On the Universality of the Non-singularity of General Ginibre and Wigner Random Matrices

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Abstract

We prove the universal asymptotically almost sure non-singularity of general Ginibre and Wigner ensembles of random matrices when the distribution of the entries are independent but not necessarily identically distributed and may depend on the size of the matrix. These models include adjacency matrices of random graphs and also sparse, generalized, universal and banded random matrices. We find universal rates of convergence and precise estimates for the probability of singularity which depend only on the size of the biggest jump of the distribution functions governing the entries of the matrix and not on the range of values of the random entries. Moreover, no moment assumptions are made about the distributions governing the entries. Our proofs are based on a concentration function inequality due to Kolmogorov, Rogozin and Kesten, which allows us to improve universal rates of convergence for the Wigner case when the distribution of the entries do not depend on the size of the matrix.

Key terms: Adjacency matrix of random graphs, banded random matrix, decoupling, concentration function, generalized Wigner ensemble, Littlewood-Offord inequality, Kolmogorov-Rogozin inequality, nondegenerate distribution, sparse random matrix.

1 Introduction and main results

Let $A_n = (\xi_{ij}^{(n)})$ be an $n \times n$ random matrix where each entry $\xi_{ij}^{(n)}$ follows a distribution $F_{ij}^{(n)}$, $1 \leq i, j \leq n$. The study of the non-singularity of such matrices has mainly been considered when $F_{ij}^{(n)} \equiv F$ and for two ensembles of random matrices, the Ginibre and Wigner. We will use the

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following terminology: An $n \times n$ random matrix $G_n = (\xi_{ij})_{1 \le i,j \le n}$ is called a Ginibre matrix if ξ_{ij}, $i, j = 1, \dots, n$ are independent random variables, and an $n \times n$ random symmetric matrix $W_n = (\xi_{ij})_{1 \le i,j \le n}$ is called Wigner matrix if $\xi_{ij} = \xi_{ji}$, $i, j = 1, \dots, n$ and ξ_{ij}, $1 \le i \le j \le n$ are independent random variables. We will not assume that the distributions of the entries have moments.

The singularity of these matrices is trivial if the distributions of ξ_{ij} are degenerate. The non-singularity is also straightforward if the entries have continuous distributions. The interesting situation occurs when some of the entries have distributions with jumps. Singularity of such matrices is a highly non trivial problem.

The study of the non-singularity of Ginibre matrices goes back to the pioneering work by J. Komlós. In [15] he considers a Ginibre random matrices $GB(n, 1/2)$, whose entries are i.i.d. Bernoulli random variables, taking values 0 or 1 with probability 1/2 each. Using a very clever 'growing rank analysis' together with the Littlewood-Offord inequality Komlós proved that $\mathbb{P}\{\text{rank}(GB(n, 1/2)) < n\} = \text{o}(1)$ as $n \to \infty$. Bollobás [3] presents the concept of 'strong rank' and together with the Littlewood-Offord inequality obtains an unpublished result due to Komlós, *viz.* $\mathbb{P}\{\text{rank}(GB(n, 1/2)) < n\} = \text{O}(n^{-1/2})$ as $n \to \infty$. Komlós [16] was also the first in considering the singularity of Ginibre matrices whose entries are i.i.d. random variables with a *common* arbitrary non-degenerate distribution, proving that the probability that such an $n \times n$ matrix is singular has order $\text{o}(1)$ as $n \to \infty$. This result was improved by Kahn, Komlós and Szemerédi [12] in the case of Ginibre matrices whose entries are i.i.d. taking values -1 or 1 with probability 1/2 each, showing that the probability of singularity is bounded above by θ^n for $\theta = .999$. The value of θ has been improved by Tao and Vu [24], [25] to $\theta = 3/4 + o(1)$ and by Bourgain, Vu and Wood [5] to $\theta = 1/\sqrt{2} + o(1)$. Slinko [21] considered Ginibre random matrices whose entries have the same uniform distribution taking values in a finite set, proving also that the probability of singularity is $\text{O}(n^{-1/2})$ as $n \to \infty$.

