', \nu'_\xi) \) both have the properties above, then there is a measurable function \( c(\xi) \) on \( \mathbb{S} \) such that

\[
0 < c(\xi) < \infty,
\]

\[
\lambda'(d\xi) = c(\xi) \lambda(d\xi),
\]

\[
c(\xi) \nu'_\xi(dr) = \nu_\xi(dr) \quad \text{for } \lambda\text{-a. e. } \xi.
\]

If \( \nu \in \mathfrak{M}_L(\mathbb{R}^d) \) has a polar decomposition \( (\lambda, \nu_\xi) \), then

\[
\nu^{(p)}(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty 1_B(r\xi) \nu_\xi^{(p)}(dr), \quad B \in \mathcal{B}(\mathbb{R}^d).
\]
If $\tilde{\nu} = \nu^{(p)}$, then $\nu = \tilde{\nu}^{(1/p)}$. For any nonnegative measurable function $f(x)$ on $\mathbb{R}^d$, (2.4)  
\int_{\mathbb{R}^d} f(x)\nu^{(p)}(dx) = \int_{\mathbb{R}^d \setminus \{0\}} f(|x|^{p-1}x)\nu(dx).

The two arcsine transformations are identical modulo some $(p)$-transformations.

**Proposition 2.5.** $\nu \in \mathcal{M}_2^2(\mathbb{R}^d)$ if and only if $\nu^{(2)} \in \mathcal{M}_1^1(\mathbb{R}^d)$, and in this case $A_2(\nu) = A_1(\nu^{(2)})$.

Also $\nu \in \mathcal{M}_1^1(\mathbb{R}^d)$ if and only if $\nu^{(1/2)} \in \mathcal{M}_2^2(\mathbb{R}^d)$, and in this case $A_1(\nu) = A_2(\nu^{(1/2)})$.

**Proof.** Equivalence of $\nu \in \mathcal{M}_2^2(\mathbb{R}^d)$ and $\nu^{(2)} \in \mathcal{M}_1^1(\mathbb{R}^d)$ follows from (2.4). We have  
\begin{align*}
A_1(\nu^{(2)})(B) &= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)dr \int_r^\infty 2\pi^{-1}(s-r)^{-1/2} \nu^{(2)}(ds) \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)dr \int_r^\infty 2\pi^{-1}(s^2-r^2)^{-1/2} \nu(\xi)(ds) \\
&= A_2(\nu)(B),
\end{align*}

proving the first half. The proof of the second half is similar. \qed

2.2. **One-to-one property.** We next show that the arcsine transformations $A_1$ and $A_2$ are one-to-one. In contrast to usual proofs for the one-to-one property by the use of Laplace transform, our proof here has a different flavor.

Let us first prove some lemmas. A measure $\sigma$ on $(0, \infty)$ is said to be locally finite on $(0, \infty)$ if $\sigma((b, c)) < \infty$ whenever $0 < b < c < \infty$. For a measure $\rho$ on $(0, \infty)$, define  
\[ A(\rho)(du) = \left( \int_{u,\infty} \pi^{-1/2}(s-u)^{-1/2} \rho(ds) \right) du, \]
if the integral in the right-hand side is the density of a locally finite measure on $(0, \infty)$. This is fractional integral of order 1/2.

**Lemma 2.6.** If (2.5)  
\[ \int_{(b,\infty)} s^{-1/2} \rho(ds) < \infty \quad \text{for all } b > 0, \]
then $A(\rho)$ is definable.
Proof. Let $0 < b < c < \infty$. We have
\[
\int_b^c du \int_u^\infty (s-u)^{-1/2} \rho(ds) = \int_b^\infty \rho(ds) \int_b^{c\wedge s} (s-u)^{-1/2}du \\
= \int_b^c \rho(ds) \int_b^s (s-u)^{-1/2}du + \int_c^\infty \rho(ds) \int_b^c (s-u)^{-1/2}du \\
= 2 \int_b^c (s-b)^{1/2} \rho(ds) + 2 \int_c^\infty ((s-b)^{1/2} - (s-c)^{1/2}) \rho(ds),
\]
which is finite, since $(s-b)^{1/2} - (s-c)^{1/2} \sim (c-b)s^{-1/2}$ as $s \to \infty$. \qed

Lemma 2.7. Suppose that $\mathfrak{A}(\rho)$ is definable. Then, for $\alpha > -1$ and $b > 0$,
\[
(2.6) \quad \int_{(b,\infty)} u^\alpha \mathfrak{A}(\rho)(du) \leq C_1 \int_{(b,\infty)} s^{\alpha+1/2} \rho(ds)
\]
and
\[
(2.7) \quad \int_{(0,b]} u^\alpha \mathfrak{A}(\rho)(du) \leq C_2 \left( \int_{(0,b]} s^{\alpha+1/2} \rho(ds) + \int_{(b,\infty)} s^{-1/2} \rho(ds) \right),
\]
where $C_1$ and $C_2$ are constants independent of $\rho$.

Proof. Let $c = \pi^{-1/2}$. We have
\[
\int_{(b,\infty)} u^\alpha \mathfrak{A}(\rho)(du) = c \int_b^\infty u^\alpha du \int_{(u,\infty)} (s-u)^{-1/2} \rho(ds) \\
= c \int_b^\infty \rho(ds) \int_b^s u^\alpha (s-u)^{-1/2}du
\]
and
\[
\int_b^s u^\alpha (s-u)^{-1/2}du = s^{-1/2} \int_b^s u^\alpha (1-s^{-1}u)^{-1/2}du \\
= s^{\alpha+1/2} \int_0^1 v^\alpha (1-v)^{-1/2}dv \sim s^{\alpha+1/2}B(\alpha+1,1/2), \quad s \to \infty.
\]
Hence (2.6) holds. We have
\[
\int_{(0,b]} u^\alpha \mathfrak{A}(\rho)(du) = c \int_0^b u^\alpha du \int_{(u,\infty)} (s-u)^{-1/2} \rho(ds) \\
= c \int_{(0,\infty)} \rho(ds) \int_0^{s\wedge b} u^\alpha (s-u)^{-1/2}du \\
= c \int_{(0,b]} \rho(ds) \int_0^s u^\alpha (s-u)^{-1/2}du + c \int_{(b,\infty)} \rho(ds) \int_0^b u^\alpha (s-u)^{-1/2}du.
\]
Notice that
\[
\int_0^s u^\alpha (s-u)^{-1/2}du = s^{\alpha+1/2}B(\alpha+1,1/2)
\]
and
\[
\int_0^b u^\alpha(s - u)^{-1/2} \, du = s^{-1/2} \int_0^b u^\alpha(1 - u/s)^{-1/2} \, du \\
\leq s^{-1/2} \int_0^b u^\alpha(1 - u/b)^{-1/2} \, du = s^{-1/2} b^{\alpha+1} B(\alpha + 1/2), \quad s > b.
\]
Thus (2.7) holds. □

Lemma 2.8. Suppose that
\[
\rho((b, \infty)) < \infty \quad \text{for all } b > 0.
\]
Then \(\mathbf{A}(\rho)\) and \(\mathbf{A}(\mathbf{A}(\rho))\) are definable and
\[
\mathbf{A}(\mathbf{A}(\rho))(du) = \rho((u, \infty)) \, du,
\]
which implies that \(\rho\) is determined by \(\mathbf{A}(\rho)\) under the condition (2.8).

