Contents

0.1 Appendix on Operator Algebras

0.1.1 C*-algebras and C^*-probability spaces

Bounded operators on a Hilbert-space

We recall initially that a (complex) Hilbert-space is a vector-space $\mathcal{H}$ over the field $\mathbb{C}$ of complex numbers, which is equipped with an inner product $\langle \cdot, \cdot \rangle$, making $\mathcal{H}$ complete with respect to the associated norm ($\|\xi\| = \langle \xi, \xi \rangle^{1/2}$ for all ξ in $\mathcal{H}$).

A linear mapping $T \colon \mathcal{H} \to \mathcal{H}$ is called a linear operator on $\mathcal{H}$. For such an operator we recall that the following conditions are equivalent:

(i) T is continuous at every point ξ of $\mathcal{H}$.

(ii) T is continuous at $0 \in \mathcal{H}$.

(iii) $\sup\{\|Tx\| \mid x \in \mathcal{H},\ \|x\| \leq 1\} < \infty$.

As a consequence of the equivalence between (i) and (iii), continuous linear mappings $T \colon \mathcal{H} \to \mathcal{H}$ are generally referred to as bounded (linear) operators on $\mathcal{H}$. The class of such operators on $\mathcal{H}$ is accordingly denoted by $\mathcal{B}(\mathcal{H})$. Pointwise addition and scalar-multiplication clearly makes $\mathcal{B}(\mathcal{H})$ into a vector space. Moreover, the composition $S \circ T$ of two elements S, T of $\mathcal{B}(\mathcal{H})$ constitutes a multiplication on $\mathcal{B}(\mathcal{H})$ which together with the linear operations turns $\mathcal{B}(\mathcal{H})$ into an algebra. Specifically this means (omitting the vector space axioms) that the multiplication and the linear operations satisfy the following conditions for any R, S, T in $\mathcal{B}(\mathcal{H})$ and β in $\mathbb{C}$:

(iv) $R(ST) = (RS)T$,

(v) $\beta(ST) = (\beta S)T = S(\beta T)$,

(vi) $R(S+T) = RS + RT$ and $(S+T)R = SR + TR$.

In (iv)-(vi) we have, as is customary, written e.g. ST instead of $S \circ T$. For T in $\mathcal{B}(\mathcal{H})$ we further define:

$$\|T\| = \sup\{\|Tx\| \mid x \in \mathcal{H},\ \|x\| \leq 1\} < \infty,$$

and it is standard to check that the mapping $T \mapsto \|T\|$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is *sub-multiplicative* in the sense that

(vii) $\|ST\| \leq \|S\|\|T\|$ for all S, T in $\mathcal{B}(\mathcal{H})$.

This norm is naturally termed the *operator-norm* on $\mathcal{B}(\mathcal{H})$. The completeness of $\mathcal{H}$ further implies that $\mathcal{B}(\mathcal{H})$ is also complete with respect to the operator norm. Altogether this means that $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ constitutes a *Banach algebra*.

We finally equip $\mathcal{B}(\mathcal{H})$ with an involution (the $*$-operation). For this we recall that for any operator T in $\mathcal{B}(\mathcal{H})$ there exists a unique operator T^* in $\mathcal{B}(\mathcal{H})$ such that

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle \qquad \text{for all } \xi, \eta \text{ in } \mathcal{H}.$$

The operator T^* is called the *adjoint* of T, and the mapping $T \mapsto T^*$ is termed the adjoint operation on $\mathcal{B}(\mathcal{H})$. It is standard to check that it satisfies the following relations:

(viii) $(T^*)^* = T$,

(ix) $(\beta S + T)^* = \overline{\beta} S^* + T^*$,

(x) $(ST)^* = T^* S^*$,

(xi) $\|T^*T\| = \|T\|^2$

for any β in $\mathbb{C}$ and S, T in $\mathcal{B}(\mathcal{H})$. Condition (xi) above is of crucial importance and it is known as the *C^*-identity*. Together with conditions (vii) and (viii) it implies for example that the involution is norm-preserving:

$$\|T\|^2 = \|T^*T\| \leq \|T^*\|\|T\|$$

for any T in $\mathcal{B}(\mathcal{H})$.

Abstract C^*-algebras

The properties listed in the previous subsection for $\mathcal{B}(\mathcal{H})$ lay the foundation for the definition of an (abstract) C^*-algebra.

Definition 1. *Let $\mathcal{A}$ be a vector space over $\mathbb{C}$*

(a) *Assume that $\mathcal{A}$ is further equipped with a multiplication $(a, b) \mapsto ab\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, such that the conditions (iv),(v) and (vi) above are satisfied (when the operators S, R, T are replaced by elements from $\mathcal{A}$). We say then that $\mathcal{A}$ forms an algebra over $\mathbb{C}$.*

(b) *Assume that $\mathcal{A}$ is an algebra over $\mathbb{C}$, which is further equipped with a mapping $a \mapsto a^*\colon \mathcal{A} \to \mathcal{A}$, satisfying conditions (viii),(ix) and (x). We then say that $\mathcal{A}$ is a $*$-algebra over $\mathbb{C}$.*

(c) *Assume that $\mathcal{A}$ is an algebra over $\mathbb{C}$, which is further equipped with a norm $\|\cdot\|$ such that condition (vii) is satisfied and such that $\mathcal{A}$ is complete in the topology induced by $\|\cdot\|$. We say then that $\mathcal{A}$ is a Banach-algebra over $\mathbb{C}$.*

(d) *A C^*-algebra is a $*$-algebra $\mathcal{A}$, which is also equipped with a norm $\|\cdot\|$ satisfying condition (xi) and such that $\mathcal{A}$ is complete in the topology induced by $\|\cdot\|$.*

The calculation (0.1.1) shows that C^*-algebras are special cases of Banach-algebras. The previous subsection showed that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra for any Hilbert-space $\mathcal{H}$ over $\mathbb{C}$. Furthermore, any (linear) subspace of $\mathcal{B}(\mathcal{H})$, which is closed under multiplication (i.e. composition of operators), closed under the adjoint operation and closed in the norm topology will form a C^*-algebra.

The C^*-algebra $\mathcal{B}(\mathcal{H})$ contains a (two-sided) neutral element with respect to the multiplication, namely the identity operator $\mathbf{1}\colon \mathcal{H} \to \mathcal{H}$. In general a C^*-algebra $\mathcal{A}$ is called *unital*, if it contains a (two-sided) neutral element with respect to the multiplication. Such a multiplicative unit is necessarily unique, and we generally denote it by $\mathbf{1}_{\mathcal{A}}$. An (orthogonal) *projection* in a C^*-algebra $\mathcal{A}$ is an element p of $\mathcal{A}$ such that $p = p^* = p^2$. When $\mathcal{A} = \mathcal{B}(\mathcal{H})$ an orthogonal projection is exactly the projection operator, which maps $\mathcal{H}$ onto a closed subspace of $\mathcal{H}$.

Example 2. *The space $C_b(\mathbb{R})$ of continuous bounded functions $f\colon \mathbb{R} \to \mathbb{C}$ constitutes a $*$-algebra under point-wise linear operations, multiplication and complex conjugation. Together with the uniform norm*

$$\|f\|_u = \sup\{|f(x)| \mid x \in \mathbb{R}\}, \qquad (f \in C_b(\mathbb{R})),$$

these operations turn $C_b(\mathbb{R})$ into a C^*-algebra. Note in particular that the constant function 1 is a multiplicative unit for $C_b(\mathbb{R})$ and that the multiplication on $C_b(\mathbb{R})$ is commutative (contrary to that of $\mathcal{B}(\mathcal{H})$, when $\dim(\mathcal{H}) \geq 2$). We say that $C_b(\mathbb{R})$ is a commutative *unital C^*-algebra.*

Consider now the subset $C_0(\mathbb{R})$, consisting of continuous functions $f\colon \mathbb{R} \to \mathbb{C}$, such that $|f(x)| \to 0$ as $|x| \to \infty$. It is easily seen that $C_0(\mathbb{R})$ is closed under the linear operations, the multiplication, complex conjugation and the uniform norm. Thus $C_0(X)$ is again a commutative C^*-algebra. Note however that $C_0(\mathbb{R})$ does not have a multiplicative neutral element. However, by embedding

$\mathbb{R}$ into its one-point compactification $\mathbb{T} = \{z \in \mathbb{R} \mid |z| = 1\}$ it is not hard to see that $C_0(\mathbb{R})$ may be identified with the subset

$$\{f \in C(\mathbb{T}) \mid f(\mathrm{i}) = 0\}$$

of the space $C(\mathbb{T})$ of all continuous functions $f\colon \mathbb{T} \to \mathbb{C}$. As for $C_b(\mathbb{R})$ the point-wise operations together with the uniform norm clearly make $C(\mathbb{T})$ into a commutative unital C^*-algebra. These considerations illustrate the general fact that a (non-unital) C^*-algebra can always be embedded as a sub-algebra (in fact an ideal) into a unital C^*-algebra (see [RLL] for the general result). In this text we are generally mainly interested in unital C^*-algebras.

A (linear) subspace $\mathcal{B}$ of a C^*-algebra is called a C^*-sub-algebra, if it is closed under the multiplication, closed under the $*$-operation and closed with respect to the operator norm. In that case $\mathcal{B}$ is clearly a C^*-algebra itself. Sub-algebras and $*$-subalgebras of algebras and $*$-algebras are defined analogously. If $\mathcal{A}$ is unital, and $\mathcal{B}$ contains the multiplicative unit of $\mathcal{A}$, then clearly $\mathcal{B}$ is unital as well. If $\mathcal{B}$ does not contain the unit of $\mathcal{A}$, it can either be non-unital, or it could have a unit of its own. The latter case may also occur if $\mathcal{A}$ is non-unital. The following examples illustrate these situations.

Example 3. *(a) Suppose $\mathcal{H}$ is a Hilbert-space with dimension greater than 2, and that p is the orthogonal projection onto a proper subspace $\mathcal{H}_0$ of $\mathcal{H}$. Then $p\mathcal{B}(\mathcal{H})p$ is a C^*-sub-algebra of $\mathcal{B}(\mathcal{H})$ with unit p, which is distinct from the unit of $\mathcal{B}(\mathcal{H})$.*

(b) Consider the class

$$\mathcal{A} = \left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \,\middle|\, f_{11}, f_{12}, f_{21}, f_{22} \in C_0(\mathbb{R}) \right\}$$

of 2×2-matrices with entries from $C_0(\mathbb{R})$. It is easily seen that $\mathcal{A}$ constitutes a C^*-algebra under the natural linear operations, multiplication and $*$-operation, and under the norm:

$$\left\| \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \right\| = \sup \left\| \begin{pmatrix} f_{11}(t) & f_{12}(t) \\ f_{21}(t) & f_{22}(t) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\|,$$

where the sup is taken over all t in $\mathbb{R}$ and all (ξ, ξ_2) in $\mathbb{R}^2$ such that $\xi_1^2 + \xi_2^2 \le 1$.

It is not hard to verify that $\mathcal{A}$ does not have a unit, but it contains the 2×2-matrices $M_2(\mathbb{C})$ as a unital C^*-sub-algebra of $\mathcal{A}$ (corresponding to the cases where $f_{11}, f_{12}, f_{21}, f_{22}$ are constant functions).

As mentioned above, $\mathcal{B}(\mathcal{H})$ and its C^*-sub-algebras form the generic examples of C^*-algebras. The fundamental Gelfand-Neumark-Theorem says that in fact any C^*-algebra may be identified with a C^*-sub-algebra of $\mathcal{B}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$. We will outline the proof of this theorem in the following. A key ingredient in the proof is the so-called GNS-construction[1], which we describe next. We introduce first some needed terminology.

- An element a of a C^*-algebra $\mathcal{A}$ is called positive, if it has the form b^*b for some element b in $\mathcal{A}$. Note that this is in accordance with the notion of a positive operator in $\mathcal{B}(\mathcal{H})$ and with the notion of a positive function in the C^*-algebras considered in Example 2.

- Let $\mathcal{A}$ be a unital C^*-algebra. A linear functional $\phi\colon \mathcal{A} \to \mathbb{C}$ is called a state, if $\phi(a) \geq 0$ for any positive element a of $\mathcal{A}$, and if $\phi(\mathbf{1}_{\mathcal{A}}) = 1$ (recall that $\mathbf{1}_{\mathcal{A}}$ denotes the multiplicative unit of $\mathcal{A}$). A state on $\mathcal{A}$ is automatically continuous with respect to the norm-topology on $\mathcal{A}$.

- Suppose $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, and let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a linear mapping. We say then that Φ is a $*$-homomorphism, if $\Phi(a^*) = \Phi(a)^*$ and $\Phi(ab) = \Phi(a)\Phi(b)$ for all a, b in $\mathcal{A}$. If Φ is, in addition, injective or bijective, we refer to Φ as a $*$-monomorphism, respectively a $*$-isomorphism. If $\mathcal{A}$ and $\mathcal{B}$ are both unital, and Φ maps the unit of $\mathcal{A}$ to that of $\mathcal{B}$, we say that Φ is unital.

Consider in the following a unital C^*-algebra $\mathcal{A}$ and a state $\phi\colon \mathcal{A} \to \mathbb{C}$. The linearity and positivity of $\mathcal{A}$ then ensures that the formula

$$\langle a, b\rangle_\phi = \phi(b^*a), \qquad (a, b \in \mathcal{A}),$$

defines a non-negative definite sesqui-linear form on $\mathcal{A}$. Generally $\langle \cdot, \cdot\rangle_\phi$ is not positive definite, unless ϕ is faithful. However, if we put

$$N_\phi = \{a \in \mathcal{A} \mid \phi(a^*a) = 0\},$$

then N_ϕ is a left ideal, which is further closed in the norm topology. Moreover, $\langle \cdot, \cdot\rangle_\phi$ gives rise to a positive definite sesqui-linear form (i.e. an inner product) on the quotient linear space $\mathcal{A}/N_\phi$ via the formula:

$$\langle [a], [b]\rangle_\phi = \langle a, b\rangle_\phi, \qquad (a, b \in \mathcal{A}), \tag{1}$$

where e.g. $[a]$ denotes the equivalence class containing a. By $\mathcal{H}_\phi$ we denote the completion of $\mathcal{A}/N_\phi$ with respect to the norm $\|\cdot\|_\phi$ associated to the inner product given in (1).