The aim of this paper is to understand the asymptotic non-singularity of more general Ginibre and Wigner ensembles. We are interested in finding universality results with respect to general distributions of the entries and also when these distributions depend on the size of the matrix.

As a first step in this direction, the results in [3], [21] were generalized by Bruneau and Germinet [4] to Ginibre random matrices whose entries follow different independent non-degenerate distributions F_{ij} which do not change with the size of the matrix. Their result gives a universal rate of convergence of $n^{-1/2}$ as follows:

Theorem 1. *[4] Let G_n be an $n \times n$ Ginibre matrix with independent entries ξ_{ij} satisfying the following property H: there exists $\rho \in (0, 1/2)$ such that for any $i, j = 1, \ldots, n$, $\mathbb{P}\left\{\xi_{ij} > x_{ij}^+\right\} > \rho$ and $\mathbb{P}\left\{\xi_{ij} < x_{ij}^-\right\} > \rho$ for some real numbers $x_{ij}^- < x_{ij}^+$, then*

$$\mathbb{P}\left\{rank(G_n) < n\right\} \leq C/\sqrt{\beta_\rho(1-\rho)n}, \tag{1}$$

where the constant C is universal (coming from the Littlewood-Offord inequality) and β_ρ is an implicit constant $0 < \beta_\rho < 1$ which goes to zero as $\rho \to 1$.

Remark 1. *a) The above theorem is proved in [4] using ideas of strong rank of [3], together with a Bernoulli representation theorem for the distribution of a random variable and the Littlewood-Offord inequality.*

b) We point out that it is possible to express (1) in terms of the size of the biggest jump of the distribution functions governing the entries. Indeed, this follows using a strong rank analysis and the Kolmogorov-Rogozin concentration inequality. This inequality, stated in Section 2, will be used repeatedly in this work. Returning to (1), taking $\kappa = \max_{1\leq i,j\leq n} \sup_{x\in\mathbb{R}} \mathbb{P}\{\xi_{i,j} = x\}$, the size of the biggest jump of F_{ij}, $i, j = 1, \ldots, n$, for $0 \geq \kappa < 1$ we have

$$\mathbb{P}\left\{rank(G_n) < n\right\} \leq \frac{C_1}{\sqrt{\beta'_\kappa(1-\kappa)n}}, \tag{2}$$

where the constant C_1 is the universal coming from Kolmogorov-Rogozins inequality.

c) We observe that the constants β_ρ and β'_κ are not universal –they might depend on the distributions F_{ij}.

d) These two results highlight the fact that the non-singularity of Ginibre matrix depends only on ρ or, equivalently, the size of the biggest jump κ. In other words, the universal property of a random matrix being non-singular depends neither on the range of values taken by the entries nor on other properties of their distribution except the size of the biggest jump.

As for Wigner random matrices, the study of their singularity was initiated by Costello, Tao and Vu [8] inspired by the work of Komlós [15].

Theorem 2. *[8] Let $W_n = (\xi_{ij})$ be an $n \times n$ Wigner matrix whose upper diagonal entries ξ_{ij} are independent random variables with common Bernoulli distribution on $\{0, 1\}$ with parameter $1/2$. Then, as $n \to \infty$,*

$$\mathbb{P}\left\{rank(W_n)) < n\right\} = O(n^{-1/8+\alpha}),$$

for any positive constant α, the implicit constant in $O(\cdot)$ depending on α.

Remark 2. *a) The proof of the above theorem in [8] required developing a quadratic Littlewood-Offord inequality. A possible generalization to distributions other than Bernoulli was also indicated in [8].*

b) Theorem 3.b below gives a better universal rate of convergence $n^{-1/4+\alpha}$, for any Wigner random matrix $W_n = (\xi_{ij})$ with independent entries which need not be identical. While the off-diagonal entries need to be non-degenerate, the diagonal entries could be degenerate.