Proof. Since (2.8) is stronger than (2.5), \(\mathbf{A}(\rho)\) is definable. Using (2.6) of Lemma 2.7, we see from Lemma 2.6 that \(\mathbf{A}(\mathbf{A}(\rho))\) is definable. Next, notice that
\[
\int_0^\infty \pi^{-1/2}(s - u)^{-1/2} \mathbf{A}(\rho)(ds) \\
= \pi^{-1} \int_0^\infty (s - u)^{-1/2} \, ds \int_{(s, \infty)} (v - s)^{-1/2} \rho(dv) \\
= \pi^{-1} \int_{(u, \infty)} \rho(dv) \int_u^v (s - u)^{-1/2} (v - s)^{-1/2} \, ds = \rho((u, \infty)),
\]
because
\[
\int_u^v (s - u)^{-1/2} (v - s)^{-1/2} \, ds = \int_0^1 s^{-1/2} (1 - s)^{-1/2} \, ds = B(1/2, 1/2) = \pi.
\]
Hence (2.9) is true. □

For the proof of the next theorem, we introduce new functions for simplicity. For any measure \(\rho\) on \((0, \infty)\) and for \(k = 1, 2\), let
\[
a_k(\rho)(r) = \int_{(0, \infty)} a_k(r; s) \rho(ds),
\]
admitting the infinite value.

Theorem 2.9. For \(k = 1, 2\), \(A_k\) is one-to-one.

Proof. Case \(k = 1\). Suppose that \(\nu, \nu' \in \mathcal{M}^1_1(\mathbb{R}^d)\) and \(A_1(\nu) = A_1(\nu')\). Let \((\lambda, \nu_\xi)\) and \((\lambda', \nu'_\xi)\) be polar decompositions of \(\nu\) and \(\nu'\), respectively. Then
\[
A_1(\nu)(B) = \int_{\Sigma} \lambda(d\xi) \int_0^\infty 1_B(r\xi) a_1(\nu_\xi)(r) \, dr,
\]
and
\[
\int_0^b u^\alpha(s - u)^{-1/2} \, du = s^{-1/2} \int_0^b u^\alpha(1 - u/s)^{-1/2} \, du \\
\leq s^{-1/2} \int_0^b u^\alpha(1 - u/b)^{-1/2} \, du = s^{-1/2} b^{\alpha+1} B(\alpha + 1/2), \quad s > b.
\]
\[ \mathcal{A}_1(\nu')(B) = \int_S \lambda'(d\xi) \int_0^\infty 1_B(r\xi) a_1(\nu'_\xi)(r)dr. \]

Hence it follows from Proposition 2.4 that there is a measurable function \( c(\xi) \) satisfying \( 0 < c(\xi) < \infty \) such that \( \lambda'(d\xi) = c(\xi)\lambda(d\xi) \) and \( a_1(\nu'_\xi)(r)dr = c(\xi)^{-1} a_1(\nu_\xi)(r)dr \) for \( \lambda \)-a.e. \( \xi \). Thus

\[
\left( \int_{r^2}^{\infty} (s - r^2)^{-1/2} \nu'_\xi(ds) \right) dr = \left( c(\xi)^{-1} \int_{r^2}^{\infty} (s - r^2)^{-1/2} \nu_\xi(ds) \right) dr.
\]

Using a new variable \( u = r^2 \), we see that

\[
\left( \int_u^{\infty} (s - u)^{-1/2} \nu'_\xi(ds) \right) du = \left( c(\xi)^{-1} \int_u^{\infty} (s - u)^{-1/2} \nu_\xi(ds) \right) du.
\]

Since \( \nu_\xi \) and \( \nu'_\xi \) satisfy (2.8), we obtain \( \nu_\xi = c(\xi)^{-1} \nu'_\xi \) for \( \lambda \)-a.e. \( \xi \) from Lemma 2.8.

It follows that \( \nu = \nu' \).

Case \( k = 2 \). Use Proposition 2.5. Then \( \mathcal{A}_2(\nu) \) equals \( \mathcal{A}_1(\nu^{(2)}) \), which determines \( \nu^{(2)} \) by Case \( k = 1 \), and \( \nu^{(2)} \) determines \( \nu = (\nu^{(2)})^{(1/2)} \).

\[ \square \]

2.3. Ranges. We will show some facts concerning the ranges of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \).

**Proposition 2.10.** The ranges of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are identical:

\[ \mathcal{R}(\mathcal{A}_1) = \mathcal{R}(\mathcal{A}_2). \]

**Proof.** This is a direct consequence of Proposition 2.5. \[ \square \]

Let us show some necessary conditions for \( \tilde{\nu} \) to belong to the range.

**Proposition 2.11.** If \( \tilde{\nu} \) is in the common range of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), then \( \tilde{\nu} \) is in \( \mathcal{M}_L(\mathbb{R}^d) \) with a polar decomposition \( (\lambda, \ell_\xi(r)dr) \) having the following properties: \( \ell_\xi(r) \) is measurable in \( (\xi, r) \) and lower semi-continuous in \( r \in (0, \infty) \), and there is \( b_\xi \in (0, \infty) \) such that \( \ell_\xi(r) > 0 \) for \( r < b_\xi \) and, if \( b_\xi < \infty \), then \( \ell_\xi(r) = 0 \) for \( r \geq b_\xi \).

**Proof.** Let \( \tilde{\nu} = \mathcal{A}_k(\nu) \) with \( \nu \in \mathcal{M}_L^k(\mathbb{R}^d) \) and \( (\lambda, \nu_\xi) \) a polar decomposition of \( \nu \). Then \( \tilde{\nu} \in \mathcal{M}_L(\mathbb{R}^d) \) with polar decomposition \( (\lambda, a_k(\nu_\xi)(r)dr) \) from the definition. Recall that

\[ a_k(\nu_\xi)(r) = 2\pi^{-1} \int_{r^2/k, \infty} (s^k - r^2)^{-1/2} \nu_\xi(ds). \]

Then our assertion is proved in the same way as Proposition 2.13 of [14]. \[ \square \]
2.4. **How big is $\mathcal{R}(A_k)$?** Several well-known and well studied classes of multivariate infinitely divisible distributions are the following. The Jurek class, the class of selfdecomposable distributions, the Goldie-Steutel-Bondesson class, the Thorin class and the class of generalized type $G$ distributions. They are characterized only by the radial component of their Lévy measures with no influence of $\Sigma$ and $\gamma$ in the Lévy-Khintchine triplet. Among them, the Jurek class is the biggest. Recently, bigger than the Jurek class have been discussed in the study of extension of selfdecomposability, (see, e.g. [7] and [13]). Then a natural question is how big $\mathcal{R}(A_k)$ is. Let $\mathcal{M}_L^U(\mathbb{R}^d)$ be the class of Lévy measures of distributions in the Jurek class. The radial component $\nu_\xi$ of $\nu \in \mathcal{M}_L^U(\mathbb{R}^d)$ satisfies that $\nu_\xi(dr) = \ell_\xi(r)dr, r > 0$, where $\ell_\xi(r)$ is measurable in $(\xi, r)$ and decreasing and right-continuous in $r > 0$. We will show below that $\mathcal{R}(A_k)$ is at least strictly bigger than $\mathcal{M}_L^U(\mathbb{R}^d)$.

**Theorem 2.12.** We have

$$\mathcal{M}_L^U(\mathbb{R}^d) \subset \mathcal{R}(A_1) = \mathcal{R}(A_2).$$

**Proof.** Let $\tilde{\nu} \in \mathcal{M}_L^U(\mathbb{R}^d)$. Equivalently, let $\tilde{\nu} \in \mathcal{M}_L(\mathbb{R}^d)$ with a polar decomposition $(\lambda, \ell_\xi(r)dr)$ such that $\ell_\xi(r)$ is measurable in $(\xi, r)$ and decreasing and right-continuous in $r > 0$. Further, we may and do assume that $\lambda$ is a probability measure and

$$\int_0^{\infty} (1 \land r^2)\ell_\xi(r)dr = c := \int_{\mathbb{R}^d} (1 \land |x|^2)\tilde{\nu}(dx).$$

Let $\rho_\xi$ be a measure on $(0, \infty)$ such that $\rho_\xi((r^2, \infty)) = \ell_\xi(r)$ for $r > 0$ and let $\eta_\xi = A(\rho_\xi)$. Lemma 2.8 says that $\eta_\xi$ is definable and

$$\rho_\xi((u, \infty)) = \int_{(u, \infty)} \pi^{-1/2}(s - u)^{-1/2}\eta_\xi(ds) \text{ for Lebesgue a.e. } u > 0.$$