Now any element a of $\mathcal{A}$ gives rise to a linear operator L_a on $\mathcal{A}/N_\phi$ via the formula:

$$L_a([b]) = [ab], \qquad (b \in \mathcal{A}).$$

For any b in $\mathcal{A}$ we note that

$$\|L_a([b])\|_\phi = \|ab\|_\phi = \phi((ab)^*(ab)) = \phi(b^*a^*ab) \leq \phi(\|a^*a\|b^*b) = \|a^*a\|\phi(b^*b) = \|a\|^2\|b\|_\phi,$$

[1] GNS stands for Gelfand, Neumark and Seagal

where we have used that the operator $\|a^*a\|b^*b - b^*a^*ab$ is positive, so that $\phi(b^*a^*ab) \le \phi(\|a^*a\|b^*b)$. The calculation above shows that L_a is uniformly continuous on $\mathcal{A}/N_\phi$ and hence it extends by continuity to a continuous operator, also denoted by L_a, on $\mathcal{H}_\phi$. The calculation above further shows that

$$\|L_a\| = \sup\{\|L_a(\xi)\|_\phi \mid \xi \in \mathcal{H}_\phi,\ \|\xi\|_\phi \le 1\} = \sup\{\|L_a([b])\|_\phi \mid b \in \mathcal{A},\ \|[b]\|_\phi \le 1\} \le \|a\|.$$

Moreover it is straightforward to check that

$$L_{za+a'} = zL_a + L_{a'}, \quad L_{aa'} = L_a L_{a'}, \quad L_{a^*} = L_a^*, \quad \text{and} \quad L_{\mathbf{1}_\mathcal{A}} = \mathbf{1}_{\mathcal{B}(\mathcal{H}_\phi)}$$

for any z in $\mathbb{C}$ and a, a' in $\mathcal{A}$. It follows altogether that if we define a mapping $\pi_\phi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$ by setting

$$\pi_\phi(a) = L_a, \qquad (a \in \mathcal{A}),$$

then π_ϕ is a unital $*$-homomorphism of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\phi)$. We note also that π_ϕ has the following key property:

$$\langle \pi_\phi(a)[\mathbf{1}_\mathcal{A}], [\mathbf{1}_\mathcal{A}]\rangle_\phi = \langle L_a[\mathbf{1}_\mathcal{A}], [\mathbf{1}_\mathcal{A}]\rangle_\phi = \langle [a], [\mathbf{1}_\mathcal{A}]\rangle = \phi(\mathbf{1}_\mathcal{A}^* a) = \phi(a), \qquad (2)$$

so that ϕ is turned into the vector state $\langle \cdot [\mathbf{1}_\mathcal{A}], [\mathbf{1}_\mathcal{A}]\rangle$ via the representation π_ϕ. We shall refer to the triplet $(\pi_\phi, \mathcal{H}_\phi, [\mathbf{1}_\mathcal{A}])$ as the *GNS-triplet associated to* ϕ.

Theorem 4 (Gelfand-Neumark)**.** *Let $\mathcal{A}$ be a unital C^*-algebra. Then there exists a Hilbert-space $\mathcal{H}$ and a unital injective $*$-homomorphism $\Phi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

Sketch of proof: Apart from the GNS-construction outlined above, the main ingredient in the proof is the fact that the class $\mathcal{S}(\mathcal{A})$ of states on $\mathcal{A}$ separate the elements of $\mathcal{A}$. To be precise: If $a, b \in \mathcal{A}$ such that $\phi(a) = \phi(b)$ for any state ϕ on $\mathcal{A}$, then $a = b$. A proof of this fact can be found in [KR97, Theorem 4.3.4].

Now let $\mathcal{H}$ be the Hilbert-space $\oplus_{\phi\in\mathcal{S}(\mathcal{A})}\mathcal{H}_\phi$, where, for each ϕ in $\mathcal{S}(\mathcal{A})$, $\mathcal{H}_\phi$ is the Hilbert-space from the GNS-construction described above. Then we may define a mapping $\Phi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be setting

$$\Phi(a) = \oplus_{\phi\in\mathcal{S}(\mathcal{A})}\pi_\phi(a), \qquad (a \in \mathcal{A}),$$

where the right hand side is the “diagonal” operator on $\mathcal{H}$ with “diagonal elements” $\pi_\phi(a)$, $\phi \in \mathcal{S}(\mathcal{A})$. Since π_ϕ is a $*$-homomorphism for each ϕ in $\mathcal{S}(\mathcal{A})$, it follows easily that so is Φ. Moreover, $\Phi(a) = 0 \in \mathcal{B}(\mathcal{H})$, if and only if $\pi_\phi(a) = 0$ for all ϕ in $\mathcal{S}(\mathcal{A})$. In that case it follows from (2) that

$$\phi(a) = \langle \pi_\phi(a)[\mathbf{1}_\mathcal{A}], [\mathbf{1}_\mathcal{A}]\rangle_\phi = 0$$

for all ϕ in $\mathcal{S}(\mathcal{A})$, and since $\mathcal{S}(\mathcal{A})$ separates the elements of $\mathcal{A}$, this implies that $a = 0$. Thus Φ is injective, as desired. ■

Spectral theory

Let d be a positive integer, and let A be an element of the C^*-algebra $M_d(\mathbb{C})$ of $d \times d$-matrices with complex entries. Recall that A is called *normal*, if $AA^* = A^*A$. In that case we recall also that A can diagonalized, i.e. $A = U\Delta U^*$ for some unitary $d \times d$ matrix (so that $U^*U = \mathbf{1}_{M_d(\mathbb{C})}$), and Δ is a $d \times d$-diagonal matrix, whose diagonal elements consist of the eigenvalues $\lambda_1, \dots, \lambda_d$ of A (counted with multiplicity). We denote by $\mathrm{spe}(A)$ the *spectrum* of A, i.e. the set of eigenvalues of A. Given any function $f\colon \mathrm{spe}(A) \to \mathbb{C}$ we then define the $d \times d$-matrix $f(A)$ by the formula:

$$f(A) = U \begin{pmatrix} f(\lambda_1) & & & 0 \\ & f(\lambda_2) & & \\ & & \ddots & \\ 0 & & & f(\lambda_d) \end{pmatrix} U^*.$$

Since U^*U is the multiplicative unit of $M_d(\mathbb{C})$ it follows immediately $f(A)$ that coincides with the natural definition, if $f(x) = x^p$ for some p in $\mathbb{N}_0$. By linearity the same observation holds in the case where f is a polynomial. We note also that the mapping $f \mapsto f(A)$ is a $*$-homomorphism from the C^*-algebra of all complex-valued functions on $\mathrm{spe}(A)$ into $M_d(\mathbb{C})$.

In the following we describe how the considerations above generalize to any normal element of unital C^*-algebra $\mathcal{A}$. For a general element a of $\mathcal{A}$ we say that a is invertible, if it has a multiplicative inverse, i.e. an element b of $\mathcal{A}$ such that $ab = ba = \mathbf{1}_{\mathcal{A}}$. We then define the spectrum $\mathrm{spe}(a)$ as

$$\mathrm{spe}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \mathbf{1}_{\mathcal{A}} \text{ is not invertible}\},$$

and we note that this definition is in accordance with the usual definition of the spectrum of a $d \times d$-matrix (considered above). We list next a few fundamental facts about the spectrum.

Proposition 5. *For an arbitrary element a of a unital C^*-algebra $\mathcal{A}$, we have the following facts:*

(i) The spectrum $\mathrm{spe}(a)$ is a nonempty, compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \le \|a\|\}$.

(ii) If p is a polynomial (in one variable), and $p(a)$ is the element of $\mathcal{A}$ defined in the intuitive way, then $\mathrm{spe}(p(a)) = \{p(\lambda) \mid \lambda \in \mathrm{spe}(a)\}$.

(iii) If a is normal, then $\|a\| = \sup\{|\lambda| \mid \lambda \in \mathbb{C}\}$.

(iv) In general $\mathrm{spe}(a^*) = \{\overline{\lambda} \mid \lambda \in \mathrm{spe}(a)\}$. In particular $\mathrm{spe}(a) \subseteq \mathbb{R}$, if a is selfadjoint.

Proof. (i) For any a in $\mathcal{A}$ such that $\|a\| < 1$, the completeness of $\mathcal{A}$ ensures that the series $\sum_{n=0}^{\infty} a^n$ converges in the norm topology to an element b of $\mathcal{A}$. A direct calculation further shows that $(\mathbf{1}_{\mathcal{A}} - a)b = b(\mathbf{1}_{\mathcal{A}} - a) = \mathbf{1}$, implying that $\mathbf{1}_{\mathcal{A}} - a$ is invertible. For a general element a of $\mathcal{A}$ we may subsequently conclude that $a - \lambda \mathbf{1}_{\mathcal{A}} = -\lambda(\mathbf{1}_{\mathcal{A}} - \lambda^{-1} a)$ is invertible whenever $|\lambda| > \|a\|$, which means that $\mathrm{spe}(a)$ is contained in the disk $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$. Since the set of invertible elements in $\mathcal{A}$ is open in the norm topology and since the mapping $\lambda \mapsto a - \lambda \mathbf{1}_{\mathcal{A}}$ is continuous in that same topology, it follows further that $\mathrm{spe}(a)$ is a closed subset of $\mathbb{C}$. Altogether $\mathrm{spe}(a)$ is a compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$.

To verify that $\mathrm{spe}(a)$ is non-empty, we apply the Hahn-Banach extension theorem to choose an norm-continuous linear functional $\rho\colon \mathcal{A} \to \mathbb{C}$, such that $\rho(a) = \|a\|$. We consider then the mapping $f\colon \mathbb{C} \setminus \mathrm{spe}(a) \to \mathbb{C}$ given by

$$f(z) = (a - z\mathbf{1}_{\mathcal{A}})^{-1}, \qquad (z \in \mathbb{C} \setminus \mathrm{spe}(a)).$$

For z, w in $\mathbb{C} \setminus \mathrm{spe}(a)$ we note that

$$\begin{aligned} \frac{f(z) - f(w)}{z - w} &= \frac{1}{z - w}(a - z\mathbf{1}_{\mathcal{A}})^{-1}\big((a - w\mathbf{1}_{\mathcal{A}}) - (a - z\mathbf{1}_{\mathcal{A}})\big)(a - w\mathbf{1}_{\mathcal{A}})^{-1} &(3)\\ &= (a - z\mathbf{1}_{\mathcal{A}})^{-1}(a - w\mathbf{1}_{\mathcal{A}})^{-1} &(4)\\ &\longrightarrow (a - w\mathbf{1}_{\mathcal{A}})^{-1} \quad \text{as } z \to w. &(5) \end{aligned}$$

It follows that f is holomorphic on $\mathbb{C} \setminus \mathrm{spe}(a)$. If $\mathrm{spe}(a) = \emptyset$, f would be an entire function, and furthermore

$$|f(z)| = |\lambda|^{-1}\big|\rho((|\lambda^{-1}|\mathbf{1}_{\mathcal{A}} - a)^{-1})\big| \longrightarrow 0,$$

as $|z| \to \infty$, since $\rho((|\lambda^{-1}|\mathbf{1}_{\mathcal{A}} - a)^{-1}) \to \rho(a^{-1})$ as $|z| \to \infty$. This contradicts Liouville's Theorem, and hence $\mathrm{spe}(a)$ cannot be empty.

(iv) Note first that if an element b of $\mathcal{A}$ is invertible, then so is b^*, and $(b^*)^{-1} = (b^{-1})^*$. Indeed, $b^*(b^{-1})^* = (b^{-1}b)^* = \mathbf{1}_{\mathcal{A}}^* = \mathbf{1}_{\mathcal{A}}$, and similarly $(b^{-1})^* b^* = \mathbf{1}_{\mathcal{A}}$. From this observation it follows immediately for any complex number λ that $\lambda \in \mathbb{C} \setminus \mathrm{spe}(a)$, if and only if $\overline{\lambda} \in \mathbb{C} \setminus \mathrm{spe}(a^*)$. In other words $\mathrm{spe}(a^*) = \{\overline{\lambda} \mid \lambda \in \mathrm{spe}(a)\}$. If $a = a^*$, it follows in particular that $\mathrm{spe}(a) \subseteq \mathbb{R}$. ■

Theorem 6. *Let $\mathcal{A}$ be a unital C^*-algebra, let a be a selfadjoint element of a, and consider the C^*-algebra $C(spe(a))$ of continuous, complex-valued functions on $spe(a)$.*

Then there exists a unique $*$-homomorphism $\Phi_a\colon C(spe(a)) \to \mathcal{A}$ with the following properties:

(i) $\Phi_a(p) = p(a)$ *(defined in the intuitive way) for any polynomial p (in one variable). In particular $\Phi(1) = \mathbf{1}_{\mathcal{A}}$ and $\Phi_a(\mathrm{id}) = a$, where 1 denotes the constant polynomium 1, and* id *denotes the identity function.*

(ii) $\|\Phi_a(f)\| = \sup\{|f(\lambda)| \mid \lambda \in spe(a)\}$ *for any function f from $C(spe(a))$. In other words Φ_a is an isometry from $C(spe(a))$ into $\mathcal{A}$.*

(iii) $\Phi_a(f)$ *is a normal element of* $\mathcal{A}$ *for any* f *in* $C(spe(a))$*.*

(iv) *If* $b \in \mathcal{A}$ *such that* $ab = ba$*, then also* $\Phi(f)b = b\Phi(f)$ *for any function* f *in* $C(spe(a))$*.*

The operator $\Phi_a(f)$ described in Theorem 6 is often denoted $f(a)$.