More recently, the Wigner matrix has been studied when the entries satisfy some restrictions. Nguyen [17] considered a Wigner matrix W_n with entries taking values -1 or 1 with probability $1/2$ each, subject to the condition that each row has exactly $n/2$ entries which are zero. He showed that the probability of W_n being singular is $\mathrm{O}(n^{-C})$, for any positive constant C, the implicit constant in $\mathrm{O}(\cdot)$ depending on C. Also recently, Vershynin [26] has considered the case of a Wigner matrix W_n whose entries satisfy the following property: the above-diagonal entries are independent and identically distributed with zero mean, unit variance and subgaussian, while the diagonal entries satisfy $\xi_{ii} \leq K\sqrt{n}$ for some K. He showed that the probability of W_n being singular is bounded above by $2\exp(-n^c)$, where c depends only on the subgaussian distribution and on K.

One of the goals of this paper is to study the non-singularity of Ginibre and Wigner matrices when the distributions of the entries $F_{ij}^{(n)}$ depend on the matrix size. This kind of random matrices appear in the study of random graphs [6], sparse matrices [7], [9] and some other models that have recently been extensively considered like the so-called generalized, universal and banded Wigner ensembles [10], [22] among other works. See also the non i.i.d. Wigner case in, for example, [2, pp 26].

One difficulty that arises in this situation is to find adequate asymptotic estimates of the probability of the singularity being zero where the involved constants in the rate of convergence do not depend on the distributions of the entries. We overcome this difficulty using a universal concentration inequality due to Kesten [14] which we express in terms of the size of the jumps of the distribution functions.

1.1 Main results We now consider Ginibre and Wigner matrix ensembles $G_n^{(n)} = \left(\xi_{ij}^{(n)}\right)_{1\leq i,j\leq n}$, and $W_n^{(n)} = \left(\xi_{ij}^{(n)}\right)_{1\leq i,j\leq n}$, where the distribution function $F_{ij}^{(n)}$ governing $\xi_{ij}^{(n)}$ is allowed to change with the size of the matrix.

One of our main conclusions is the non-singularity of the above Ginibre and Wigner ensembles. More specifically, given a collection of non-degenerate distribution functions $\{F_{ij}^{(n)} : i,j \geq 1,\ n \geq 1\}$ and a subsequence $\{m_n : n \geq 1\}$ we study the singularity of the $m_n \times m_n$ matrix with independent entries $\xi_{kl}^{(n)}$ being governed by the distribution function $F_{kl}^{(n)}$ for every $1 \leq k, l \leq m_n$. Let us denote by κ_n the biggest jump of the distribution functions $F_{ij}^{(n)}$, $1 \leq i, j \leq m_n$, i.e., if $\kappa_{i,j} = \sup_{x \in \mathbb{R}} \mathbb{P}\{\xi_{i,j}^n = x\}$, then

$$\kappa_n = \max_{1 \leq i,j \leq n} \{\kappa_{i,j}\}. \tag{3}$$

We give a sufficient condition for $m_n = n$ in terms of the sequence of biggest jumps $(\kappa_n)_{n \geq 1}$.

Theorem 3. *(Universality of non-singularity of Ginibre and Wigner ensembles) With the notation as above, let $G_r^{(n)}$ and $W_r^{(n)}$ be the $r \times r$ Ginibre and Wigner matrices respectively, each with entries $\xi_{i,j}^{(n)}$, $1 \leq i \leq j \leq r$. Assume that $\kappa_n < \kappa \in [0,1)$ for all n*

a) As $n \to \infty$

$$\mathbb{P}\left\{ rank(G_n^{(n)}) < n \right\} = O\left(n^{-1/2}\right) \tag{4}$$

where the implicit constant in $O(\cdot)$ depends on κ.

b) For any $\varepsilon \in (0,1)$

$$\mathbb{P}\left\{ rank(W_n^{(n)}) < n \right\} = O\left(n^{-(1-\varepsilon)/4}\right), \tag{5}$$

where the implicit constant in $O(\cdot)$ depends on ε and κ.