Note that $\eta_\xi(E)$ is measurable in $\xi$ for each $E \in B((0, \infty))$. We have, for $B \in B(\mathbb{R}^d)$,

$$\tilde{\nu}(B) = \int_{S} \lambda(d\xi) \int_0^{\infty} 1_B(r\xi)\rho_\xi((r^2, \infty))dr$$

$$= \int_{S} \lambda(d\xi) \int_0^{\infty} 1_B(r\xi)dr \int_{(r^2, \infty)} \pi^{-1/2}(s - r^2)^{-1/2}\eta_\xi(ds)$$

$$= \int_{S} \lambda(d\xi) \int_0^{\infty} 1_B(r\xi)(\pi^{1/2}/2)a_1(\eta_\xi)(r)dr.$$

We claim that

(2.10) $$\int_{S} \lambda(d\xi) \int_0^{\infty} (1 \land u)\eta_\xi(du) < \infty.$$
This will ensure that \((\lambda, (\pi^{1/2}/2)\eta_\xi(dr))\) is a polar decomposition of some \(\nu \in \mathcal{M}_L(\mathbb{R}^d)\) and that \(\tilde{\nu} = A_1(\nu)\). First, notice that
\[
c = \int_0^\infty (1 \wedge r^2)\rho_\xi((r^2, \infty))dr = \frac{1}{2} \int_0^\infty (1 \wedge u)\rho_\xi((u, \infty))u^{-1/2}du
\]
\[
= \frac{1}{2} \int_1^1 u^{1/2}\rho_\xi((u, \infty))du + \frac{1}{2} \int_1^\infty u^{-1/2}\rho_\xi((u, \infty))du
\]
\[
\geq \frac{1}{2}\rho_\xi((1, \infty)) + \frac{1}{2} \int_1^\infty u^{-1/2}\rho_\xi((u, \infty))du.
\]
Then, use (2.6) of Lemma 2.7 with \(\alpha = 0\) to obtain
\[
\int_{(1, \infty)} \eta_\xi(du) = \int_{(1, \infty)} A(\rho_\xi)(du) \leq C_1 \int_{(1, \infty)} s^{1/2}\rho_\xi(ds)
\]
\[
= C_1\rho_\xi((1, \infty)) + \frac{C_1}{2} \int_1^\infty s^{-1/2}\rho_\xi((s, \infty))ds \leq 3cC_1.
\]
Similarly, using (2.7) of Lemma 2.7 with \(\alpha = 1\),
\[
\int_{[0,1]} u \eta_\xi(du) = \int_{[0,1]} u A(\rho_\xi)(du)
\]
\[
\leq C_2 \left( \int_{[0,1]} s^{3/2}\rho_\xi(ds) + \int_{(1, \infty)} s^{-1/2}\rho_\xi(ds) \right)
\]
\[
\leq C_2 \left( \frac{3}{2} \int_{[0,1]} s^{1/2}\rho_\xi((s,1])ds + \int_{(1, \infty)} s^{1/2}\rho_\xi(ds) \right) \leq 6cC_2.
\]
Hence (2.10) is true. It follows that \(\mathcal{M}_L^U(\mathbb{R}^d) \subset \mathcal{R}(A_1)\).

To see the inclusion is strict, let \(\delta_1\) be Dirac measure at 1 and \(\lambda\) a probability measure on \(S\). Consider \(\eta \in \mathcal{R}(A_1)\) defined by
\[
\eta(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\alpha_1(\delta_1)(r)dr
\]
\[
= \int_S \lambda(d\xi) \int_0^1 1_B(r\xi)2\pi^{-1}(1 - r^2)^{-1/2}dr.
\]
Then \(\eta \not\in \mathcal{M}_L^U(\mathbb{R}^d)\), since the radial component has density strictly increasing on \((0, 1)\).

2.5. \(A_1\) and \(A_2\) as (modified) Upsilon transformations. Barndorff-Nielsen, Rosiński and Thorbjørnsen [4] considered general Upsilon transformations, (see also [3] and [13]). Given a measure \(\tau\) on \((0, \infty)\), a transformation \(\Upsilon_\tau\) from measures on \(\mathbb{R}^d\) into \(\mathcal{M}_L(\mathbb{R}^d)\) is called an Upsilon transformation associated to \(\tau\) (or with dilation measure
Theorem 2.13. Let $k = 1, 2$. Then for $\nu \in \mathcal{M}_k^d(\mathbb{R}^d)$

$$
(2.12) \quad \mathcal{A}_k(\nu)(B) = \int_0^1 \nu^{(k/2)}(u^{-1}B)2\pi^{-1}(1 - u^2)^{-1/2}du, \quad B \in \mathcal{B}(\mathbb{R}^d).
$$

Proof. Let $(\lambda, \nu_\xi)$ be a polar decomposition of $\nu \in \mathcal{M}_k^d(\mathbb{R}^d)$. Then with $c = 2\pi^{-1}$

$$
\mathcal{A}_k(\nu)(B) = c \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)dr \int_{(r^2/s^2, \infty)} (s^k - r^2)^{-1/2}\nu_\xi(ds)
$$

$$
= c \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(ds) \int_0^{s^{k/2}} 1_B(r\xi)(s^k - r^2)^{-1/2}dr
$$

$$
= c \int_S \lambda(d\xi) \int_0^\infty \nu_\xi(ds) \int_0^1 1_B(us^{k/2}\xi)(1 - u^2)^{-1/2}du
$$

$$
= c \int_0^1 (1 - u^2)^{-1/2}du \int_S \lambda(d\xi) \int_0^\infty 1_B(us^{k/2}\xi)\nu_\xi(ds)
$$

$$
= c \int_0^1 (1 - u^2)^{-1/2}du \int_{\mathbb{R}^d} \lambda(dx) \int_0^\infty 1_B(ux)\nu^{(k/2)}(dx),
$$

which shows (2.12). \qed

Corollary 2.14. The transformation $\mathcal{A}_2$ is an Upsilon transformation with dilation measure $\tau(du) = a_1(u; 1)du$. In other words, the expression $\tilde{\nu} = \mathcal{A}_2(\nu)$ for $\nu \in \mathcal{M}_2^d(\mathbb{R}^d)$ is written as $\tilde{\nu}(B) = E[\nu(A^{-1}B)], \quad B \in \mathcal{B}(\mathbb{R}^d)$, where $A$ is a random variable with arcsine density $a_1(u; 1)$.

Remark 2.15. The mapping $\mathcal{A}_1$ is not an Upsilon transformation for any dilation measure $\tau$. This remarkable result will be proved in Section 3.6, as a byproduct of Theorem 3.7 shown in Section 3.5.
3. Type A distributions on $\mathbb{R}^d$

3.1. Definition and stochastic integral representation via arcsine transformations.

**Definition 3.1.** A probability distribution in $I(\mathbb{R}^d)$ is said to be a *type A distribution* on $\mathbb{R}^d$ if its Lévy measure $\nu$ belongs to $\mathcal{R}(\mathcal{A}_1) = \mathcal{R}(\mathcal{A}_2)$. There is no restriction on $\Sigma$ and $\gamma$ in its Lévy-Khintchine triplet. We denote by $A(\mathbb{R}^d)$ the class of all type A distributions on $\mathbb{R}^d$.

In the following, we study a probabilistic interpretation of type A distributions, since they have been defined by an analytic way in terms of their Lévy measures above. One probabilistic interpretation is a representation by stochastic integral with respect to Lévy processes. The problem is what the integrand is. We start with this section to answer this question.