Proof of Theorem 6: Assigning the intuitive meaning to $p(a)$ for any polynomial p (in one variable), the mapping $\Phi_0 \colon p \mapsto p(a)$ is clearly a $*$-homomorphism from the $*$-algebra of polynomials into $\mathcal{A}$. In particular this entails that $p(a)$ is a normal element of $\mathcal{A}$ for any polynomial p, and hence Proposition 5(ii)-(iii) imply that

$$\|p(a)\| = \sup\{|\zeta| \mid \zeta \in \operatorname{spe}(p(a))\} = \sup\{|p(\lambda)| \mid \lambda \in \operatorname{spe}(a)\} = \|p\|, \qquad (6)$$

where the last norm is that of p considered as an element of the C^*-algebra $C(\operatorname{spe}(a))$.

By Proposition 5(i)-(iv), $\operatorname{spe}(a)$ is a non-empty compact subset of $\mathbb{R}$, and hence The Weierstrass Approximation Theorem implies that the $*$-algebra of polynomials is dense in $C(\operatorname{spe}(a))$. From this, (6) and the completeness of $\mathcal{A}$, it follows by a standard argument (extension by continuity) that Φ_0 may be extended to a mapping $\Phi_a \colon C(\operatorname{spe}(a)) \to \mathcal{A}$, which, automatically satisfies that

$$\|\Phi_a(f)\| = \|f\|,\ \Phi_a(\alpha f + g) = \alpha\Phi_a(f) + \Phi_a(g),\ \Phi_a(fg) = \Phi_a(f)\Phi_a(g),\ \Phi_a(\overline{f}) = \Phi_a(f)^*$$

for any f, g in $C(\operatorname{spe}(a))$ and α in $\mathbb{C}$. In other words Φ_a is an isometric $*$-homomorphism, so that (i) and (ii) are satisfied.

Since $p(a)$ is normal for any polynomial p, and since the norm limit of a sequence of normal elements in $\mathcal{A}$ is clearly normal as well, it follows that $\Phi_a(f)$ is normal for any f in $C(\operatorname{spe}(a))$. Similarly, if $b \in \mathcal{A}$ such that $ab = ba$, then clearly $p(a)b = bp(a)$ for any polynomial p, and hence by approximating $\Phi_a(f)$ by elements of the form $p(a)$, it follows that also $b\Phi_a(f) = \Phi_a(f)b$ for any f in $C(\operatorname{spe}(a))$. This verifies (iii) and (iv) and completes the proof. ∎

Corollary 7. *Let* $\mathcal{A}$ *be a unital* C^**-algebra, and let* ϕ *be a state on* $\mathcal{A}$*. Let further* a *be a selfadjoint element of* $\mathcal{A}$*. Then there exists a unique Borel-probability measure* μ_a *on* $spe(a)$*, such that*

$$\int_{spe(a)} f(t)\,\mu_a(\mathrm{d}t) = \phi(f(a)), \qquad \text{for any function } f \text{ in } C(spe(a)).$$

Here $f(a) = \Phi_a(f)$ *in the notation of Theorem 6.*

Proof. By the properties of the mapping $\Phi_a \colon a \mapsto f(a)$ described in Theorem 6, it follows that the formula

$$\Lambda(f) = \phi(f(a)), \qquad (f \in C(\operatorname{spe}(a)),$$

defines a positive, linear functional on $C(\mathrm{spe}(a))$. Hence, by the Riesz representation theorem (see e.g. [Ru]), there exists a Borel-measure on $\mathrm{spe}(a)$, such that

$$\int_{\mathrm{spe}(a)} f(t)\,\mu_a(\mathrm{d}t) = \Lambda(f) = \phi(f(a)),$$

for any function f in $C(\mathrm{spe}(a))$. This condition clearly implies that μ is a probability measure (since $\phi(\mathbf{1}_{\mathcal{A}}) = 1$), and it determines μ_a uniquely as a Borel-measure on $\mathrm{spe}(a)$. ■

Definition 8. *Let $\mathcal{A}$ be a unital C^*-algebra.*

*(a) For any state ϕ on $\mathcal{A}$, we say that the pair $(\mathcal{A}, \phi)$ constitutes a C^*-*probability space*.*

(b) If $(\mathcal{A}, \phi)$ is a C^*-probability space, and a is a selfadjoint element of $\mathcal{A}$, then the measure μ_a described in Corollary 7 is called the spectral distribution of *a with respect to ϕ.*

In the remaining part of this subsection our objective is to extend Theorem 6 from selfadjoint elements to general normal elements of a unital C^*-algebra. To achieve this, one generally passes via a version of Gelfand's Theorem, which we describe next.

Suppose $\mathcal{A}$ is a unital C^*-algebra. A non-zero linear functional $\phi\colon \mathcal{A} \to \mathbb{C}$ is called a *character* of $\mathcal{A}$, if it is multiplicative, i.e. if $\phi(ab) = \phi(a)\phi(b)$ for any a, b in $\mathcal{A}$. The set of characters of $\mathcal{A}$ is denoted by $\Delta = \Delta(\mathcal{A})$.

Proposition 9. *Let $\mathcal{A}$ be a unital C^*-algebra.*

(i) There is a one-to-one correspondence between the character set Δ of $\mathcal{A}$ and the set $\mathbb{C}J$ of maximal ideals of $\mathcal{A}$ given by

$$\omega \mapsto \{a \in \mathcal{A} \mid \omega(a) = 0\}, \qquad (\omega \in \Delta).$$

In particular Δ is non-empty.

(ii) The character set Δ is compact in the weak topology induced by the family of linear functionals $\{\hat{a} \mid a \in \mathcal{A}\}$ given by

$$\hat{a}(\omega) = \omega(a), \qquad (\omega \in \Delta, a \in \mathcal{A}).$$

(iii) For any element a of $\mathcal{A}$, we have that

$$\mathit{spe}(a) = \{\omega(a) \mid \omega \in \Delta\}.$$

Theorem 10 (Gelfand)**.** *Let $\mathcal{A}$ be a unital and* commutative *C^*-algebra, and let $C(\Delta)$ denote the set of complex valued functions on Δ which are continuous in the topology described in Proposition 9(ii).*

Then the mapping

$$\Gamma\colon a \mapsto \hat{a}\colon \mathcal{A} \to C(\Delta)$$

is a $*$-isomorphism of $\mathcal{A}$ onto $C(\Delta)$.

The mapping $\Gamma\colon \mathcal{A} \to C(\Delta)$ described in Theorem 10 is called *the Gelfand transform*. Gelfand's theorem shows in particular that any unital commutative C^*-algebra may identified with the continuous complex-valued functions on a compact Hausdorff space, and thus the theory of commutative C^*-algebras may be seen as a sub-discipline of classical topology. As a consequence, the theory of general (non-commutative) C^*-algebras is sometimes referred to as "non-commutative topology".

Proof of Theorem 10 It is clear that Γ is a linear mapping, and for a, b in $\mathcal{A}$, and ω in Δ we note further that

$$[\Gamma(ab)](\omega) = \omega(ab) = \omega(a)\omega(b) = [\Gamma(a)\Gamma(b)](\omega),$$

verifying that Γ preserves multiplication as well.

To see that Γ also preserves the adjoint operation, we note first that if $a = a^*$, then it follows from Proposition 5(iv) and Proposition 9(iii) that $\omega(a) \in \mathbb{R}$ for all ω in Δ. For a general element a of $\mathcal{A}$, we may write a as $a = a' + \mathrm{i}\, a''$, where $a' = \frac{1}{2}(a + a^*)$ and $a'' = \frac{1}{2\,\mathrm{i}}(a - a^*)$, which are both selfadjoint elements of $\mathcal{A}$. For any ω in Δ, we may thus conclude that

$$[\Gamma(a^*)](\omega) = \omega(a^*) = \omega(a' - \mathrm{i}\, a'') = \omega(a) - \mathrm{i}\, \omega(a'') = \overline{\omega(a) + \mathrm{i}\, \omega(a'')} = \overline{\omega(a)} = \overline{[\Gamma(a)](\omega)}$$

as desired.

We note next that Γ is an isometry. Indeed, for any a in $\mathcal{A}$ it follows from Proposition 9(iii) and Proposition 5(iii) that

$$\|\Gamma(a)\|^2 = \|\Gamma(a)\overline{\Gamma(a)}\| = \||\hat{a}|^2\| = \sup\{|\hat{a}(\omega)|^2 \mid \omega \in \Delta\} = \sup\{\overline{\omega(a)}\omega(a) \mid \omega \in \Delta\}$$

$$= \sup\{\omega(a^*a) \mid \omega \in \Delta\} = \sup\{|\lambda| \mid \lambda \in \mathrm{spe}(a^*a)\} = \|a^*a\|^2 = \|a\|^2.$$

In particular Γ is injective, and it remains to show that it is surjective as well. We note first that the range $\Gamma(\mathcal{A}) = \{\hat{a} \mid a \in \mathcal{A}\}$ is clearly a $*$-subalgebra of $C(\Delta)$, which separates the points of Δ: If $\omega, \omega' \in \Delta$ such that $\hat{a}(\omega) = \hat{a}(\omega')$ for all a in $\mathcal{A}$, then clearly $\omega = \omega'$. Note finally that $\Gamma(\mathcal{A})$ does not vanish identically at any point of Δ, since $\hat{\mathbf{1}}_{\mathcal{A}}(\omega) = \omega(\mathbf{1}_{\mathcal{A}}) \in \mathrm{spe}(\mathbf{1}_{\mathcal{A}}) = \{1\}$ for any ω in Δ according to Proposition 9(iii). The considerations above together with the Stone-Weierstrass Theorem imply that $\Gamma(\mathcal{A})$ is dense in $C(\Delta)$. At the same time the completeness of $\mathcal{A}$ and the fact that Γ is an isometry imply that $\Gamma(\mathcal{A})$ is complete and hence closed in $C(\Delta)$. Therefore $\Gamma(\mathcal{A}) = C(\Delta)$ as desired, and this completes the proof. ∎

Theorem 11. *Let $\mathcal{A}$ be a unital C^*-algebra, and let a be a normal element of $\mathcal{A}$. Let further $\mathcal{A}_0$ denote the smallest subalgebra of $\mathcal{A}$ containing a and $\mathbf{1}_{\mathcal{A}}$.*

Then there exists a unique $*$-isomomorphism $\Phi_a\colon C(spe(a)) \to \mathcal{A}_0$ from $C(spe(a))$ onto $\mathcal{A}_0$, such that $\Phi_a(\mathrm{id}) = a$ and $\Phi_a(1) = \mathbf{1}_{\mathcal{A}}$, where id *is the identity function on spe(a) and* 1 *denotes the constant function* 1 *on spe(a).*

Proof. Note first that $\mathcal{A}_0$ is the norm closure of the linear span of $\mathbf{1}_{\mathcal{A}}$ and all elements of the form $a^{p_1}(a^*)^{q_1}a^{p_2}(a^*)^{q_2}\cdots a^{p_n}(a^*)^{q_n}$, where $n \in \mathbb{N}$, and $p_1, q_1, \ldots, p_n, q_n \in \{0, 1, 2, \ldots\}$. Since a is normal, we note further that $\mathcal{A}_0$ is abelian, and hence Gelfand's theorem asserts the existence of a $*$-isomorphism Γ from $\mathcal{A}_0$ onto $C(\Delta_0)$, where Δ_0 denotes the character set of $\mathcal{A}_0$. Using Proposition 6 one may verify that the spectrum of a considered as an element of $\mathcal{A}_0$ is the same as that of a considered as an element of $\mathcal{A}$ (denoted $\mathrm{spe}(a)$ in the theorem). Hence Proposition 9(iii) shows that $\mathrm{spe}(a) = \{\omega(a) \mid \omega \in \Delta_0\}$. We note further that a character ω on $\mathcal{A}_0$ is uniquely determined by the number $\omega(a)$, since this number determines the values of ω on the norm dense subspace of $\mathcal{A}_0$ described above. Thus, in this setting, the mapping $\omega \mapsto \omega(a)$ is a bijection between Δ_0 and $\mathrm{spe}(a)$, which is clearly continuous and hence a homeomorphism (since Δ_0 and $\mathrm{spe}(a)$ are compact). From these considerations it follows that we may define a $*$-isomorphism $\Psi\colon C(\mathrm{spe}(a)) \to C(\Delta_0)$ by setting

$$[\Psi(f)](\omega) = f(\omega(a)), \qquad (\omega \in \Delta_0,\ f \in C(\mathrm{spe}(a))).$$

We may subsequently define a $*$-isomorphism $\Phi_a\colon C(\mathrm{spe}(a)) \to \mathcal{A}_0$ by setting $\Phi_a = \Gamma^{-1} \circ \Psi$. It follows then that

$$\Phi_a(1) = \Gamma^{-1}(1) = \mathbf{1}_{\mathcal{A}}.$$

Note further that $\Psi(\mathrm{i}d)$ is the function $\omega \mapsto \omega(a)$ on Δ_0, also known as $\hat{a}$. Therefore

$$\Phi_a(\mathrm{i}d) = \Gamma^{-1}(\hat{a}) = a.$$

To see, finally, that Φ_a is unique, we note that a $*$-isomorphism on $C(\mathrm{spe}(a))$ is uniquely determined by its values on $\mathrm{i}d$ and 1, since the polynomials in $1, \mathrm{i}d$ and $\overline{\mathrm{i}d}$ are dense in $C(\mathrm{spe}(a))$ according to the Stone-Weierstrass Theorem, and since $*$-isomorphisms are automatically norm-continuous. This completes the proof. ■

0.1.2 von Neumann algebras and W^*-probability spaces

The strong and weak operator topologies

Throughout this subsection we consider a fixed Hilbert space $\mathcal{H}$. In the previous subsections we have considered the $*$-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators equipped with the (operator-) norm topology. There are an abundance of other natural (weaker) topologies on $\mathcal{B}(\mathcal{H})$, which all play a significant role in the theory of operator algebras. When $\mathcal{H}$ is finite dimensional these generally coincide with the norm topology. In the present context we shall restrict ourselves to considering the two perhaps most basic topologies on $\mathcal{B}(\mathcal{H})$ (apart from the norm topology).