Remark 3. *a) The proof of Lemma 6 in section 3, where Theorem 3.a is given, highlights the fact that the probability that a Ginibre matrix has small rank is small. This is due only to the independence of entries.*

b) The bound $n^{-1/4+\alpha}$ in (5) improves the rate $n^{-1/8+\alpha}$ in Theorem 2 of Costello, Tao and Vu [8].

We now turn to Theorem 3. A natural question is to understand what happens when $\kappa_n \to 1$.

Proposition 1. *Let $\kappa_n \in [0,1]$ there is a Ginibre matrix G_{m_n} of size m_n, whose entries have the same distribution, such that the biggest jump of G_{m_n} is κ_n and*

$$\mathbb{P}\{G_{m_n} \text{ has full rank }\} \to 1 \qquad n \to \infty.$$

In the following examples we can see that if $\kappa_n \to 1$ at some appropite rate, we can have different behavior for the probability of singularity.

Let $GB(n,p)$ and $WB(n,p)$ denote the $n \times n$ Ginibre and Wigner matrices whose entries have Bernoulli distribution on $\{0,1\}$ with parameter p. Let ZGB_n (ZWB_n) be the event that the first row of $GB(n,1/n)$, ($WB(n,1/n)$) contains only zeros, then

$$\mathbb{P}\{ZGB_n\} = \left(1 - \frac{1}{n}\right)^n, \qquad \mathbb{P}\{ZWB_n\} = \left(1 - \frac{1}{n}\right)^n,$$

hence

$$e^{-1} \leq \lim_{n\to\infty} \mathbb{P}\{\operatorname{rank}(GB(n,1/n)) < n\},$$

$$e^{-1} \leq \lim_{n\to\infty} \mathbb{P}\{\operatorname{rank}(WB(n,1/n)) < n\}.$$

However, if $\alpha \in (0,1)$ there is a constant $C_\alpha > 0$

$$\mathbb{P}\{\operatorname{rank}(WB(n,n^\alpha/n)) < n\} \leq n^{-C_\alpha}. \tag{6}$$

In the Ginibre case it is not clear what happens in the case $\kappa = n^\alpha/n$, but if $\gamma \in (0,1)$

$$\mathbb{P}\{\operatorname{rank}(GB(n,n^\alpha/n)) > \gamma n\} \to 1 \text{ as } n \to \infty. \tag{7}$$

Furthermore, as an application of the Wigner case, we obtain an estimation of the probability that the adjacency matrix of a sparse random graph (not necessarily an Erdös-Rényi graph) is non-singular. Costello and Vu [6] have analyzed the adjacency matrices of sparse Erdös-Rényi graphs, where each entry is equal to 1 with the same probability $p(n)$ which tends to 0 as n goes to infinity (see also Costello and Vu [7] where a generalization of [6] is considered in which each entry takes the value $c \in \mathbb{C}$ with probability p and zero with probability $1-p$, and the diagonal entries are possibly non-zero). It is proved in [6] that when $c\ln(n)/n \leq p(n) \leq 1/2$, $c > 1/2$, then with probability $1-\mathrm{O}((\ln\ln(n))^{-1/4})$, the rank of the adjacency matrix equals the number of non-isolated vertices. Now we consider the following model extension of Erdös-Rényi graphs, where vertices i and j are linked with a probability that depends on i and j and the number of vertices. Furthermore, the rate of convergence is an improvement of the one given in [6] for $c\ln n/n^\beta \leq p(n) \leq 1/2$ with $c > 0$ and $\beta \in (0,1)$. From the proof of Theorem 3.b in section 4, if $\kappa_n = 1 - p(n)$, we have as $n \to \infty$

$$\frac{\kappa_n^{\frac{3}{8}n - \frac{1}{2}n^{1-\varepsilon}}}{\kappa_n(1-\kappa_n)} \leq \left(\frac{\kappa_n^2}{n^{1-\varepsilon}(1-\kappa_n)}\right)^{1/4} \leq \left(\frac{(1-c(\ln n/n^\beta))^2}{n^{1-\varepsilon-\beta}\ln n}\right)^{1/4} \to 0$$

if $\varepsilon + \beta < 1$.