Let $T \in (0, \infty)$ and let $f(t)$ be a square integrable function on $[0, T]$. Then the stochastic integral $\int_0^T f(t)dX_t^{(\mu)}$ is defined for any $\mu \in I(\mathbb{R}^d)$ and is infinitely divisible. Define the stochastic integral mapping $\Phi_f$ based on $f$ as

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^T f(t) dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d).$$

If $\mu \in I(\mathbb{R}^d)$ has the Lévy-Khintchine triplet $(\Sigma, \nu, \gamma)$, then $\tilde{\mu} = \Phi_f(\mu)$ has the Lévy-Khintchine triplet $(\tilde{\Sigma}, \tilde{\nu}, \tilde{\gamma})$ expressed as

\begin{align*}
(3.1) \quad \tilde{\Sigma} &= \int_0^T f(t)^2 \Sigma dt, \\
(3.2) \quad \tilde{\nu}(B) &= \int_0^T dt \int_{\mathbb{R}^d} 1_B(f(t)x) \nu(dx), \quad B \in \mathcal{B}(\mathbb{R}^d), \\
(3.3) \quad \tilde{\gamma} &= \int_0^T f(t)ds \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1 + |f(t)x|^2} - \frac{1}{1 + |x|^2}\right) \nu(dx)\right).
\end{align*}

(See Proposition 2.17 and Corollary 2.19 of [11] and Proposition 2.6 of [12].)

Let us characterize the class $A(\mathbb{R}^d)$ as the range of a stochastic integral mapping.

**Theorem 3.2.** Let

\begin{align*}
(3.4) \quad \Phi_{\cos}(\mu) &= \mathcal{L}\left(\int_0^1 \cos(2^{-1}\pi t) dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d).
\end{align*}

Then $\Phi_{\cos}$ is a one-to-one mapping and

\begin{align*}
(3.5) \quad A(\mathbb{R}^d) = \Phi_{\cos}(I(\mathbb{R}^d)).
\end{align*}
Let \( \mathcal{A}_2 \) and hence, by Corollary 2.14, \( \tilde{\nu}(B) = \int_0^1 \nu(u^{-1}B)2\pi^{-1}(1 - u^2)^{-1/2}du, \quad B \in \mathcal{B}(\mathbb{R}^d) \) with some \( \nu \in \mathcal{M}_L^2(\mathbb{R}^d) \). Let \( s = g(u) = \int_u^1 2\pi^{-1}(1 - v^2)^{-1/2}dv = 2\pi^{-1} \arccos(u) \) for \( 0 < u < 1 \). Then \( u = \cos(2^{-1}\pi t) \) for \( 0 < t < 1 \). Thus
\[
\tilde{\nu}(B) = -\int_0^1 dg(u) \int_{\mathbb{R}^d} 1_B(ux)\nu(dx) = \int_0^1 dt \int_{\mathbb{R}^d} 1_B(x\cos(2^{-1}\pi t))\nu(dx).
\]
That is, (3.2) is satisfied with \( T = 1 \) and \( f(t) = \cos(2^{-1}\pi t) \). Using \( \nu \), we can find \( \Sigma \) and \( \gamma \) satisfying (3.1) and (3.3). Let \( \mu \) be the distribution in \( I(\mathbb{R}^d) \) with the Lévy-Khintchine triplet \( (\Sigma, \nu, \gamma) \). Then \( \tilde{\mu} = \Phi_{\cos}(\mu) \). Hence \( A(\mathbb{R}^d) \subset \Phi_{\cos}(I(\mathbb{R}^d)) \).

Conversely, suppose that \( \tilde{\mu} = \Phi_{\cos}(\mu) \) for some \( \mu \in I(\mathbb{R}^d) \). The Lévy-Khintchine triplets \( (\tilde{\Sigma}, \tilde{\nu}, \tilde{\gamma}) \) and \( (\Sigma, \nu, \gamma) \) of \( \tilde{\mu} \) and \( \mu \) are related by (3.1)—(3.3) with \( T = 1 \) and \( f(s) = \cos(2^{-1}\pi s) \). Then a similar calculus shows that (3.1) holds. Hence \( \tilde{\nu} \in \mathcal{N}(\mathcal{A}_2) \) and \( \tilde{\mu} \in A(\mathbb{R}^d) \), showing that \( \Phi_{\cos}(I(\mathbb{R}^d)) \subset A(\mathbb{R}^d) \).

The mapping \( \Phi_{\cos} \) is one-to-one, since \( \nu \) is determined by \( \tilde{\nu} \) (Theorem 2.9 with \( k = 2 \)) and \( \Sigma \) and \( \gamma \) are determined by \( \tilde{\Sigma}, \tilde{\gamma}, \) and \( \nu \).

3.2. \( \Upsilon^0 \)-transformation. For later use, we introduce a transformation \( \Upsilon^0 \). Define
\[
\Upsilon^0(\nu)(B) = \int_0^\infty \nu(ux)\,e^{-u}\,du, \quad B \in \mathcal{B}(\mathbb{R}^d).
\]
Let \( \mathcal{M}_L^R(\mathbb{R}^d) \) be the class of Lévy measures of the Goldie-Steutel-Bondesson class \( B(\mathbb{R}^d) \). In [2], it is shown that \( \Upsilon^0(\mathcal{M}_L(\mathbb{R}^d)) = \mathcal{M}_L^R(\mathbb{R}^d) \). This is the transformation of Lévy measures associated with the stochastic integral mapping \( \Upsilon \) from \( I(\mathbb{R}^d) \) into \( I(\mathbb{R}^d) \) and it is known that \( \Upsilon(I(\mathbb{R}^d)) = B(\mathbb{R}^d) \) (see [2]). Both \( \Upsilon^0 \) and \( \Upsilon \) are one-to-one. For \( \nu \in \mathcal{M}_L(\mathbb{R}^d) \) with a polar decomposition \((\lambda, \nu_\xi)\), we have the expression
\[
\Upsilon^0(\nu)(B) = \int \lambda(d\xi) \int_0^\infty 1_B(r\xi) \Upsilon^0(\nu_\xi)(dr), \quad B \in \mathcal{B}(\mathbb{R}^d),
\]
where \( \Upsilon^0 \) in the right-hand side acts on \( \mathcal{M}_L^2((0, \infty)) \).

**Proposition 3.3.** Let \( \nu \in \mathcal{M}_L(\mathbb{R}^d) \). Then \( \Upsilon^0(\nu) \in \mathcal{M}_L^1(\mathbb{R}^d) \) if and only if \( \nu \in \mathcal{M}_L^1(\mathbb{R}^d) \).

**Proof.** Notice that
\[
\int_{|x| \leq 1} |x| \Upsilon^0(\nu)(dx) = \int_0^\infty e^{-u}du \int_{|ux| \leq 1} |ux| \nu(dx)
\]
\[
\int_0^\infty ue^{-u}du \int_{|x| \leq 1/u} |x|\nu(dx) = \int_{\mathbb{R}^d} |x|\nu(dx) \int_0^{1/|x|} ue^{-u}du
\]
\[
\leq \int_{|x| \leq 1} |x|\nu(dx) \int_0^\infty ue^{-u}du + \int_{|x| > 1} 2^{-1}|x|^{-1}\nu(dx),
\]
\[
\geq \int_{|x| \leq 1} |x|\nu(dx) \int_0^1 ue^{-u}du,
\]
to see the equivalence. \qed

3.3. A representation of completely monotone functions. In [9], the class of generalized type \(G\) distributions on \(\mathbb{R}^d\), denoted by \(G(\mathbb{R}^d)\), is defined as follows. \(\mu \in G(\mathbb{R}^d)\) if and only if the radial component \(\nu_\xi\) of the Lévy measure of \(\mu\) satisfies
\[
\nu_\xi(dr) = g(\xi^2)rdr, \quad g(\xi^2) \text{ is a completely monotone function on } (0, \infty).
\]
\(\mathcal{ML}_G(\mathbb{R}^d)\) denotes the class of all Lévy measures of \(\mu \in G(\mathbb{R}^d)\). We use the following result when dealing with \(G(\mathbb{R}^d)\). It is a result on the arcsine transformation representation of a function \(g(r^2)\) when \(g\) is completely monotone on \((0, \infty)\).

**Proposition 3.4.** Let \(g(u)\) be a real-valued measurable function on \((0, \infty)\). Then the following three conditions are equivalent.