The strong operator topology: Any element x of $\mathcal{H}$, gives rise to a seminorm N_x on $\mathcal{B}(\mathcal{H})$ defined by

$$N_x(T) = \|Tx\|, \qquad (T \in \mathcal{B}(\mathcal{H})).$$

The strong operator topology is the topology on $\mathcal{B}(\mathcal{H})$ induced by the family $\{N_x \mid x \in \mathcal{H}\}$. Specifically, for any T in $\mathcal{B}(\mathcal{H})$, x in $\mathcal{H}$ and ϵ in $(0, \infty)$ we put

$$U(T, x, \epsilon) := \{S \in \mathcal{B}(\mathcal{H}) \mid N_x(S - T) < \epsilon\} = \{S \in \mathcal{B}(\mathcal{H}) \mid \|Sx - Tx\| < \epsilon\}. \quad (7)$$

Then for any fixed T in $\mathcal{B}(\mathcal{H})$ a neighborhood basis at T in the strong operator topology is given by

$$\Big\{\textstyle\bigcap_{j=1}^{N} U(T, x_j, \epsilon) \;\Big|\; N \in \mathbb{N},\ x_1, \dots, x_N \in \mathcal{H},\ \epsilon > 0\Big\}.$$

Given a net $(T_l)_{l\in\Lambda}$ in $\mathcal{B}(\mathcal{H})$ and another operator T in $\mathcal{B}(\mathcal{H})$, it follows that $T_l \to T$ in the strong operator (written $T_l \overset{\text{so}}{\to} T$), if and only if $\|T_l x - Tx\| \to 0$ for any fixed vector x in $\mathcal{H}$.

Obviously the seminorms N_x, $x \in \mathcal{H}$, separate the elements of $\mathcal{B}(\mathcal{H})$ in the sense that $N_x(T - S) = 0$ for all x in $\mathcal{H}$, if and only if $T = S$. As a consequence, the strong operator topology is in particular a Hausdorff topology. Note also that the vector space operations are continuous in the strong operator topology. If $\mathcal{H}$ is infinite dimensional, the multiplication on $\mathcal{B}(\mathcal{H})$ (considered as a mapping of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$) is not continuous in the strong operator topology. Multiplication is strong operator continuous, however, if one of the arguments (factors) is fixed, or if the left factor is restricted to a bounded subset of $\mathcal{B}(\mathcal{H})$. When $\mathcal{H}$ is infinite dimensional, the adjoint operation is not continuous in the strong operator topology.

The weak operator topology: Any two vectors x, y in $\mathcal{H}$ give rise to a bounded linear functional $\omega_{x,y} \colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by the expression:

$$\omega_{x,y}(T) = \langle Tx, y\rangle, \qquad (T \in \mathcal{B}(\mathcal{H})).$$

The weak operator topology is the weak topology on $\mathcal{B}(\mathcal{H})$ induced by the family $\{\omega_{x,y} \mid x, y \in \mathcal{H}\}$. For T in $\mathcal{B}(\mathcal{H})$, x, y in $\mathcal{H}$ and ϵ in $(0, \infty)$ we put

$$\begin{aligned} V(T, x, y, \epsilon) &= \{S \in \mathcal{B}(\mathcal{H}) \mid |\omega_{x,y}(S - T)| < \epsilon\} \\ &= \{S \in \mathcal{B}(\mathcal{H}) \mid |\langle Sx, y\rangle - \langle Tx, y\rangle| < \epsilon\}. \end{aligned} \quad (8)$$

Then a neighborhood basis at T for the weak operator topology is given by

$$\Big\{\textstyle\bigcap_{j=1}^{N} V(T, x_j, y_j, \epsilon) \;\Big|\; N \in \mathbb{N},\ x_1, y_1 \dots, x_N, y_N \in \mathcal{H},\ \epsilon > 0\Big\}.$$

Given a net $(T_l)_{l\in\Lambda}$ in $\mathcal{B}(\mathcal{H})$ and another operator T in $\mathcal{B}(\mathcal{H})$, it follows that $T_l \to T$ in the weak operator (written $T_l \overset{\text{wo}}{\to} T$), if and only if $\langle T_l x, y\rangle \to \langle Tx, y\rangle$ in $\mathbb{C}$ for any x, y in $\mathcal{H}$. From this it follows immediately that the adjoint operation is continuous in the weak operator topology, and in particular this demonstrates that the weak operator topology and the strong operator topology are distinct, when $\mathcal{H}$ is infinite dimensional. Multiplication is continuous with respect to the weak operator topology, if one of the arguments (factors) is fixed, but not jointly continuous in both arguments (when $\mathcal{H}$ is infinite dimensional).

If $\langle Tx, y\rangle = 0$ for all x, y in $\mathcal{H}$, then $Tx = 0$ for all x in $\mathcal{H}$, i.e. $T = 0$. Thus the weak operator topology is a Hausdorff topology.

Proposition 12. *Let $\mathcal{H}$ be a Hilbert space, and consider the strong operator and weak operator topologies on $\mathcal{B}(\mathcal{H})$.*

(i) The norm topology is stronger than the the strong operator topology which in turn is stronger than the weak operator topology. The three topologies are distinct, unless $\mathcal{H}$ is finite dimensional.

(ii) For any convex subset $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$ the strong operator closure $\mathcal{C}^{-\mathrm{so}}$ and the weak operator closure $\mathcal{C}^{-\mathrm{wo}}$ coincide.

(iii) The closed unit ball $\{T \in \mathcal{B}(\mathcal{H}) \mid \|T\| \leq 1\}$ in $\mathcal{B}(\mathcal{H})$ is compact in the weak operator topology.

Proof. To prove (i), suppose V is a subset of $\mathcal{B}(\mathcal{H})$, which is open in the weak operator topology, and let T be an element of V. Then there exist N in $\mathbb{N}$, $x_1, y_1, \dots, x_N, y_n$ in $\mathcal{H}$ and ϵ in $(0, \infty)$ such that (with notation from (8)) $T \in \bigcap_{j=1}^N V(T, x_j, y_j, \epsilon) \subseteq V$. Choosing a positive δ such that $\delta\|y_j\| \leq \epsilon$ for all j in $\{1, \dots, N\}$, it follows then from the Cauchy-Schwarz inequality that (with notation from (7))

$$T \in \bigcap_{j=1}^N U(T, x_j, \delta) \subseteq \bigcap_{j=1}^N V(T, x_j, y_j, \epsilon) \subseteq V,$$

which verifies that V is open in the strong operator toplogy as well. Next, if γ is chosen in $(0, \infty)$ so small that $\gamma\|x_j\| \leq \delta$ for all j in $\{1, \dots, N\}$, then we have the inclusion

$$\{S \in \mathcal{B}(\mathcal{H}) \mid \|S - T\| < \gamma\} \subseteq \bigcap_{j=1}^N U(T, x_j, \delta),$$

which similarly implies that the norm topology is stronger than the strong operator topology.

If $\mathcal{H}$ is infinite dimensional, then the strong operator and weak operator topologies are distinct, since, as mentioned previously, the adjoint operation is continuous in the latter topology but not in the former. Since the adjoint operation is norm-continuous and the norm topology is (obviously) stronger than the strong operator topology, it follows further that all three topologies are distinct. This completes the proof of (i).

For the proofs of (ii) and (iii) we refer to Theorems 5.1.3 and 5.1.4 in [KR97]. ■

Definition 13. *Let $\mathcal{A}$ be a Hilbert space, and let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, which contains the unit of $\mathcal{B}(\mathcal{H})$. We say then that $\mathcal{A}$ is a* von Neumann algebra*, if $\mathcal{A}$ is closed in the weak operator topology.*

It follows from Proposition 12(i) that a von Neumann algebra is also closed in the norm topology (and is hence a C^*-algebra) and in the strong operator topology. Conversely, any $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, which contains the unit of $\mathcal{B}(\mathcal{H})$ and is closed in the *strong* operator topology is in fact a von Neumann algebra according to Proposition 12(ii).

Example 14. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, and consider the associated Hilbert space $L^2(\mu)$. We shall argue in the following that $L^\infty(\mu)$ may be considered as a von Neumann algebra acting on this Hilbert space.*

Note first that every f in $L^\infty(\mu)$ may be identified with an operator M_f in $\mathcal{B}(L^2(\mu))$ defined as follows:

$$M_f g = fg, \qquad (g \in L^2(\mu)).$$

More precisely our aim then is to show that

$$\mathcal{A} := \{M_f \mid f \in L^\infty(\mu)\}$$

is a von Neumann algebra acting on $L^2(\mu)$. Clearly $\mathcal{A}$ is a $*$-sub-algebra of $\mathcal{B}(\mathcal{H})$, and the problem is, henceforth, to show that $\mathcal{A}$ is closed in the strong operator topology. Consider thus a net (f_ℓ) of functions in $L^\infty(\mu)$, such that $M_{f_\ell} \overset{\text{so}}{\to} T$ for some operator T in $\mathcal{B}(\mathcal{H})$. Since $1_\Omega \in L^2(\mu)$, it follows then in particular that

$$f_\ell = M_{f_\ell} 1_\Omega \longrightarrow T1_\Omega \quad \textit{in} \quad L^2(\mu).$$

Putting $f := T1_\Omega \in L^2(\mu)$, we want to show that $f \in L^\infty(\mu)$, and that $T = M_f$. For g in $L^\infty(\mu) \subseteq L^2(\mu)$ we first note that

$$Tg = \lim_\ell M_{f_\ell} g = \lim_\ell f_\ell g = \lim_\ell M_g f_\ell = M_g f = gf, \tag{9}$$

since $f_\ell \to f$ in $L^2(\mu)$. We then put

$$\|f\|_\infty = \textit{ess sup}(f) = \inf\{K \geq 0 \mid |f| \leq K \ \mu\text{-}\textit{n.o.}\}.$$

Assume for a moment that $\|f\|_\infty > 0$. For arbitrary a in $[0, \|f\|_\infty)$ we may then consider the function:

$$g_a = (\mu(\{|f| > a\})^{-1/2} 1_{\{|f|>a\}} \in L^\infty(\mu) \subseteq L^2(\mu).$$

Since $\|g_a\|_2 = 1$, it follows from (9) that

$$\|T\|^2 \geq \|Tg_a\|_2^2 = \|fg_a\|_2^2 = \int_\Omega |fg_a|^2 \, \mathrm{d}\mu = (\mu(\{|f| > a\})^{-1} \int_\Omega |f|^2 1_{\{|f|>a\}} \, \mathrm{d}\mu$$
$$\geq (\mu(\{|f| > a\})^{-1} \int_\Omega a^2 1_{\{|f|>a\}} \, \mathrm{d}\mu = a^2.$$

Since a was chosen arbitrarily in $[0, \|f\|_\infty)$, it follows that $\|f\|_\infty \leq \|T\|$, and in particular $f \in L^\infty(\mu)$. Obviously these conclusions remain valid if $\|f\|_\infty = 0$. The calculation (9) shows subsequently that

$$Tg = M_f g, \qquad \textit{for all } g \textit{ in } L^\infty(\mu).$$

Since $L^\infty(\mu)$ is dense in $L^2(\mu)$, and since T and M_f are both bounded, we may thus conclude that $M_f = T$, as desired.

Remark 15. *As emphasized before, Gelfand's theorem (Theorem 10) shows in particular that any commutative C^*-algebra my be identified with $C(\Delta)$ for some compact Hausdorff space Δ. The example above shows that any finite measure space gives rise to the commutative von Neumann algebra $L^\infty(\mu)$ (considered as a set of multiplication operators), and in fact one may show that any commutative von Neumann algebra acting on a separable Hilbert space is $*$-isomorphic to $L^\infty(\mu)$ for some finite measure space $(\Omega, \mathcal{F}, \mu)$ (see e.g. [Zhu, Theorem 22.6]). For that reason the theory of general (non-commutative) von Neumann algebras is sometimes referred to as "non-commutative measure theory". The passage from $C(\Delta)$ to $L^\infty(\mu)$ gives a usefull intuitive idea of the difference between C^*-algebras and von Neumann algebras. More specifically, any measurable subset A of Ω, such that $0 < \mu(A) < \mu(\Omega)$, gives rise to the non-trivial projection 1_A in $L^\infty(\mu)$, and so $L^\infty(\mu)$ generally contains an abundance of projections. By contrast, a C^*-algebra may very well contain no non-trivial projections; for example $C_0(\mathbb{R})$ (cf. Example 2).*

We end this section by presenting two of the most fundamental theorems on von Neumann algebras: von Neumann's double commutant theorem and Kaplansky's density theorem. For proofs of these results we refer to [KR97, Section 5.3]. In order to state the double commutant theorem, we introduce for any subset M of $\mathcal{B}(\mathcal{H})$ its *commutant* M' defined by

$$M' = \{S \in \mathcal{B}(\mathcal{H}) \mid ST = TS \text{ for all } T \text{ in } M\}.$$

Note in particular that M is always contained in its own double commutant: $M \subseteq (M')'$.