Proposition 2. *Let $\{p_{ij} \in (0,1) : i,j = 1,2,\ldots\}$ be a double sequence of positive numbers with $p_n^* = \min_{1\le i\le j\le n}\{p_{ij}\} \in [c\ln n/n^\beta, 1/2]$, $c > 0$, and $\varepsilon + \beta < 1$, $\varepsilon, \beta \in (0,1)$, then there is a random graph with n vertices such that the vertex i is linked with the vertex j with probability p_{ij}, $1 \le i < j \le n$, and if A_n is the adjacency matrix, we have as $n \to \infty$*

$$\mathbb{P}\{rank(A_n) < n\} \le Cn^{-(1-\varepsilon-\beta)/4}, \tag{8}$$

for some constant $C > 0$.

Remark 4. *a) In many applications of random matrices one considers ensembles of the form $G_n^{(n)} = a_n^{-1}G_n$ and $W_n^{(n)} = a_n^{-1}W_n$ where $a_n \to \infty$ as $n \to \infty$ and the non-degenerate distributions of the entries of G_n and W_n do not depend on matrix size n. In this case $\kappa_n = \kappa < 1$ for all $n \ge 1$, if the distribution is not degenerated. However the ensembles $G_n^{(n)}$ and $W_n^{(n)}$ are asymptotically almost surely non-singular. In fact, this holds for any sequence $a_n \to \infty$ and the rate of convergence to zero of the probability of singularity is not affected by the rate of convergence of a_n if the distributions of the entries have discrete support.*

b) The case $a_n = 1/\sqrt{n}$ is the set-up of problems of random matrices appearing in the study of the asymptotic spectral distributions [1], [2], geometric functional analysis [23], [18] and restricted isometries [20], among others.

c) Finally, the results in the Ginibre case have a straightforward extension to non-square $n \times m$ random matrices whose entries are independent random variables and have distributions with jumps.

2 Preliminaries on Concentration Inequalities

In this section we present the Kolmogorov-Rogozin concentration inequalities that we use for the proofs of our main results on non-singularity. We express these inequalities in terms of the size of biggest jump of the non-degenerate distribution functions.

The Lévy concentration function $Q(\xi;\lambda)$ of a random variable ξ is defined by

$$Q(\xi;\lambda) = \sup_{x\in\mathbb{R}} \mathbb{P}\{\xi \in [x, x+\lambda]\}, \quad \lambda > 0.$$

Let $\xi_1, \xi_2, \ldots$ be independent random variables and $S_n = \sum_{i=1}^n \xi_i$. An expression that relates the concentration function of S_n to the concentration functions of the summands ξ_i was given by Kolmogorov-Rogozin; see [13].

Lemma 1 (Kolmogorov-Rogozin Inequality)**.** *There exists a universal constant C such that for any independent random variables $\xi_1, \ldots, \xi_n$ and any real numbers $0 < \lambda_1, \ldots, \lambda_n \le L$, one has*

$$Q(S_n; L) \le CL \left\{ \sum_{i=1}^{n} \lambda_i^2 \left[1 - Q(X_i; \lambda_i)\right] \right\}^{-1/2}.$$

Kesten [14] obtained the following refinement of the above inequality.

Lemma 2. *For the constant C of the Kolmogorov-Rogozin inequality and any independent random variables $\xi_1, \ldots, \xi_n$, and real numbers $0 < \lambda_1, \ldots, \lambda_n \le 2L$, one has*

$$Q(S_n; L) \le 4 \cdot 2^{1/2}(1 + 9C)L \frac{\sum_{i=1}^{n} \lambda_i^2 \left[1 - Q(\xi_i; \lambda_i)\right] Q(\xi_i, L)}{\left\{\sum_{i=1}^{n} \lambda_i^2 \left[1 - Q(\xi_i; \lambda_i)\right]\right\}^{3/2}}.$$

For the study of non-singularity of random matrices, one has to find an estimate of the probability that a polynomial of independent random variables equals a real number. In the case of Ginibre and Wigner matrices the polynomial are of degree one and two respectively. Our first goal is to write Kesten inequality in terms of the size of the biggest jump and then obtain the corresponding linear and quadratic concentration inequalities.