(a) The function \(g(u)\) is completely monotone on \((0, \infty)\) and satisfies
\[
\int_0^\infty (1 \wedge r^2)g(r^2)dr < \infty. \tag{3.7}
\]

(b) There exists a completely monotone function \(h(s)\) on \((0, \infty)\) satisfying
\[
\int_0^\infty (1 \wedge s)h(s)ds < \infty \tag{3.8}
\]
such that
\[
g(r^2) = \int_0^\infty a_1(r; s)h(s)ds, \quad r > 0.
\]

(c) There exists a measure \(\rho\) on \((0, \infty)\) satisfying
\[
\int_0^\infty (1 \wedge s)\rho(ds) < \infty
\]
such that
\[
g(r^2) = a_1(\Upsilon_0(\rho))(r), \quad r > 0. \tag{3.9}
\]

**Proof.** (a) \(\Rightarrow\) (b): From Bernstein’s theorem, there exists a measure \(Q\) on \([0, \infty)\) such that
\[
g(u) = \int_{[0,\infty)} e^{-uv}Q(dv), \quad u > 0. \tag{3.10}
\]
It follows from (3.7) that \( Q(\{0\}) = 0 \), since \( Q(\{0\}) = \lim_{n \to \infty} g(u) \). We need the fact that the one-dimensional Gaussian density \( \phi(x; t) \) of mean 0 and variance \( t \) is the arcsine transform of the exponential distribution with mean \( t > 0 \). More precisely,\[
\phi(x; t) = (2\pi t)^{-1/2} e^{-x^2/(2t)} = \int_0^\infty e^{-s/t} a(x; s) ds, \quad t > 0, \quad x \in \mathbb{R}.
\]
This is the well-known Box-Muller method to generate normal random variables.

Using (3.11), we have
\[
g(r^2) = \int_{(0, \infty)} e^{-r^2v} Q(dv)
\]
\[
= \int_{(0, \infty)} v^{1/2} Q(dv) \int_{r^2/2}^\infty e^{-2sv} 2\pi^{-1/2} (2s - r^2)^{-1/2} ds
\]
\[
= \int_{(0, \infty)} v^{1/2} Q(dv) \int_{r^2/2}^\infty e^{-sv} \pi^{-1/2} (s - r^2)^{-1/2} ds.
\]
\[
= \int_{r^2}^\infty \pi^{-1/2} (s - r^2)^{-1/2} ds \int_{(0, \infty)} e^{-sv} v^{1/2} Q(dv)
\]
\[
= \int_0^\infty a_1(r; s) h(s) ds,
\]
where
\[
h(s) = 2^{-1} \pi^{1/2} \int_{(0, \infty)} e^{-sv} v^{1/2} Q(dv).
\]

Applying Theorem 2.2 for \( d = 1 \), we see (3.8) from (3.7).

(b) \( \Rightarrow \) (c): Since \( h(s) \) is completely monotone satisfying (3.8), there is \( \rho \in M_2(\mathbb{R}) \) such that \( h(s) ds = \Upsilon^0(\rho) \) (see Theorem A of [2]). Since \( \Upsilon^0(\rho) \) is concentrated on \((0, \infty)\), \( \rho \) is concentrated on \((0, \infty)\). Using Proposition 3.3 we see that \( \int_{(0, 1]} s \rho(ds) < \infty \).

(c) \( \Rightarrow \) (a): It follows from Proposition 3.3 that \( \int_{(0, 1]} \Upsilon^0(\rho)(ds) < \infty \). Hence it follows from (3.9) that \( g(r^2) \) satisfies (3.7) (use Theorem 2.2 for \( d = 1 \)). Finally let us prove that \( g(u) \) is completely monotone. There is a completely monotone function \( h(s) \) such that \( \Upsilon^0(\rho)(ds) = h(s) ds \) (see Theorem A of [2] again). Hence from Bernstein’s theorem we can find a measure \( R \) on \([0, \infty)\) such that
\[
h(s) = \int_{(0, \infty)} e^{-sv} R(dv), \quad s > 0.
\]
We have \( R(\{0\}) = 0 \) since \( \int_1^\infty h(s) ds < \infty \). Thus
\[
g(r^2) = \int_0^\infty a_1(r; s) h(s) ds = \int_{r^2}^\infty 2\pi^{-1} (s - r^2)^{-1/2} ds \int_{(0, \infty)} e^{-sv} R(dv)
\]
\[
= \int_{(0, \infty)} R(dv) \int_{r^2}^{\infty} 2\pi^{-1}(s - r^2)^{-1/2}e^{-sv}ds
\]
\[
= \int_{(0, \infty)} e^{-r^2v}2\pi^{-1/2}v^{-1/2}R(dv),
\]
where the last equality is from the same calculus as in the proof that (a) \(\Rightarrow\) (b). Now we see that \(g(u)\) is completely monotone. \(\square\)

3.4. A representation of \(G(\mathbb{R}^d)\) in terms of \(A_1\). We now give an alternative representation for Lévy measures of distributions in \(G(\mathbb{R}^d)\).

**Theorem 3.5.** Let \(\tilde{\mu}\) be an infinitely divisible distribution on \(\mathbb{R}^d\) with the Lévy-Khintchine triplet \((\tilde{\Sigma}, \tilde{\nu}, \tilde{\gamma})\). Then the following three conditions are equivalent.

(a) \(\tilde{\mu} \in G(\mathbb{R}^d)\).

(b) \(\tilde{\nu} = A_1(\nu)\) with some \(\nu \in \mathcal{M}_B^\rho(\mathbb{R}^d) \cap \mathcal{M}_L^1(\mathbb{R}^d)\).

(c) \(\tilde{\nu} = A_1(\Upsilon_0(\rho))\) with some \(\rho \in \mathcal{M}_L^1(\mathbb{R}^d)\).

In condition (b) or (c), the representation of \(\tilde{\nu}\) by \(\nu\) or \(\rho\) is unique.

**Proof.** (a) \(\Rightarrow\) (b): By definition of \(G(\mathbb{R}^d)\), the Lévy measure \(\tilde{\nu}\) of \(\tilde{\mu}\) has polar decomposition \((\lambda, g_\xi(r^2)dr)\) where \(g_\xi(u)\) is measurable in \((\xi, u)\) and completely monotone in \(u > 0\). Hence, by Proposition 3.4, for each \(\xi\) we can find a completely monotone function \(\ell_\xi(s)\) such that \(\int_0^\infty (1 \wedge s)\ell_\xi(s)ds < \infty\) and

\[
g_\xi(r^2) = \int_0^\infty a_1(r; s)\ell_\xi(s)ds, \quad r > 0.
\]

The measure \(Q_\xi\) in the representation (3.10) of \(g_\xi(u)\) has the property that \(Q_\xi(E)\) is measurable in \(\xi\) for every Borel set \(E\) in \([0, \infty)\) (see Remark 3.2 of [2]). Hence, for any nonnegative function \(f(s, v)\) measurable in \((s, v)\), \(\int_0^\infty f(s, v)Q_\xi(dv)\) is measurable in \((\xi, s)\). Hence the function \(h_\xi(s)\) defined as in (3.12) is measurable in \((\xi, s)\). Thus we have

\[
\tilde{\nu}(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty 1_B(r\xi)dr \int_0^\infty a_1(r; s)h_\xi(s)ds.
\]

Now, an argument similar to the proof of Theorem 2.2 shows that

\[
\int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty (1 \wedge s)h_\xi(s)ds < \infty.
\]

Thus, letting \(\nu\) denote the Lévy measure with polar decomposition \((\lambda, h_\xi(s)ds)\), we see that \(\tilde{\nu} = A_1(\nu)\) and \(\nu \in \mathcal{M}_B^\rho(\mathbb{R}^d) \cap \mathcal{M}_L^1(\mathbb{R}^d)\).
(b) ⇒ (c): It follows from \( \nu \in \mathcal{M}_L^B(\mathbb{R}^d) \) that \( \nu = \Upsilon^0(\rho) \) for some unique \( \rho \in \mathcal{M}_L^B(\mathbb{R}^d) \) (Theorem A of [2]). Since \( \nu \in \mathcal{M}_L^1(\mathbb{R}^d) \), we have \( \rho \in \mathcal{M}_L^1(\mathbb{R}^d) \) from Proposition 3.3.