Theorem 16 (Double Commutant Theorem)**.** *Let $\mathcal{H}$ be a Hibert space, and let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, which contains the unit of $\mathcal{B}(\mathcal{H})$.*

Then the closures of $\mathcal{A}$ in the weak operator and strong operator topologies both coincide with the double commutant of $\mathcal{A}$:

$$\mathcal{A}^{-\mathrm{wo}} = \mathcal{A}^{-\mathrm{so}} = (\mathcal{A}')'.$$

In particular Theorem 16 implies that a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, containing the unit of $\mathcal{B}(\mathcal{H})$, is a von Neumann algebra, if and only if it equals its own double commutant. Moreoever, for any (non-empty) subset M of $\mathcal{B}(\mathcal{H})$, which is closed under the adjoint operation, the smallest von Neumann algebra in $\mathcal{B}(\mathcal{H})$ containing M is the double commutant $(M')'$. Indeed, it is easy to check directly from the definition that M' is a von Neumann algebra, and hence $(M')'$ is a von Neumann algebra as well, which obviously contains M. And if $\mathcal{A}$ is any von Neumann algebra in $\mathcal{B}(\mathcal{H})$ containing M, we have that $\mathcal{A}' \subseteq M'$ and hence $(M')' \subseteq (\mathcal{A}')' = \mathcal{A}^{-\mathrm{wo}} = \mathcal{A}$.

Theorem 17 (Kaplansky's Densisty Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the closed unit ball of $\mathcal{A}^{-\mathrm{so}}$ coincides with the strong operator closure of the closed unit ball of $\mathcal{A}$:*

$$\{T \in \mathcal{A}^{-\mathrm{so}} \mid \|T\| \leq 1\} = \{T \in \mathcal{A} \mid \|T\| \leq 1\}^{-\mathrm{so}}.$$

In view of Proposition 12(ii) Kaplansky's density theorem holds, verbatim, if the appearing closures in the strong operator topology are replaced by closures in the weak operator topology.

Spectral theory for unbounded operators

Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, and let $\mathcal{A}$ be a state on $\mathcal{A}$. In Corollary 7 we saw that any selfadjoint operator T in $\mathcal{A}$ gives rise to a Borel-probability measure μ_T (the spectral distribution of T) concentrated on the spectrum $\mathrm{spe}(T)$. In particular μ_T is compactly supported. Conversely, given any compactly supported Borel-probability measure μ on $\mathbb{R}$, we may equip the C^*-algebra $C(\mathrm{supp}(\mu))$ with the state ϕ given by

$$\phi(f) = \int_{\mathrm{supp}(\mu)} f(x)\,\mu(\mathrm{d}x), \qquad (f \in C(\mathrm{supp}(\mu)).$$

Then μ may be identified as the spectral distribution of the function $\mathrm{i}d(x) = x$ with respect to ϕ. In order to express in operator terms e.g. the additive free addtitive convolution of arbitrary (not necessarilly compactly supported) probability measures, it is necessary to have the ability to interpret an arbitrary probability measure μ on $\mathbb{R}$ as the spectral distribution of some selfadjoint Hilbert space operator T . From the foregoing discussion, T cannot be bounded, if μ has unbounded support, and hence we are forced to consider unbounded operators in $\mathcal{H}$, by which we mean arbitrary linear mappings $T\colon \mathcal{D}(T) \to \mathcal{H}$ defined on some subspace $\mathcal{D}(T)$ (the domain of T) of $\mathcal{H}$. For such linear mappings T and S we say that

- T is *densely defined*, if $\mathcal{D}(T)$ is dense in $\mathcal{H}$,
- T is *closed*, if the graph $\mathcal{G}(T) = \{(h, Th) \mid h \in \mathcal{D}(T)\}$ of T is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$,
- a is *preclosed*, if the norm closure $\overline{\mathcal{G}(a)}$ is the graph of a (uniquely determined) operator, denoted $[a]$, in $\mathcal{H}$,
- a is *affiliated with* $\mathcal{A}$, if $au = ua$ for any unitary operator u in the commutant $\mathcal{A}'$,
- T is contained in S, written $T \subseteq S$, if $\mathbb{G}(T) \subseteq \mathbb{G}(S)$.

For a densely defined operator T in $\mathcal{H}$, the *adjoint operator* T^* has domain

$$\mathcal{D}(T^*) = \Big\{\eta \in \mathcal{H} \;\Big|\; \sup\{|\langle T\xi, \eta\rangle| \mid \xi \in \mathcal{D}(T),\ \|\xi\| \le 1\} < \infty\Big\},$$

and is determined by the condition

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle, \qquad (\xi \in \mathcal{D}(T),\ \eta \in \mathcal{D}(T^*)).$$

We say that T is *selfadjoint* if $T = T^*$ (in particular this requires that $\mathcal{D}(T^*) = \mathcal{D}(T)$).

In order to introduce spectral distributions of unbounded selfadjoint operators, we need an analog of Theorem 6 for unbounded operators. To obtain this we first introduce resolutions of the identity.

Definition 18 (Resolutions of the identity)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathcal{H}$ be a Hilbert space. An ($\mathcal{F}$-)resolution of the identity (in $\mathcal{B}(\mathcal{H})$) is a mapping $E\colon \mathcal{F} \to \mathcal{B}(\mathcal{H})$ with the following properties:*

(R1) $E(\emptyset) = 0$, *and* $E(\Omega) = \mathbf{1}_{\mathcal{B}(\mathcal{H})}$.

(R2) For any set M from $\mathcal{F}$ the operator $E(M)$ is an orthogonal projection in $\mathcal{B}(\mathcal{H})$.

(R3) $E(M \cap M') = E(M)E(M')$ *for any sets M, M' from $\mathcal{F}$.*

(R4) For any x in $\mathcal{H}$ the set-function $E_x\colon \mathcal{F} \to [0, \|x\|^2]$ defined by

$$E_x(M) = \langle E(M)x, x\rangle, \qquad (M \in \mathcal{F})$$

is a measure on $(\Omega, \mathcal{F})$.

Note that condition (R4) Definition 18 further ensures that the formula

$$E_{x,y}(M) = \langle E(M)x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \langle E(M)(x+\mathrm{i}^k\, y), x+\mathrm{i}^k\, y\rangle = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k\, E_{x+\mathrm{i}^k\, y}(M), \quad (M \in \mathcal{F}),$$

defines a complex measure on $(\Omega, \mathcal{F})$. The conditions (R1)-(R4) additionally imply that an $\mathcal{F}$-resolution E of the identity has the following properties:

(R5) $E(M \cup M') = E(M) + E(M')$, when $M, M' \in \mathcal{F}$ and $M \cap M' = \emptyset$.

(R6) If M, M' are sets from $\mathcal{F}$ such that $M \subseteq M'$, then $E(M) \le E(M')$ in the sense that

$$\langle E(M)x, x\rangle \le \langle E(M')x, x\rangle, \quad \text{for all } x \text{ in } \mathcal{H}.$$

In particular $E(M) = 0$, if $M \subseteq M'$ and $E(M') = 0$.

(R7) If $(M_n) \subseteq \mathcal{F}$ such that $E(M_n) = 0$ for all n, then also $E(\bigcup_{n\in\mathbb{N}} M_n) = 0$.

(R8) If (M_n) is a sequence of disjoint sets from $\mathcal{F}$, then

$$E\big(\textstyle\bigcup_{n\in\mathbb{N}} E_n\big) = \lim_{N\to\infty} \displaystyle\sum_{n=1}^{N} E(M_n) =: \sum_{n=1}^{\infty} E(M_n),$$

where the limit is in the strong operator topology on $\mathcal{B}(\mathcal{H})$.

For any measurable function $f\colon \Omega \to \mathbb{C}$ we put (cf. property (R4))

$$\mathcal{D}_f = \{x \in \mathcal{H} \mid f \in \mathbb{L}^2(E_x)\} = \big\{x \in \mathcal{H} \;\big|\; \textstyle\int_\Omega |f|^2\, \mathrm{d}E_x < \infty\big\}.$$

It turns out that $\mathcal{D}_f$ is always a dense susbspace of $\mathcal{H}$. This is (implicitly) part of the following theorem, which shows how one may develop an integration theory with respect to a resolution of the identity.

Theorem 19. *Let $(\Omega, \mathcal{F})$ be a measurable space, let $\mathcal{H}$ be a Hilbert space, and let E be an $\mathcal{F}$-resolution of the identity. For any measurable function $f\colon \Omega \to \mathbb{C}$ there exists a (generally unbounded) densely defined operator $\Psi(f)$ in $\mathcal{H}$ with the following properties:*

(i) The domain $\mathcal{D}(\Psi(f))$ of $\Psi(f)$ equals $\mathcal{D}_f$.

(ii) For any x in $\mathcal{D}_f$ it holds that

$$\langle \Psi(f)x, y\rangle = \int_\Omega f \,\mathrm{d}E_{x,y} \quad \text{for all } y \text{ in } \mathcal{H}.$$

(iii) $\Psi(f)^* = \Psi(\overline{f}), \quad$ *and* $\quad \Psi(f)^*\Psi(f) = \Psi(|f|^2) = \Psi(f)\Psi(f)^*.$

(iv) If $g\colon \Omega \to \mathbb{C}$ is another measurable function, it holds that

$$\Psi(f)\Psi(g) \subseteq \Psi(fg), \quad \text{and} \quad \mathcal{D}(\Psi(f)\Psi(g)) = \mathcal{D}_g \cap \mathcal{D}_{fg}.$$

The operator $\Psi(f)$ described in Theorem 19 is often denoted by $\int_\Omega f\,\mathrm{d}E$ or $\int_\Omega f(\omega)\,E(\mathrm{d}\omega)$. For the proof of the theorem we refer to [Ru91, Theorem 13.24]. To illustrate a key aspect of the proof we restrict ourselves to considering a non-negative, bounded measurable function f. In this case it holds in particular that $\mathcal{D}_f = \mathcal{H}$. One may then observe that the formula

$$B(x,y) = \int_\Omega f\,\mathrm{d}E_{x,y} \qquad (x, y \in \mathcal{H}),$$

defines a bounded sesquilinear form on $\mathcal{H}$. It is well-known (see e.g. [Ru91, Theorem 12.8]) that any such sesquilinear form is given in the form $(x,y) \mapsto \langle Tx, y\rangle$, where T is a uniquely determined operator in $\mathcal{B}(\mathcal{H})$. We can then define $\Psi(f) = T$ in the considered situation. The remaining part of the proof essentially amounts to approximation arguments.

A (possibly unbounded) operator T in a Hilbert space $\mathcal{H}$ is called *invertible*, if there exists an operator S in $\mathcal{B}(\mathcal{H})$, such that

$$ST \subseteq TS = \mathbf{1}_{\mathcal{B}}.$$

In particular this condition entails that T be injective. The spectrum $\mathrm{spe}(T)$ of T is subsequently defined as

$$\mathrm{spe}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}.$$

The analog of Theorem 6 for unbounded operators may now be stated as follows:

Theorem 20. *Let T be a selfadjoint (possibly unbounded) operator in a Hilbert space $\mathcal{H}$. Then there exists a unique resolution E^T of the identity defined on the measure space $(spe(T), \mathcal{B}(spe(T)))$ such that*

$$\int_{spe(T)} t\,E^T(\mathrm{d}t) = T,$$

or equivalently

$$\int_{spe(T)} t\, E^T_{x,y}(\mathrm{d}t) = \langle Tx, y\rangle \quad \textit{for all } x \textit{ in } \mathcal{D}(T) \textit{ and } y \textit{ in } \mathcal{H}.$$

For the proof of Theorem 20 we refer to [Ru91, Theorem 13.30]. For a selfadjoint operator T in $\mathcal{H}$, it follows from Theorem 19 that the mapping

$$\Psi_T(f) = \int_{\mathrm{spe}(T)} f(t)\, E^T(\mathrm{d}t), \tag{10}$$

defined for all Borel-functions f, has properties similar to those of the mapping Φ_T, described in Theorem 6, when T is an operator in $\mathcal{B}(\mathcal{H})$. In this case, the mapping Ψ_T extends Φ_T from continuous functions to general Borel functions. One often writes $f(T)$ instead of $\Psi_T(f)$. If f is a bounded function, it follows from Theorem 19(ii) that $f(T) \in \mathcal{B}(\mathcal{H})$. One may further verify that T is affiliated with $\mathcal{A}$, if and only if $f(T) \in \mathcal{A}$ for any *bounded* Borel-function $f\colon \mathrm{spe}(T) \to \mathbb{C}$.

Definition 21. *Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.*

(a) A state $\phi\colon \mathcal{A} \to \mathbb{C}$ is called normal*, if the restriction of ϕ to the closed unit ball $\{T \in \mathcal{A} \mid \|T\| \le 1\}$ of $\mathcal{A}$ is continuous in the trace topology induced by the weak operator topology on $\mathcal{B}(\mathcal{H})$.*

(b) A state $\tau\colon \mathcal{A} \to \mathbb{C}$ is called a trace *or a* tracial state*, if it satisfies that $\tau(ab) = \tau(ba)$ for all elements a, b of $\mathcal{A}$.*

*(c) For any normal, faithful trace $\tau\colon \mathcal{A} \to \mathbb{C}$, the pair $(\mathcal{A}, \tau)$ is called a W^*-*probability space

Proposition 22. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space, and let T be a selfadjoint operator in $\mathcal{H}$ affiliated with $\mathcal{A}$. Then there exists a unique probability measure μ_T on $(spe(T), \mathcal{B}(spe(T)))$ such that*

$$\tau(\Psi_T(f)) = \tau\Big(\int_{spe(T)} f\, \mathrm{d}E_T\Big) = \int_{spe(T)} f\, \mathrm{d}\mu_T \tag{11}$$

for any bounded Borel function $f\colon spe(T) \to \mathbb{C}$.