We first discuss the relation between the size of the biggest jump of a non-degenerate distribution F and its corresponding Lévy concentration function. Let D_F be the set of discontinuities of F and κ its biggest jump, i.e., $\kappa = \sup_{x \in \mathbb{R}} \mathbb{P}\{\xi = x\}$ where ξ has distribution function F.

We note the following:

1. There exists $x_\kappa \in \mathbb{R}$ such that $\mathbb{P}\{\xi = x_\kappa\} = \kappa$.

2. Let $p_i = \mathbb{P}\{\xi = x_i\}$, $i \in \mathbb{N}$, then $\sum_{i \ge 1} p_i \le 1$, i.e., for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $\sum_{i \ge n} p_i \le \varepsilon$ for all $n \ge N(\varepsilon)$.

3. If F is a discrete distribution $\left(\sum_{i \in \mathbb{N}} p_i = 1\right)$ and x_κ is not an accumulation point of D_F, there exists $\delta_1 > 0$ with

$$\sup_{x \in \mathbb{R}} \mathbb{P}\{\xi \in [x, x + \delta_1]\} = \kappa.$$

Otherwise, if F is not discrete or x_κ is an accumulation point of D_F, there exists some $\Delta > 0$, which may be taken as small as desired, such that, for Δ fixed, there is $\delta_2 > 0$ with

$$\sup_{x\in\mathbb{R}} \mathbb{P}\{\xi \in [x, x+\delta_2]\} = \kappa + \Delta < 1.$$

We define κ_Δ, for $\Delta \in (0,1)$ fixed, by $\kappa_\Delta := \kappa$ if F is discrete and x_κ is not an accumulation point of D_F and otherwise, $\kappa_\Delta := \kappa + \Delta$. So, we have that there is $\delta > 0$ such that

$$\sup_{x\in\mathbb{R}} \mathbb{P}\{\xi \in [x, x+\delta]\} = \kappa_\Delta. \tag{9}$$

4. We fix $\Delta \in (0,1)$ and $\delta > 0$ that satisfies (9). If $a \in \mathbb{R}$ with $|a| \geq 1$, then

$$\sup_{x\in\mathbb{R}} \mathbb{P}\{a\xi \in [x, x+\delta]\} \leq \kappa_\Delta.$$

Indeed, if $\sup_{x\in\mathbb{R}} \mathbb{P}\{a\xi \in [x, x+\delta]\} > \kappa_\Delta$, then there exists some $x^* \in \mathbb{R}$ such that $\mathbb{P}\{a\xi \in [x^*, x^*+\delta]\} > \kappa_\Delta$, but

$$\delta \geq |a\xi - x^*| = |a| \left|\xi - \frac{x^*}{a}\right| \geq \left|\xi - \frac{x^*}{a}\right|,$$

which is a contradiction to the definition of κ_Δ. So, we have that

$$Q(\beta\xi, \delta) \leq \kappa_\Delta \text{ for } |\beta| \geq 1.$$

Now let $\xi_1, \ldots, \xi_n$ be independent random variables with distribution functions $F_1, \ldots, F_n$, respectively. For each ξ_i we consider $\kappa(i), \kappa_\Delta(i) < 1$ defined as above. We first prove the following concentration inequality in terms of the biggest jumps of the distribution functions.

Lemma 3 (Linear Concentration Inequality)**.** *Let $\xi_1, \ldots, \xi_n$ be independent random variables with non-degenerate distributions $F_1, \ldots, F_n$, respectively, and let $\alpha_1, \ldots, \alpha_n$ be real numbers with $\alpha_i \neq 0$, $i = 1, \ldots, n$. Then*

$$\sup_{x\in\mathbb{R}} \mathbb{P}\left\{\sum_{i=1}^n \alpha_i\xi_i = x\right\} = O\left(\frac{\sum_{i=1}^n (1-\kappa(i))\kappa_\Delta(i)}{\{\sum_{i=1}^n [1-\kappa_\Delta(i)]\}^{3/2}}\right),$$

where the implicit constant in $O(\cdot)$ does not depend on F_i, $i = 1, \ldots, n$.