(c) ⇒ (a): It follows from \( \rho \in \mathcal{M}_L^1(\mathbb{R}^d) \) that \( \Upsilon^0(\rho) \in \mathcal{M}_L^1(\mathbb{R}^d) \) (Proposition 3.3). Let \((\lambda, \nu_\xi)\) be polar decomposition of \( \nu = \Upsilon^0(\rho) \). Then \( \nu_\xi(ds) = \ell_\xi(s)ds \) where \( \ell_\xi(s) \) is measurable in \((\xi, s)\) and completely monotone in \( s > 0 \). Define \( g_\xi(u) \) by

\[
g_\xi(r^2) = \int_0^\infty a_1(r^2; s, s)\ell_\xi(s)ds.
\]

Then \( g_\xi(u) \) is measurable in \((\xi, u)\). It follows from Proposition 3.4 that \( g_\xi(u) \) is completely monotone in \( u > 0 \). Hence \( \tilde{\nu} \in \mathcal{M}_L^{\tilde{G}}(\mathbb{R}^d) \) and \( \tilde{\mu} \in G(\mathbb{R}^d) \).

**3.5. \( G(\mathbb{R}^d) \) as image of \( A(\mathbb{R}^d) \) under a stochastic integral mapping.** Following [8], we define the transformation \( \Upsilon_{\alpha, \beta}(\nu) \) for \( \alpha < 2 \) and \( 0 < \beta \leq 2 \). For a measure \( \nu \) on \( \mathbb{R}^d \) with \( \nu(\{0\}) = 0 \) define

\[
\Upsilon_{\alpha, \beta}(\nu)(B) = \int_0^\infty \nu(s^{-1}B)\beta s^{-\alpha-1}e^{-s^\alpha}ds, \quad B \in \mathcal{B}(\mathbb{R}^d),
\]

whenever the right-hand side gives a measure in \( \mathcal{M}_L(\mathbb{R}^d) \). This definition is different from that of [8] in the constant factor \( \beta \). A special case with \( \beta = 1 \) coincides with the transformation of Lévy measures in the stochastic integral mapping \( \Psi_\alpha \) studied by Sato [12]. Of particular interest in this work is the mapping \( \Upsilon_{-2, 2} \). Notice that \( \Upsilon_{-1, 1} = \Upsilon^0 \).

**Proposition 3.6.** \( \Upsilon_{-2, 2}(\nu) \) is definable if and only if \( \nu \in \mathcal{M}_L(\mathbb{R}^d) \). The mapping \( \Upsilon_{-2, 2} \) is one-to-one.

**Proof.** Let \( \tilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)2se^{-s^2}ds \). Then

\[
\int_{\mathbb{R}^d} f(x)\tilde{\nu}(dx) = \int_0^\infty 2se^{-s^2}ds \int_{\mathbb{R}^d} f(sx)\nu(dx)
\]

for all nonnegative measurable functions \( f \). Hence

\[
\int_{\mathbb{R}^d} (1 \land |x|^2)\tilde{\nu}(dx) = \int_0^\infty 2se^{-s^2}ds \int_{\mathbb{R}^d} (1 \land |sx|^2)\nu(dx)
\]

\[
= \int_0^\infty 2se^{-s^2}ds \left( \int_{|x| \leq 1/s} |sx|^2\nu(dx) + \int_{|x| > 1/s} \nu(dx) \right)
\]

\[
= \int_{\mathbb{R}^d} |x|^2\nu(dx) \int_0^{1/|x|} 2s^3e^{-s^2}ds + \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty 2se^{-s^2}ds.
\]

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Observe that \( \int_0^{1/|x|} 2s^3e^{-s^2}ds \) is convergent as \(|x| \downarrow 0\) and \(\sim 2^{-1}|x|^{-4}\) as \(|x| \to \infty\) and \(\int_{1/|x|}^{\infty} 2se^{-s^2}ds \) is \(\sim e^{-1/|x|^2}\) as \(|x| \downarrow 0\) and convergent as \(|x| \to \infty\). Then we see that \(\int_{\mathbb{R}^d}(1 \wedge |x|^2)\tilde{\nu}(dx)\) is finite if and only if \(\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(dx)\) is finite. To prove that \(\Upsilon_{-2,2}\) is one-to-one, make a similar argument to the proof of Proposition 4.1 of \cite{[12]}. \(\square\)

The following result is needed in showing the characterization of \(G(\mathbb{R}^d)\) in terms of type \(A\) distributions. However, it also shows that \(\mathcal{A}_1\) and \(\Upsilon^0\) are not commutative, while \(\mathcal{A}_2\) and \(\Upsilon^0\) are commutative, both being Upsilon transformations with domain equal to \(\mathfrak{M}_L(\mathbb{R}^d)\).

**Theorem 3.7.** It holds that

\[ \Upsilon_{-2,2}(\mathcal{A}_1(\rho)) = \mathcal{A}_1(\Upsilon^0(\rho)) \quad \text{for } \rho \in \mathfrak{M}_L^1(\mathbb{R}^d). \]

**Proof.** Suppose that \(\rho \in \mathfrak{M}_L^1(\mathbb{R}^d)\) with polar decomposition \((\lambda, \rho_\xi)\). Let \(\nu = \mathcal{A}_1(\rho)\) and \(\tilde{\nu} = \Upsilon_{-2,2}(\nu)\). Then \(\nu\) has polar decomposition \((\lambda, \nu_\xi)\) with \(\nu_\xi(ds) = a_1(\rho_\xi)(s)ds\). From Theorem 2.6 (ii) in \cite{[8]}, \(\tilde{\nu}\) has polar decomposition \((\lambda, \tilde{\nu}_\xi)\) given by

\[ \tilde{\nu}_\xi(dr) = rg_\xi(r^2)dr \]

with

\[ g_\xi(r^2) = 2 \int_0^\infty s^{-2}e^{-r^2/s^2}\nu_\xi(ds). \]

Using (3.13) and (3.14) we have

\[
rg_\xi(r^2) = 2r \int_0^\infty e^{-r^2/s^2}s^{-2}a_1(\rho_\xi)(s)ds \\
= \int_0^\infty e^{-t}(-t/2)a_1(\rho_\xi)(t^{-1/2})dt \\
= \int_0^\infty e^{-t}dt \int_0^\infty a_1(t^{-1/2}; s)\rho_\xi(ds) \\
= \int_0^\infty e^{-t}dt \int_0^\infty a_1(r; ts)\rho_\xi(ds) \\
= \int_0^\infty a_1(r; u)\Upsilon^0(\rho_\xi)(du),
\]

since

\[
\int_0^\infty f(u)\Upsilon^0(\rho_\xi)(du) = \int_0^\infty e^{-t}dt \int_0^\infty f(ts)\rho_\xi(ds)
\]

for every nonnegative measurable function \(f\). It follows that

\[ \tilde{\nu}(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)a_1(\Upsilon^0(\rho_\xi))(dr), \quad B \in \mathcal{B}(\mathbb{R}^d). \]
Using (3.6), we see that \( \tilde{\nu} = A_1(\Upsilon^0(\rho)) \).

The following result shows that \( G(\mathbb{R}^d) \) is the class of distributions of stochastic integrals with respect Lévy processes with type A distribution at time 1. This is a multivariate and not necessarily symmetric generalization of (1.3).