Proof. For any Borel subset M of $\mathrm{spe}(T)$, we put

$$\mu_T(M) = \tau(E^T(M)),$$

where E^T is the resolution of the identity described in Theorem 20. If (M_n) is a sequence of disjoint Borel-subsets of $\mathrm{spe}(T)$, it follows from (R8) that $E^T(\bigcup_{n\in\mathbb{N}} M_n) = \sum_{n=1}^{\infty} E^T(M_n)$, where the series converges in the strong and hence in the weak operator topology. Since τ is normal, this implies that

$$\mu_T\Big(\bigcup_{n\in\mathbb{N}} M_n\Big) = \sum_{n=1}^{\infty} \mu_T(M_n),$$

so that μ_T is a Borel measure on $\mathrm{spe}(T)$.

The formula (11) subsequently follows immediately, if f is a simple function (i.e. a finite linear combination of indicator functions for Borel sets). For a general bounded Borel function $f\colon \mathrm{spe}(T) \to \mathbb{C}$, we choose a sequence (f_n) of simple functions such that $f_n \to f$ pointwise as $n \to \infty$, and such that $|f_n| \le |f|$ for all n. By linearity we may assume that $|f| \le 1$. Then by Theorem 19(ii) and dominated convergence it follows that $\Psi_T(f_n) \to \Psi_T(f)$ in the weak operator topology, and since τ is normal, we may thus conclude that

$$\int_{\mathrm{spe}(T)} f \, \mathrm{d}\mu_T = \lim_{n\to\infty} \int_{\mathrm{spe}(T)} f_n \, \mathrm{d}\mu_T = \lim_{n\to\infty} \tau(\Psi_T(f_n)) = \tau(\Psi_T(f)),$$

which completes the proof. ■

The probability measure μ_T described in Proposition 22 is called the spectral distribution of T with respect to τ. Conversely, any Borel probability measure μ on $\mathbb{R}$ may be considered as the spectral distribution of some unbounded selfadjoint operator (as claimed previously). Indeed, given μ we may consider the von Neumann algebra $L^\infty(\mu)$ acting on the Hilbert space $L^2(\mu)$ as described in Example 14. Consider further the operator T in $L^2(\mu)$ defined by

$$[Tf](t) = tf(t), \qquad (t \in \mathbb{R}),$$

with domain consisting of those functions f in $L^2(\mu)$ for which the function $t \mapsto tf(t)$ is again in $L^2(\mu)$. Then T is selfadjoint and $\mathrm{spe}(T) = \mathrm{supp}(\mu)$. For any bounded Borel function $f\colon \mathbb{R} \to \mathbb{C}$, $\Psi_T(f)$ is the operator M_f described in Example 14, so in particular T is affiliated with $L^\infty(\mu)$. If we define

$$\tau(f) = \int_{\mathbb{R}} f \, \mathrm{d}\mu, \qquad (f \in L^\infty(\mu)),$$

then τ is a normal trace on $L^\infty(\mu)$, and $\tau(\Psi_T(f)) = \int_{\mathbb{R}} f \, \mathrm{d}\mu$ for any bounded Borel function f on $\mathbb{R}$. Hence μ is the spectral distribution of T, as desired.

The $*$-algebra of operators affiliated with a von Neumann algebra

In the following we consider throughout a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$. By $\overline{\mathcal{A}}$ we denote the set of closed, densely defined operators in $\mathcal{H}$, which are affiliated with $\mathcal{A}$. In general, dealing with unbounded operators is somewhat unpleasant, compared to the bounded case, since one needs constantly to take the domains into account. However, the following two important propositions allow us to deal with operators in $\overline{\mathcal{A}}$ in a quite relaxed manner.

Proposition 23 (cf. [Ne74]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. If $a, b \in \overline{\mathcal{A}}$, then $a + b$ and ab are densely defined, preclosed operators affiliated with $\mathcal{A}$, and their closures $[a + b]$ and $[ab]$ belong to $\overline{\mathcal{A}}$. Furthermore, $a^* \in \overline{\mathcal{A}}$.*

By virtue of the proposition above, the adjoint operation may be restricted to an involution on $\overline{\mathcal{A}}$, and we may define operations, the *strong sum* and the *strong product*, on $\overline{\mathcal{A}}$, as follows:

$$(a, b) \mapsto [a + b], \quad \text{and} \quad (a, b) \mapsto [ab], \quad (a, b \in \overline{\mathcal{A}}).$$

Proposition 24 (cf. [Ne74]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Equipped with the adjoint operation and the strong sum and product, $\overline{\mathcal{A}}$ is a $*$-algebra.*

The effect of the above proposition is, that w.r.t. the adjoint operation and the strong sum and product, we can manipulate with operators in $\overline{\mathcal{A}}$, without worrying about domains etc. So, for example, we have rules like

$$[[a+b]c] = [[ac]+[bc]], \quad [a+b]^* = [a^* + b^*], \quad [ab]^* = [b^* a^*],$$

for operators a, b, c in $\overline{\mathcal{A}}$. Note, in particular, that the strong sum of two selfadjoint operators in $\overline{\mathcal{A}}$ is again a selfadjoint operator. In the following, we shall omit the brackets in the notation for the strong sum and product, and it will be understood that all sums and products are formed in the strong sense.

Remark 25. *If $a_1, a_2 \dots, a_r$ are selfadjoint operators in $\overline{\mathcal{A}}$, we say that $a_1, a_2, \dots, a_r$ are freely independent if, for any bounded Borel functions $f_1, f_2, \dots, f_r \colon \mathbb{R} \to \mathbb{R}$, the bounded operators $f_1(a_1), f_2(a_2), \dots, f_r(a_r)$ in $\mathcal{A}$ are freely independent. Given any two probability measures μ_1 and μ_2 on $\mathbb{R}$, it follows from a free product construction (see [VoDyNi92]), that one can always find a W^*-probability space $(\mathcal{A}, \tau)$ and selfadjoint operators a and b affiliated with $\mathcal{A}$, such that $\mu_1 = L\{a\}$ and $\mu_2 = L\{b\}$. As noted above, for such operators $a + b$ is again a selfadjoint operator in $\overline{\mathcal{A}}$, and, as was proved in [BeVo93, Theorem 4.6], the (spectral) distribution $L\{a + b\}$ depends only on μ_1 and μ_2. We may thus define the free additive convolution $\mu_1 \boxplus \mu_2$ of μ_1 and μ_2 to be $L\{a + b\}$.*

Next, we shall equip $\overline{\mathcal{A}}$ with a topology; the so called measure topology, which was introduced by Segal in [Se53] and later studied by Nelson in [Ne74]. For any positive numbers ϵ, δ, we denote by $N(\epsilon, \delta)$ the set of operators a in $\overline{\mathcal{A}}$, for which there exists an orthogonal projection p in $\mathcal{A}$, satisfying that

$$p(\mathcal{H}) \subseteq \mathcal{D}(a), \quad \|ap\| \leq \epsilon \quad \text{and} \quad \tau(p) \geq 1 - \delta. \tag{12}$$

Definition 26. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. The measure topology on $\overline{\mathcal{A}}$ is the vector space topology on $\overline{\mathcal{A}}$ for which the sets $N(\epsilon, \delta)$, $\epsilon, \delta > 0$, form a neighbourhood basis for 0.*

It is clear from the definition of the sets $N(\epsilon, \delta)$ that the measure topology satisfies the first axiom of countability. In particular, all convergence statements can be expressed in terms of sequences rather than nets.

Proposition 27 (cf. [Ne74]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and consider the $*$-algebra $\overline{\mathcal{A}}$. We then have*

(i) Scalar-multiplication, the adjoint operation and strong sum and product are all continuous operations w.r.t. the measure topology. Thus, $\overline{\mathcal{A}}$ is a topological $*$-algebra w.r.t. the measure topology.

(ii) The measure topology on $\overline{\mathcal{A}}$ is a complete Hausdorff topology.

We shall note, next, that the measure topology on $\overline{\mathcal{A}}$ is, in fact, the topology for convergence in probability. Recall first, that for a closed, densely defined operator a in $\mathcal{H}$, we put $|a| = (a^*a)^{1/2}$. In particular, if $a \in \overline{\mathcal{A}}$, then $|a|$ is a selfadjoint operator in $\overline{\mathcal{A}}$ (see [KR97, Theorem 6.1.11]), and we may consider the probability measure $L\{|a|\}$ on $\mathbb{R}$.

Definition 28. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. We say then that $a_n \to a$ in probability, as $n \to \infty$, if $|a_n - a| \to 0$ in distribution, i.e. if $L\{|a_n - a|\} \to \delta_0$ weakly.*

If a and a_n, $n \in \mathbb{N}$, are *selfadjoint* operators in $\overline{\mathcal{A}}$, then, as noted above, $a_n - a$ is selfadjoint for each n, and $L\{|a_n - a|\}$ is the transformation of $L\{a_n - a\}$ by the mapping $t \mapsto |t|$, $t \in \mathbb{R}$. In this case, it follows thus that $a_n \to a$ in probability, if and only if $a_n - a \to 0$ in distribution, i.e. if and only if $L\{a_n - a\} \to \delta_0$ weakly.

From the definition of $L\{|a_n - a|\}$, it follows immediately that we have the following characterization of convergence in probability:

Lemma 29. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let a and a_n, $n \in \mathbb{N}$, be operators in $\overline{\mathcal{A}}$. Then $a_n \to a$ in probability, if and only if*

$$\forall \epsilon > 0\colon \tau\big[1_{]\epsilon,\infty[}(|a_n - a|)\big] \to 0, \quad \text{as } n \to \infty.$$

Proposition 30 (cf. [Te81])**.** *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Then for any positive numbers ϵ, δ, we have*

$$N(\epsilon, \delta) = \big\{a \in \overline{\mathcal{A}} \bigm| \tau\big[1_{]\epsilon,\infty[}(|a|)\big] \le \delta\big\}, \tag{13}$$

where $N(\epsilon, \delta)$ is defined via (12). In particular, a sequence a_n in $\overline{\mathcal{A}}$ converges, in the measure topology, to an operator a in $\overline{\mathcal{A}}$, if and only if $a_n \to a$ in probability.

Proof. The last statement of the proposition follows immediately from formula (13) and Lemma 29. To prove (13), note first that by considering the polar decomposition of an operator a in $\overline{\mathcal{A}}$ (cf. [KR97, Theorem 6.1.11]), it follows that $N(\epsilon, \delta) = \{a \in \overline{\mathcal{A}} \mid |a| \in N(\epsilon, \delta)\}$. From this, the inclusion $\supseteq$ in (13) follows easily. Regarding the reverse inclusion, suppose $a \in N(\epsilon, \delta)$, and let p be a projection in $\mathcal{A}$, such that (12) is satisfied with a replaced by $|a|$. Then, using spectral theory, it can be shown that the ranges of the projections p and $1_{]\epsilon,\infty[}(|a|)$ only have 0 in common. This implies that $\tau[1_{]\epsilon,\infty[}(|a|)] \le \tau(\mathbf{1} - p) \le \delta$. We refer to [Te81] for further details. ∎

Finally, we shall need the fact that convergence in probability implies convergence in distribution, also in the non-commutative setting. The key point in the proof given below is that weak convergence can be expressed in terms of the Cauchy transform (cf. [Ma92, Theorem 2.5]).

Proposition 31. *Let (a_n) be a sequence of selfadjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, and assume that a_n converges in probability, as $n \to \infty$, to a selfadjoint operator a affiliated with $(\mathcal{A}, \tau)$. Then $a_n \to a$ in distribution too, i.e. $L\{a_n\} \xrightarrow{\text{w}} L\{a\}$, as $n \to \infty$.*

Proof. Let x, y be real numbers such that $y > 0$, and put $z = x + \mathrm{i}y$. Then define the function $f_z\colon \mathbb{R} \to \mathbb{C}$ by

$$f_z(t) = \frac{1}{t-z} = \frac{1}{(t-x)-\mathrm{i}y}, \quad (t \in \mathbb{R}),$$

and note that f_z is continuous and bounded with $\sup_{t\in\mathbb{R}} |f_z(t)| = y^{-1}$. Thus, we may consider the bounded operators $f_z(a_n), f_z(a) \in \mathcal{A}$. Note then that (using strong products and sums),

$$\begin{aligned} f_z(a_n) - f_z(a) &= (a_n - z\mathbf{1})^{-1} - (a - z\mathbf{1})^{-1} \\ &= (a_n - z\mathbf{1})^{-1}\big((a - z\mathbf{1}) - (a_n - z\mathbf{1})\big)(a - z\mathbf{1})^{-1} \qquad (14) \\ &= (a_n - z\mathbf{1})^{-1}(a - a_n)(a - z\mathbf{1})^{-1}. \end{aligned}$$

Now, given any positive numbers ϵ, δ, we may choose N in $\mathbb{N}$, such that $a_n - a \in N(\epsilon, \delta)$, whenever $n \geq N$. Moreover, since $\|f_z(a_n)\|, \|f_z(a)\| \leq y^{-1}$, we have that $f_z(a_n), f_z(a) \in N(y^{-1}, 0)$. Using then the rule: $N(\epsilon_1, \delta_1)N(\epsilon_2, \delta_2) \subseteq N(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$, which holds for all ϵ_1, ϵ_2 in $]0, \infty[$ and δ_1, δ_2 in $[0, \infty[$ (see [Ne74, Formula 17']), it follows from (14) that $f_z(a_n) - f_z(a) \in N(\epsilon y^{-2}, \delta)$, whenever $n \geq N$. We may thus conclude that $f_z(a_n) \to f_z(a)$ in the measure topology, i.e. that $L\{|f_z(a_n) - f_z(a)|\} \xrightarrow{\mathrm{w}} \delta_0$, as $n \to \infty$. Using now the Cauchy-Schwarz inequality for τ, it follows that

$$\big|\tau(f_z(a_n) - f_z(a))\big|^2 \leq \tau(|f_z(a_n) - f_z(a)|^2)\cdot\tau(\mathbf{1}) = \int_0^\infty t^2 \, L\{|f_z(a_n) - f_z(a)|\}(\mathrm{d}t) \to 0,$$

as $n \to \infty$, since $\mathrm{supp}(L\{|f_z(a_n) - f_z(a)|\}) \subseteq [0, 2y^{-1}]$ for all n, and since $t \mapsto t^2$ is a continuous bounded function on $[0, 2y^{-1}]$.