**Theorem 3.8.** Let

\[
\Psi_{-2,2}(\mu) = \mathcal{L} \left( \int_0^1 \left( \log \frac{1}{t} \right)^{1/2} dX_t^{(\mu)} \right), \quad \mu \in I(\mathbb{R}^d).
\]

Then \( \Psi_{-2,2} \) is one-to-one and

\[
G(\mathbb{R}^d) = \Psi_{-2,2}(A(\mathbb{R}^d)) = \Psi_{-2,2}(\overline{\text{Phi}}_\text{cos}(I(\mathbb{R}^d))),(3.15)
\]

where \( \overline{\text{Phi}}_\text{cos} \) is defined by (3.4). In other words, for any \( \tilde{\mu} \in G(\mathbb{R}^d) \) there exists a Lévy process \( \{X_t^{(\mu)} : t \geq 0\} \) with type A distribution \( \mu \) at time 1 such that

\[
\tilde{\mu} = \mathcal{L} \left( \int_0^1 \left( \log t^{-1} \right)^{1/2} dX_t^{(\mu)} \right).
\]

**Proof.** Let \( g(t) = \int_t^{\infty} 2ue^{-u^2} du = e^{-t^2} \). Then the inverse function of \( g \) is \( f(t) = (\log t^{-1})^{1/2} \) which is square-integrable on \((0, 1)\). Thus, \( \Psi_{-2,2}(\mu) \) is definable for all \( \mu \). Suppose that \( \tilde{\mu} \in G(\mathbb{R}^d) \) with triplet \((\tilde{\Sigma}, \tilde{\nu}, \tilde{\gamma})\). Then it follows from Theorems 3.5 and 3.7 that

\[
\tilde{\nu} = A_1(\Upsilon^0(\rho)) = \Upsilon_{-2,2}(A_1(\rho))
\]

for some \( \rho \in \mathcal{M}_1^0(\mathbb{R}^d) \). Let \( \nu = A_1(\rho) \). Since \( \nu = \Upsilon_{-2,2}(\nu) \), we have (3.2) for the function \( f(s) = (\log s^{-1})^{1/2} \) and \( T = 1 \). Choose \( \Sigma \) and \( \gamma \) satisfying (3.1) and (3.3). Let \( \mu \in I(\mathbb{R}^d) \) having triplet \((\Sigma, \nu, \gamma)\). Then \( \mu \in A(\mathbb{R}^d) \) and \( \tilde{\mu} = \Psi_{-2,2}(\mu) \). Conversely, we can see that if \( \mu \in A(\mathbb{R}^d) \), then \( \Psi_{-2,2}(\mu) \in G(\mathbb{R}^d) \). Thus the first equality in (3.15) is proved. The second equality follows from (3.5) of Theorem 3.2. The one-to-one property of \( \Psi_{-2,2} \) follows from that of \( \Upsilon_{-2,2} \) in Proposition 3.6. \( \square \)

**Remark 3.9.** (a) The two representations of \( \tilde{\mu} \in G(\mathbb{R}^d) \) in Theorems 3.5 and 3.8 are related in the following way. Theorem 3.8 shows that \( \tilde{\mu} \in G(\mathbb{R}^d) \) if and only if \( \tilde{\mu} = \Upsilon_{-2,2}(\overline{\text{Phi}_\text{cos}}(\mu)) \) for some \( \mu \in I(\mathbb{R}^d) \). This \( \mu \) has Lévy measure \( \rho^{(1/2)} \) if \( \rho \) is the Lévy measure in the representation of \( \tilde{\mu} \) in Theorem 3.5 (c). For the proof, use Proposition 2.5, Theorems 3.2 and 3.7.

(b) We have another representation of the class \( G(\mathbb{R}^d) \). We introduce the mapping \( \mathcal{G} \) as follows. Let \( h(t) = \int_t^{\infty} e^{-u^2} du, t > 0 \), and denote its inverse function by \( h^*(s) \).
For $\mu \in I(\mathbb{R}^d)$, we define

$$G(\mu) = \mathcal{L}\left(\int_0^{\sqrt{\pi}/2} h^*(s) dX^{(\mu)}_s\right).$$

It is known that $G(\mathbb{R}^d) = G(I(\mathbb{R}^d))$, see Theorem 2.4 (5) in [9]. This suggests us that $G$ is decomposed into

\begin{equation}
G = \Psi_2 \circ \Phi_{\cos} = \Phi_{\cos} \circ \Psi_2
\end{equation}

with the same domain $I(\mathbb{R}^d)$, where $\circ$ means composition of mappings. This is verified as follows. By Corollary 2.14, $A_2$ is an Upsilon transformation and $A_2$ corresponds to $\Phi_{\cos}$ (see (3.5)). Also, $\Upsilon_{-2,2}$ corresponds to the Upsilon transformation with the dilation measure $\tau(dx) = x^2 e^{-x^2} dx$. By Proposition 4.1 in [4], we have the second equality in (3.17).

3.6. $A_1$ is not an Upsilon transformation. By Theorem 3.7, we obtain the following remarkable result.

**Theorem 3.10.** The transformation $A_1$ is not an Upsilon transformation $\Upsilon_\tau$ for any dilation measure $\tau$.

**Proof.** Suppose that there is a measure $\tau$ on $(0, \infty)$ such that

$$A_1(\rho)(B) = \int_0^\infty \rho(u^{-1}B)\tau(du) \quad \text{for } B \in \mathcal{M}_1^1(\mathbb{R}^d).$$

Then, we can show that

$$A_1(\Upsilon^0(\rho)) = \Upsilon^0(A_1(\rho)) \quad \text{for } \rho \in \mathcal{M}_1^1(\mathbb{R}^d).$$

Indeed, for any nonnegative measurable function $f$

$$\int_{\mathbb{R}^d} f(x) A_1(\rho)(dx) = \int_0^\infty \tau(du) \int_{\mathbb{R}^d} f(ux)\rho(dx),$$

$$\int_{\mathbb{R}^d} f(y) \Upsilon^0(\rho)(dy) = \int_0^\infty e^{-v}dv \int_{\mathbb{R}^d} f(vy)\rho(dy),$$

and

$$A_1(\Upsilon^0(\rho))(B) = \int_0^\infty \tau(du) \int_0^\infty e^{-v}dv \int_{\mathbb{R}^d} 1_B(uvx)\rho(dx) = \Upsilon^0(A_1(\rho))(B).$$

Then, it follows from Theorem 3.7 that

$$\Upsilon_{-2,2}(A_1(\rho)) = \Upsilon^0(A_1(\rho)) \quad \text{for } \rho \in \mathcal{M}_1^1(\mathbb{R}^d).$$
Let \( \tilde{\rho} = A_1(\rho) \). If \( \int_{\mathbb{R}^d} |x| \rho(dx) < \infty \), then

\[
\int_{\mathbb{R}^d} x Y^0(\tilde{\rho})(dx) = \int_0^\infty e^{-u} du \int_{\mathbb{R}^d} u x \tilde{\rho}(dx) = \int_{\mathbb{R}^d} x \tilde{\rho}(dx)
\]

and

\[
\int_{\mathbb{R}^d} x Y_{-2,2}(\tilde{\rho})(dx) = \int_0^\infty 2ue^{-u^2} du \int_{\mathbb{R}^d} u x \tilde{\rho}(dx) \\
= \int_0^\infty 2u^2e^{-u^2} du \int_{\mathbb{R}^d} x \tilde{\rho}(dx) = 2^{-1} \pi^{1/2} \int_{\mathbb{R}^d} x \tilde{\rho}(dx).
\]

Hence \( Y_{-2,2}(\tilde{\rho}) \neq Y^0(\tilde{\rho}) \) whenever \( \int_{\mathbb{R}^d} x \tilde{\rho}(dx) \neq 0 \) (for example, choose \( \rho = \delta_{e_1}, e_1 = (1,0,...,0) \)). This is a contradiction. Hence the measure \( \tau \) does not exist. \( \square \)

4. Examples

We conclude this paper with examples for Theorems 3.5 and 3.7, where the modified Bessel function \( K_0 \) plays an important role in the Lévy measure of infinitely divisible distributions. We only consider the one-dimensional case of Lévy measures concentrated on \((0, \infty)\). Multivariate extensions are possible by using the polar decomposition.