Finally, let G_n and G denote the Cauchy transforms for $L\{a_n\}$ and $L\{a\}$ respectively. From what we have established above, it follows then that

$$G_n(z) = -\tau(f_z(a_n)) \longrightarrow -\tau(f_z(a)) = G(z), \quad \text{as } n \to \infty,$$

for any complex number $z = x + \mathrm{i}y$ for which $y > 0$. By [Ma92, Theorem 2.5], this means that $L\{a_n\} \xrightarrow{\mathrm{w}} L\{a\}$, as desired. ■

Conditional expectations in operator algebras

Definition 32. *Let $\mathcal{A}$ be an algebra with unit $\mathbf{1}$, and let $\mathcal{B}$ be a sub-algebra of $\mathcal{A}$, such that $\mathbf{1} \in \mathcal{B}$. Let further $\mathbb{E}_{\mathcal{B}}\colon \mathcal{A} \to \mathcal{B}$ be a linear mapping, and consider the two conditions:*

(a) $\mathbb{E}_{\mathcal{A}}(b) = b$ for all b in $\mathcal{B}$.

(b) $\mathbb{E}_{\mathcal{B}}(b_1 a b_2) = b_1\mathbb{E}_{\mathcal{B}}(a)b_2$ for any a in $\mathcal{A}$ and b_1, b_2 in $\mathcal{B}$.

If $\mathbb{E}_{\mathcal{B}}$ satisfies condition (a), then it is called a projection *of $\mathcal{A}$ onto $\mathcal{B}$, and if $\mathbb{E}_{\mathcal{B}}$ satisfies both (a) and (b), then it is called a* conditional expectation *of $\mathcal{A}$ onto $\mathcal{B}$.*

In the C^*-algebra setting, i.e. when $\mathcal{B}$ is a unital C^*-subalgebra of a unital C^*-algebra $\mathcal{A}$, one generally further requires that conditional expectations be *contractive* (with respect to the C^*-norm), i.e. that

$$\|\mathbb{E}_{\mathcal{B}}(a)\| \leq \|a\| \quad \text{for all } a \text{ in } \mathcal{A}.$$

In this context we have the following theorem due to J. Tomiyama:

Theorem 33. *Let $\mathcal{A}$ be a C^*-algebra with unit $\mathbf{1}$, and let $\mathcal{B}$ be a C^*-sub-algebra of $\mathcal{A}$, such that $\mathbf{1} \in \mathcal{B}$.*

Then any contractive projection $\mathbb{E}_{\mathcal{A}}$ from $\mathcal{A}$ onto $\mathcal{B}$ is automatically a conditional expectation.

For a proof of Theorem 33 we refer to [BO08, Theorem 1.5.10], where the theorem is proved even for non-unital C^*-algebras, and where it is proved additionally that a contractive projection from a C^*-algebra $\mathcal{A}$ onto a C^*-subalgebra $\mathcal{B}$ is automatically *completely positive*.

In the context of W^*-probability spaces, we have the following fundamental result (see [BO08, Lemma 1.5.11]).

Theorem 34. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space, and let $\mathcal{B}$ be a von Neumann sub-algebra of $\mathcal{A}$. Then there exists a contractive conditional expectation $\mathbb{E}_{\mathcal{B}} \colon \mathcal{A} \to \mathcal{B}$, which further satisfies the following two conditions:*

(i) $\mathbb{E}_{\mathcal{B}}$ is normal (i.e. continuous with respect to the ultra-weak topology on $\mathcal{A}$).

(ii) $\mathbb{E}_{\mathcal{B}}$ is trace-preserving, i.e. $\tau(\mathbb{E}_{\mathcal{B}}(a)) = \tau(a)$ for any a in $\mathcal{A}$.

0.1.3 Free products of C^*-probability spaces and W^*-probability spaces

In this subsection we establish, loosely speaking, that any family $(\mathcal{A}_i, \tau_i)_{i\in I}$ of C^*-probability spaces can always be embedded into one C^*-probability space $(\mathcal{A}, \tau)$, in such a way that the $\mathcal{A}_i$'s form a free family of subalgebras of $\mathcal{A}$ with respect to τ. The C^*-probability space $(\mathcal{A}, \tau)$ is referred to as the reduced free product of the $(\mathcal{A}_i, \tau_i)$, $i \in I$. The word "reduced" is included to distinguish $\mathcal{A}$ from the universal free product of the $\mathcal{A}_i$'s in the category of unital C^*-algebras. At the end of this subsection, we briefly discuss a similar construction in the case where the C^*-probability spaces $(\mathcal{A}_i, \tau_i)$ are actually W^*-probability spaces, in which case the reduced free product may also be defined as a W^*-probability space.

The C^*-algebra $\mathcal{A}$ described above is constructed as a subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space free product of the GNS-spaces associated to the family $(\mathcal{A}_i, \tau_i)_{i\in I}$. We start thus by giving the formal definition of the free product of Hilbert spaces. This definition involves the notion of tensor products of Hilbert spaces for which we refer to [KR97, Section 2.6].

Definition 35. *Let I be an index set, and for each i in I let $\mathcal{H}_i$ be a Hilbert space, and let ξ_i be a specific "distinguished" unit vector ξ_i in $\mathcal{H}_i$. For each i denote by $\mathcal{H}_i^\circ$ the orthogonal complement in $\mathcal{H}_i$ of ξ_i, i.e.*

$$\mathcal{H}_i^\circ = \mathcal{H}_i \ominus (\mathbb{C}\xi_i) = \{\eta_i \in \mathcal{H}_i \mid \eta_i \perp \xi_i\}.$$

Then the Hilbert space free product $*_{i\in I}(\mathcal{H}_i, \xi_i)$ of the pairs $(\mathcal{H}_i, \xi_i)$ (or just of the $\mathcal{H}_i$'s, if the ξ_i's are understood) is the Hilbert space $\mathcal{H}$ given as:

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n\in\mathbb{N}} \bigoplus_{i_1\neq i_2, i_2\neq i_3,\dots,i_{n-1}\neq i_n} \mathcal{H}_{i_1}^\circ \otimes \mathcal{H}_{i_2}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ.$$

Here the appearing vector ξ is merely notation to distinguish a specific copy of the complex numbers. It is assumed that $\|\xi\| = 1$, and ξ will always be considered as the distinguished unit vector in $\mathcal{H}$.

For a given family $(\mathcal{A}_i)_{i\in I}$ of unital C^*-algebras represented on Hilbert spaces $(\mathcal{H}_i)_{i\in I}$ we describe next a fundamental construction (due to D.V. Voiculescu) of a family $(\lambda_i)_{i\in I}$ of representations $\lambda_i\colon \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space free product of the $\mathcal{H}_i$'s. The construction is presented in two steps.

Constructions

(1) For each i in an index set I, let $\mathcal{H}_i$ be a Hilbert space equipped with a distinguished unit vector ξ_i, and consider the Hilbert space free product $\mathcal{H} = *_{i\in I}(\mathcal{H}_i, \xi_i)$. For each i in I we consider further the following sub-Hilbert space of $\mathcal{H}$:

$$\mathcal{H}(i) = \mathbb{C}\xi \oplus \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{i_1\neq i_2, i_2\neq i_3,\dots,i_{n-1}\neq i_n \\ i_1\neq i}} \mathcal{H}_{i_1}^\circ \otimes \mathcal{H}_{i_2}^\circ \otimes \cdots \otimes \mathcal{H}_{i_n}^\circ.$$

We may then, for each fixed i, define a unique mapping $V_i\colon \mathcal{H}_i \otimes \mathcal{H}(i) \to \mathcal{H}$, by the requirements:

(i) $V_i(\xi_i \otimes \xi) = \xi$.

(ii) $V_i(x \otimes \xi) = x$ for all x in $\mathcal{H}_i^\circ$.

(iii) $V_i(\xi_i \otimes (x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})) = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$, whenever $n \in \mathbb{N}$, $i_1, i_2, \dots, i_n \in I$ such that $i_1 \neq i$, $i_1 \neq i_2$, $i_2 \neq i_3, \dots, i_{n-1} \neq i_n$, and $x_{i_1} \in \mathcal{H}_{i_1}^\circ, \dots, x_{i_n} \in \mathcal{H}_{i_n}^\circ$.

(iv) $V_i(x_i \otimes (x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n})) = x_i \otimes x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$, whenever $n \in \mathbb{N}$, $i_1, i_2, \dots, i_n \in I$ such that $i_1 \neq i$, $i_1 \neq i_2$, $i_2 \neq i_3, \dots, i_{n-1} \neq i_n$, and $x_i \in \mathcal{H}_i^\circ$, $x_{i_1} \in \mathcal{H}_{i_1}^\circ, \dots, x_{i_n} \in \mathcal{H}_{i_n}^\circ$.

It is not hard to check that V_i maps a dense subspace of $\mathcal{H} \otimes \mathcal{H}(i)$ isometrically onto a dense subspace of $\mathcal{H}$, and hence, by continuity, V_i is a unitary operator from $\mathcal{H} \otimes \mathcal{H}(i)$ onto $\mathcal{H}$.

(2) Let I be an index set, and for each i in I let $\mathcal{A}_i$ be a unital C^*-algebra. Consider further, for each i, a representation of $\mathcal{A}_i$ on a Hilbert-space $\mathcal{H}_i$, i.e. a unital $*$-homomorphism $\pi_i\colon \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$. Assume also that each $\mathcal{H}_i$ comes equipped with a distinguished unit vector ξ_i, and consider the Hilbert space free product $\mathcal{H} = *_{i\in I}(\mathcal{H}_i, \xi_i)$ as well as the Hilbert spaces $\mathcal{H}(i)$ introduced in 0.1.3. For each i in I we consider then the representation $\lambda_i\colon \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}_i$ on $\mathcal{H}$ given by

$$\lambda_i(a) = V_i(\pi_i(a) \otimes I_{\mathcal{H}(i)})V_i^*, \qquad (a \in \mathcal{A}_i),$$

where $I_{\mathcal{H}(i)}$ denotes the identity operator on $\mathcal{H}(i)$ and V_i is the unitary operator from $\mathcal{H} \otimes \mathcal{H}(i)$ onto $\mathcal{H}$ constructed in 0.1.3. If we assume that π_i is faithful (and hence an isometry), it follows easily that so is λ_i. Indeed for any a in $\mathcal{A}_i$, we have that

$$\|\lambda_i(a)\| = \|V_i(\pi_i(a) \otimes I_{\mathcal{H}(i)})V_i^*\| = \|\pi_i(a) \otimes I_{\mathcal{H}(i)}\| = \|\pi_i(a)\| = \|a\|$$

(for the third equality we refer to [KR97, 2.6(16)]).

Proposition 36. *For each i in some index set I, let $(\mathcal{A}_i, \tau_i)$ be a C^*-probability space, and let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS-triplet associated to τ_i (see the paragraph preceding Theorem 4).*

Consider further the Hilbert space free product $\mathcal{H} = *_{i\in I}(\mathcal{H}_i, \xi_i)$, and for each i let $\lambda_i\colon \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$ be the representation of $\mathcal{A}_i$ on $\mathcal{H}$ introduced in 0.1.3.

Finally let $\mathcal{A}$ denote the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\bigcup_{i\in I}\lambda_i(\mathcal{A}_i)$ (i.e. the smallest C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing $\lambda_i(\mathcal{A}_i)$ for all i), and equip $\mathcal{A}$ with the vector state $\omega_\xi = \langle \cdot\xi, \xi\rangle$ associated to the distinguished vector ξ in $\mathcal{H}$.

Then the following statements hold:

(i) For each i the representation λ_i is faithful, and $\lambda_i(\mathcal{A}_i)$ is a C^*-subalgebra of $\mathcal{A}$.

(ii) For each i we have that $\tau_i = \omega_\xi \circ \lambda_i$.

(iii) The C^*-subalgebras $\lambda_i(\mathcal{A}_i)$, $i \in I$, are free in $\mathcal{A}$ with respect to ω_ξ.

The C^*-probability space $(\mathcal{A}, \tau)$ introduced in the proposition above is referred to as the reduced free product of the C^*-probability spaces $(\mathcal{A}_i, \tau_i)$, $i \in I$.

Proof. (i) For each i the state τ_i is faithful, and hence the associated GNS-representation $\pi_i\colon \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$ is faithful. As explained in 0.1.3, this further entails faithfulness of λ_i, and consequently it follows easily that $\lambda_i(\mathcal{A}_i)$ is a C^*-subalgebra of $\mathcal{A}$ [2].

[2] In fact the range of a C^*-algebra by a $*$-homomorphism (faithful or not) will always be a C^*-algebra (see [KR97, Theorem 4.1.9]).

(ii) For any i in I and a in $\mathcal{A}_i$, the definitions of λ_i and V_i yield that (cf. (i) in 0.1.3)

$$\begin{aligned}\omega_\xi \circ \lambda_i(a) &= \big\langle V_i(\pi_i(a) \otimes I_{\mathcal{H}(i)})V_i^*\xi, \xi\big\rangle = \big\langle (\pi_i(a) \otimes I_{\mathcal{H}(i)})V_i^*\xi, V_i^*\xi\big\rangle \\ &= \big\langle (\pi_i(a) \otimes I_{\mathcal{H}(i)})\xi_i \otimes \xi, \xi_i \otimes \xi\big\rangle = \big\langle \pi_i(a)\xi_i \otimes \xi, \xi_i \otimes \xi\big\rangle \\ &= \langle \pi_i(a)\xi_i, \xi_i\rangle \cdot \langle \xi, \xi\rangle = \tau_i(a),\end{aligned}$$

where the last identity follows from the properties of the GNS-representation associated to τ_i.

(iii) Consider elements $i_1, \dots, i_n$ in I, such that $i_1 \neq i_2$, $i_2 \neq i_3, \dots, i_{n-1} \neq i_n$, and let $a_1, \dots, a_n$ be elements of $\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_n}$, respectively, such that

$$\omega_\xi(\lambda_{i_1}(a_1)) = \cdots = \omega_\xi(\lambda_{i_n}(a_n)) = 0.$$

We must show, that

$$\omega_\xi(\lambda_{i_1}(a_1) \cdots \lambda_{i_n}(a_n)) = 0.$$

For each j in $\{1, \dots, n\}$, it follows from (ii), that

$$\omega_{\xi_{i_j}}(\pi_{i_j}(a_j)) = \tau_i(a_i) = \omega_\xi \circ \lambda_i(a) = 0, \quad \text{i.e.} \quad \pi_{i_j}(a_j)\xi_{i_j} \in \mathcal{H}^0_{i_j}.$$

From this together with the definitions of the λ_i's and the V_i's, we see that (cf. (i) and (ii) in 0.1.3)

$$\begin{aligned}\lambda_{i_n}(a_n)\xi &= V_{i_n}(\pi_{i_n}(a_n) \otimes I_{\mathcal{H}(i_n)})V_{i_n}^*\xi = V_{i_n}(\pi_{i_n}(a_n) \otimes I_{\mathcal{H}(i_n)})(\xi_{i_n} \otimes \xi) \\ &= V_{i_n}(\pi_{i_n}(a_n)\xi_{i_n} \otimes \xi) = \pi_{i_n}(a_n)\xi_{i_n},\end{aligned}$$

and hence, since $i_{n-1} \neq i_n$, that (cf. (iii) and (iv) in 0.1.3)

$$\begin{aligned}\lambda_{i_{n-1}}(a_{n-1})\lambda_{i_n}(a_n)\xi &= \lambda_{i_{n-1}}(a_{n-1})\pi_{i_n}(a_n)\xi_{i_n} \\ &= V_{i_{n-1}}(\pi_{i_{n-1}}(a_{n-1}) \otimes I_{\mathcal{H}(i_{n-1})})(\xi_{i_{n-1}} \otimes \pi_{i_n}(a_n)\xi_{i_n}) \\ &= V_{i_{n-1}}\big(\pi_{i_{n-1}}(a_{n-1})\xi_{n-1} \otimes \pi_{i_n}(a_n)\xi_{i_n}\big) \\ &= \pi_{i_{n-1}}(a_{n-1})\xi_{n-1} \otimes \pi_{i_n}(a_n)\xi_{i_n}.\end{aligned}$$

Continuing like this, we conclude after n steps that

$$\lambda_{i_1}(a_1) \cdots \lambda_{i_n}(a_n)\xi = \pi_{i_1}(a_1)\xi_{i_1} \otimes \cdots \otimes \pi_{i_n}(a_n)\xi_{i_n} \in \mathcal{H}^0_{i_1} \otimes \cdots \otimes \mathcal{H}^0_{i_n},$$

so that, in particular, $\lambda_{i_1}(a_1) \cdots \lambda_{i_n}(a_n)\xi$ is orthogonal to ξ. ■

Remark 37. (a) *The upshot of Proposition 36 is the following: If we identify $\mathcal{A}_i$ with its image $\lambda_i(\mathcal{A}_i)$ under the faithful representation λ_i, then we can embed all the $\mathcal{A}_i$'s into one large C^*-probability space $(\mathcal{A}, \omega_\xi)$ inside which they become free. Furthermore, statement (ii) asserts in particular that when we identify a selfadjoint element a from $\mathcal{A}_i$ by its image $\lambda_i(a)$ in $\mathcal{A}$, then the spectral distribution μ_a remains unchanged. Indeed for any continuous function f on $spe(a) = spe(\lambda_i(a))$ we have that*

$$\int_{spe(a)} f \, \mathrm{d}\mu_a = \tau_i(f(a)) = \omega_\xi \circ \lambda_i(f(a)) = \omega_\xi \circ f(\lambda_i(a)) = \int_{spe(\lambda_i(a))} f \, \mathrm{d}\mu_{\lambda_i(a)},$$

where the spectral distribution $\mu_{\lambda_i(a)}$ is formed with respect to the vector state ω_ξ.

(b) *Suppose that μ_1 and μ_2 are two compactly supported probability measures on $\mathbb{R}$. Then by considering the C^*-probability spaces $(C(\mathrm{supp}(\mu_i)), \mathbb{E}_{\mu_i})$, $i = 1, 2$, it follows from the considerations in (a), that we can always find a C^*-probability space $(\mathcal{A}, \tau)$, which contains two freely independent selfadjoint elements a and b, such that $\mu_a = \mu$ and $\mu_b = \nu$. Specifically a and b may be chosen as the identity functions on $\mathrm{supp}(\mu_a)$ and $\mathrm{supp}(\mu_b)$, respectively, considered as elements of $(\mathcal{A}, \tau)$. The existence of a and b with the described properties is crucial e.g. for the definition of the free additive convolution $\mu_1 \boxplus \mu_2$ as the spectral distribution of $a + b$.*

As mentioned in the previous remark, Proposition 36 makes it possible to define $\mu_1 \boxplus \mu_2$ for two compactly supported probability measures as the spectral distribution of the sum of two freely independent selfadjoint operators with spectral distributions μ_1 and μ_2. In order to give a similar description of $\mu_1 \boxplus \mu_2$ in case μ_1 and/or μ_2 have unbounded support, one needs to pass to W^*-probability spaces and a result corresponding to Proposition 36 for such probability spaces, and where the C^*-probability space $(\mathcal{A}, \tau)$ is in fact a W^*-probability space. Fortunately, such a result can be obtained by repeating most of the argumentation leading to Proposition 36. As von Neumann algebras are always given as subalgebras of the algebra of all bounded operators on a Hilbert space, there is no need to invoke the GNS-representation, provided that the considered states are already given as vector states. Assuming the latter, one may thus replace the GNS-representation by the identity representation in all the considerations above, and with this adjustment essentially the same construction as above leads to the following result:

Proposition 38. *For each i in some index set I, let $(\mathcal{A}_i, \tau_i)$ be a W^*-probability space, where $\mathcal{A}_i$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}_i$ with distinguished unit vector ξ_i. Assume in addition that $\tau_i = \langle \cdot \xi_i, \xi_i \rangle$ for all i.*

Consider further the Hilbert space free product $\mathcal{H} = *_{i \in I}(\mathcal{H}_i, \xi_i)$, and for each i let $\lambda_i \colon \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$ be the representation of $\mathcal{A}_i$ on $\mathcal{H}$ given by

$$\lambda_i(a) = V_i(a \otimes I_{\mathcal{H}(i)})V_i^*, \qquad (a \in \mathcal{A}_i),$$

where V_i is the unitary operator constructed in 0.1.3.

Finally let $\mathcal{A}$ denote the von Neumann-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\bigcup_{i \in I} \lambda_i(\mathcal{A}_i)$, i.e. (cf. Theorem 16)

$$\mathcal{A} = \Big(\bigcup_{i \in I} \lambda_i(\mathcal{A}_i)\Big)'',$$

and equip $\mathcal{A}$ with the vector state $\omega_\xi = \langle \cdot \xi, \xi \rangle$ associated to the distinguished vector ξ in $\mathcal{H}$.

Then the following statements hold:

(i) *For each i the representation λ_i is faithful and continuous from $\mathcal{A}_i$ in its ultra weak topology to $\mathcal{B}(\mathcal{H})$ in its weak topology. Furthermore $\lambda_i(\mathcal{A}_i)$ is a von Neumann-subalgebra of $\mathcal{A}$.*

(ii) *For each i we have that $\tau_i = \omega_\xi \circ \lambda_i$.*

(iii) *The von Neumann-subalgebras $\lambda_i(\mathcal{A}_i)$, $i \in I$, are free in $\mathcal{A}$ with respect to ω_ξ.*

For a detailed proof of Proposition 38 we refer to [VoDyNi92]. The W^*-probability space $(\mathcal{A}, \tau)$ introduced in the above proposition is referred to as the reduced free product of the W^*-probability spaces $(\mathcal{A}_i, \tau_i)$, $i \in I$.

Remark 39. *In continuation of Remark 37(b), consider two probability measures μ_1, μ_2 on $\mathbb{R}$ with (possibly) unbounded support. Then by application of Proposition 38 to the W^*-probability spaces $(L^\infty(\mu_i), \mathbb{E}_{\mu_i})$, $i = 1, 2$, we obtain a W^*-probability space $(\mathcal{A}, \tau)$ containing $L^\infty(\mu_1)$ and $L^\infty(\mu_2)$ as two free von Neumann subalgebras. The corresponding two copies of the identity function on $\mathbb{R}$ then serve as to freely independent operators affiliated with $(\mathcal{A}, \tau)$ with spectral distributions μ_1 and μ_2, and $\mu_1 \boxplus \mu_2$ may be realized as the spectral distribution of their sum. Condition (ii) in Proposition 38 further ensures (as in Remark 37(a)) that*

$$\int_{\mathbb{R}} f(t)\, \mu_i(\mathrm{d}t) = \tau(f(a_i)), \qquad (i = 1, 2),$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$.

Bibliography

[An00] M. Anchelevich, *Free stochastic measures via noncrossing partitions*, Adv. Math. **155** (2000), 154-179.

[Ap] D. Applebaum ....

[Be] J. Bertoin....

[BeMaSp] S. T. Belinschi, T. Mai and R. Speicher, *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem*, J. reine angew. Math. (to appear).

[BeVo93] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J., **42** (1993), 733-773.

[Bi98] P. Biane, *Processes with free increments*, Math. Zeitschrift **227** (1998), 143-174.

[BNPAT13] O.E. Barndorff-Nielsen, V. Perez Abreu and S. Thorbjørnsen (2013): Lévy Mixing. *Alea– Latin American Journal for Probability and Math. Statistics* **10**, 1013-1062.

[BNSh01a] O.E. Barndorff-Nielsen and N. Shephard, *Non-Gaussian OU based models and some of their uses in financial economics (with Discussion)*, J. R. Statist. Soc. B **63** (2001), 167-241.

[BNSh01b] O.E. Barndorff-Nielsen and N. Shephard, *Modelling by Lévy processes for financial econometrics*, in O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick (Eds.): *Lévy Processes - Theory and Applications*, Boston: Birkhäuser (2001), 283-318.

[BNT] , O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Classical and Free Infinite Divisibility and Lévy Processes.*, In: Quantum Independent Increment Processes II. Quantum Lévy processes, classical probability and applications to physics, U. Franz and M. Schürmann (Eds.), Springer Verlag (2006), 33-160.

[BiSp98] P. Biane and R. Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Probab. Theory Related Fields **112** (1998), 373-409.

[BO08] Nathanial P. Brown and Narutaka Ozawa, *C^*-algebras and Finite Dimensional Approximations*, Graduate Studies in Mathematics **88**, American Mathematical Society (2008).

[JuVe83] Z.J. Jurek and W. Verwaat, *An integral representation for self-decomposable Banach space valued random variables*, Z. Wahrscheinlichkeitstheorie verw. Geb. **62** (1983), 247-262.

[KR97] R.V. Kadison and J.R. Ringrose (1997): *Fundamentals of the theory of operator algebras vol I-II*, Graduate Studies in Mathematics, 16, American Mathematical Society, Providence.

[Lu75] E. Lukacs, *Stochastic Convergence (second edition)*, Academic Press (1975).

[Ma92] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. **106**, (1992), 409-438.

[Ne74] E. Nelson, *Notes on Non-commutative Integration*, J. Funct. Anal. **15** (1974), 103-116.

[RR89] B.S. Rajput and J. Rosiński (1989): Spectral representations of infinitely divisble processes. *Probab. Theory Related Fields* **112**, 451-487.

[Ru] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Inc. (1987)

[Ru91] W. Rudin, *Functional Analysis (second edition)*, McGraw-Hill Inc. (1991).

[RLL] M. Rørdam, N.J. Laustsen and F. Larsen *K-theory for operator algebras.....* Cambridge University Press....

[Sa99] Sato, K. (1999): *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

[Se53] I.E. Segal, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401-457; correction **58** (1953), 595-596.

[Sp90] R. Speicher, *A new example of 'independence' and 'white noise'*, Probab. Th. Rel. Fields **84** (1990), 141-159.

[Te81] M. Terp (1981): *L^p Spaces associated with von Neumann Algebras*, Lecture notes, University of Copenhagen.

[UW67] Urbanik, K. and Woyczynski W. A. (1967): Random integrals and Orlics spaces. *Bull. Acad. Polon. Sci.* **15,** 161-169.

[VoDyNi92] D.V. Voiculescu, K.J. Dykema and A. Nica, *Free Random Variables*, CRM Monographs Series, vol. 1, A.M.S. (1992).

[Vo93] D.V. Voiculescu *The analogues of entropy and of Fisher's information measure in free probability* Int. Comm. Math. Phys. **155** (1993), 71-92.

[Vo98] D.V. Voiculescu, *Lectures on Free Probability*, Lecture notes from the 1998 Saint-Fluor Summer School on Probability Theory.

[Zhu] K. Zhu, *An introduction to operator algebras*, Studies in Advanced Mathematics, CRC press (1993).