By the well-known formula for the modified Bessel functions we have

\[
K_0(x) = \frac{1}{2} \int_0^\infty e^{-t-x^2/(4t)} t^{-1/2} dt, \quad x > 0.
\]

An alternative expression is

\[
K_0(x) = \int_1^\infty (t^2 - 1)^{-1/2} e^{-xt} dt, \quad x > 0,
\]

see (3.387.3) in [5, pp 350]. It follows that \( K_0(x) \) is completely monotone on \((0, \infty)\) and that \( \int_0^\infty K_0(x) dx = \pi/2 \).

The Laplace transform of \( K_0 \) in \( x > 0 \) is

\[
\varphi_{K_0}(s) := \int_0^\infty e^{-sx} K_0(x) dx = \begin{cases} 
(1 - s^2)^{-1/2} \arccos(s), & 0 < s < 1 \\
1, & s = 1 \\
(1 - s^2)^{-1/2} \log(s + (s^2 - 1)^{1/2}), & s > 1,
\end{cases}
\]

see (6.611.9) in [5, pp 695].

The following is an example of \( \nu \) and \( \tilde{\nu} \) in Theorem 3.5(b).
Example 4.1. Let
\[ \tilde{\nu}(dx) = K_0(x)1_{(0,\infty)}(x)dx \]
and
\[ (4.3) \quad \nu(dx) = 4^{-1/2}x^{-1/2}e^{-x/2}1_{(0,\infty)}(x)dx. \]
Then \( \nu \in \mathcal{M}_B^B(\mathbb{R}) \cap \mathcal{M}_L^1(\mathbb{R}) \), and \( \tilde{\nu} = A_1(\nu) \in \mathcal{M}_G^G(\mathbb{R}) \).

The proof is as follows. Since the function \( x^{-1/2}e^{-x/2} \) is completely monotone on \((0,\infty)\) and \( \int_0^1 x\nu(dx) < \infty \), we have \( \nu \in \mathcal{M}_B^B(\mathbb{R}) \cap \mathcal{M}_L^1(\mathbb{R}) \). Theorem 2.13, with \( k = 1 \), gives that for \( B \in \mathcal{B}(\mathbb{R}) \)
\[ A_1(\nu)(B) = \int_0^1 \nu^{(1/2)}(u^1B)2\pi^{-1}(1-u^2)^{-1/2}du \]
\[ = \int_0^1 2\pi^{-1}(1-u^2)^{-1/2}du \int_0^\infty 1_{u^{-1}B}(s^{1/2})\nu(ds) \]
\[ = \int_0^1 2^{-1}(1-u^2)^{-1/2}du \int_0^\infty 1_B(uss^{1/2})e^{-s^{1/2}}ds \]
\[ = \int_0^1 (1-u^2)^{-1/2}du \int_0^\infty 1_B(r)e^{-r^2/2}dr \]
\[ = \int_0^\infty 1_B(r)dr \int_1^\infty (y^2-1)^{-1/2}e^{-ry}dy \]
\[ = \int_0^\infty 1_B(r)K_0(r)dr = \tilde{\nu}(B). \]

The fact that \( \tilde{\nu} \in \mathcal{M}_G^G(\mathbb{R}) \) can also be shown directly, since \( K_0(x^{1/2}) \) is again completely monotone in \( x \in (0,\infty) \).

It follows from \( \tilde{\nu} \in \mathcal{M}_G^G(\mathbb{R}) \) that \( \tilde{\nu} \) is the Lévy measure of some generalized type \( G \) distribution \( \tilde{\mu} \) on \( \mathbb{R} \). Using (4.2), we find that this \( \tilde{\mu} \) is supported on \([0,\infty)\) if and only if it has Laplace transform
\[ \int_{[0,\infty)} e^{-sx}\tilde{\mu}(dx) = \exp\left\{-\gamma_0s + \varphi_{K_0}(s) - 2^{-1}\pi\right\} \]
for some \( \gamma_0 \geq 0 \).

Remark 4.2. \( A_1(\nu) \) in Example 4.1 actually belongs to a smaller class \( \mathcal{M}_L^B(\mathbb{R}) \). Therefore, in connection to Theorem 3.5, it might be interesting to find a necessary and sufficient condition on \( \nu \) for that \( \tilde{\mu} \in B(\mathbb{R}^d) \). The \( \nu \) in Example 4.1 also belongs to a smaller class than \( \mathcal{M}_B^B(\mathbb{R}) \cap \mathcal{M}_L^1(\mathbb{R}) \). It belongs to the class of Lévy measures of distributions in \( \mathcal{R}(\Psi_{-1/2}) \) studied in Theorem 4.2 of [12].

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We now give an example of \( \rho \) in Theorem 3.5 (c).

**Example 4.3.** Consider the following Lévy measure in \( \mathfrak{M}_B^0(\mathbb{R}) \):

\[
\rho(dx) = 4^{-1/2}x^{-1/2}e^{-x/4}1_{(0,\infty)}(x)dx.
\]

Then \( \nu \) in (4.3) satisfies \( \nu = \Upsilon_0(\rho) \).

To prove this, we compute the Upsilon transformation \( \Upsilon_0 \) of \( \rho \) as follows:

\[
\Upsilon_0(\rho)(dx) = \int_0^\infty \rho(u^{-1}dx)e^{-u}du
\]

\[
= 4^{-1/2}x^{-1/2} \left( \int_0^\infty u^{-1/2}e^{-u-x/(4u)}du \right) dx.
\]

By formula (3.475.15) in [5], pp 369, we have

\[
\int_0^\infty u^{-1/2}e^{-u-x/(4u)}du = \pi^{1/2}e^{-x/2}.
\]

Hence, \( \Upsilon_0(dx) = 4^{-1/2}x^{-1/2}e^{-x/4} dx \) and from (4.3) we have \( \nu = \Upsilon_0(\rho) \).

Since \( A_1(\nu) = A_1(\Upsilon_0(\rho)) = \Upsilon_{-2,2}(A_1(\rho)) \) by Theorem 3.7, \( A_1(\rho) \) is also of interest.

**Example 4.4.** Let \( \rho \) be as in (4.4). Then

\[
A_1(\rho)(dx) = 2^{-1}x^{-1/2}e^{-x^2/8}K_0(x^2/8)1_{(0,\infty)}(x)dx.
\]

The proof is as follows. We have

\[
A_1(\rho)(B) = \int_0^1 \rho^{(1/2)}(u^{-1}B)2\pi^{-1}(1-u^2)^{-1/2}du
\]

\[
= \int_0^1 2\pi^{-1}(1-u^2)^{-1/2}du \int_0^\infty 1_{u^{-1}B}(s^{1/2})\rho(ds)
\]

\[
= 2^{-1/2} \int_0^1 (1-u^2)^{-1/2}du \int_0^\infty 1_{u^{-1}B}(s^{1/2})s^{-1/2}e^{-s/4}ds
\]

\[
= \pi^{-1/2} \int_0^\infty 1_{B}(r)dr \int_0^1 u^{-1}(1-u^2)^{-1/2}e^{-r^2/(4u^2)}du
\]

\[
= 2^{-1/2} \int_0^\infty 1_{B}(r)dr \int_1^\infty y^{-1/2}(y-1)^{-1/2}e^{-r^2/4y}dy.
\]

Use (3.383.3) in [5], pp 347 to obtain

\[
\int_1^\infty y^{-1/2}(y-1)^{-1/2}e^{-r^2/4y}dy = e^{-r^2/8}K_0(r^2/8).
\]

Thus we obtain (4.3).
Remark 4.5. The $\rho$ in (4.4) also belongs to $\mathcal{M}_B^p(\mathbb{R}) \cap \mathcal{M}_L^1(\mathbb{R})$. Therefore $\mathcal{A}_1(\rho)$ itself is another example of the Lévy measure of a generalized type $G$ distribution on $\mathbb{R}$.

References