Proof. Let $a = \min_{1\leq i\leq n}\{|\alpha_i|\}$ and $\delta = \min_{1\leq i\leq n}\{\delta_i\}$, where $\delta_i > 0$ satisfies $\kappa_\Delta(i) = Q(\xi_i, \delta_i)$,

$i = 1, \dots, n$. We have for $x \in \mathbb{R}$

$$\mathbb{P}\left\{\sum_{i=1}^{n} \alpha_i \xi_i = x\right\} = \mathbb{P}\left\{\sum_{i=1}^{n} \frac{\alpha_i}{a} \xi_i = \frac{x}{a}\right\} = \mathbb{P}\left\{\sum_{i=1}^{n} \alpha_i' \xi_i = x'\right\},$$

where $\alpha_i/a = \alpha_i'$ and $x/a = x'$. Now,

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^{n} \alpha_i' \xi_i = x'\right\} &\leq \sup_{y \in \mathbb{R}} \mathbb{P}\left\{\sum_{i=1}^{n} \alpha_i' \xi_i \in [y, y+\delta]\right\} \\ &\leq 4 \cdot 2^{1/2}(1+9C) \frac{\sum_{i=1}^{n}(1-\kappa(i))\kappa_\Delta(i)}{\left\{\sum_{i=1}^{n}\left[1-\kappa_\Delta(i)\right]\right\}^{3/2}},\end{aligned}$$

the last expression following from Lemma 2. ■

Remark 5. *a) If $\kappa_\Delta(i) < \kappa < 1$ for all i,*

$$\sup_{x \in \mathbb{R}} \mathbb{P}\left\{\sum_{i=1}^{n} \alpha_i \xi_i = x\right\} = O\left(\frac{\kappa}{\sqrt{(1-\kappa)^3 n}}\right).$$

b) Lemma 3 holds when r many of the random variables $\xi_1, \dots, \xi_n$ are degenerate for some $1 \leq r < n$, in this situation n is changed by $n - r$. Contribution to the bound of the concentration inequality is provided only by the non-degenerate random variables.

In order to prove the so-called Quadratic Concentration Inequality, we recall the decoupling argument.

Lemma 4 (Decoupling)**.** *Let $X \in \mathbb{R}^{m_1}$ and $Y \in \mathbb{R}^{m_2}$ be independent random variables, with $m_1 + m_2 = n$, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Borel function. Let X' be a variable independent of X and Y, but having the same distribution as X. For any interval I of $\mathbb{R}$, we have*

$$\mathbb{P}^2\left\{\varphi(X, Y) \in I\right\} \leq \mathbb{P}\left\{\varphi(X, Y) \in I, \varphi(X', Y) \in I\right\}.$$

A quadratic Littlewood-Offord inequality for independent $\{0, 1\}$-Bernoulli random variables with probability $1/2$ was proved in [8]. The result below is for independent random variables not necessarily identically distributed and without any assumption on their moments.

Lemma 5 (Quadratic Concentration Inequality)**.** *Let $\xi_1, \dots, \xi_n$ be independent random variables with non-degenerate distributions $F_1, \dots, F_n$, respectively, and $(c_{ij})_{1 \leq i,j \leq n}$ be a symmetric $n \times n$*

array of constants. Suppose $S_1 \sqcup S_2$ is a partition of $\{1,2,\ldots,n\}$ such that for each $j \in S_2$, the set $N_j := \{i \in S_1 : c_{ij} \neq 0\}$ is non-empty. Let

$$\varphi = \varphi\{\xi_1,\ldots,\xi_n\} = \sum_{1\le i,j\le n} c_{ij}\xi_i\xi_j$$

be the quadratic form whose coefficients are c_{ij}. Then

$$\mathbb{P}\{\varphi = x\} = O\left(\left[\frac{1}{|S_2|}\sum_{j\in S_2}\left(\frac{\sum_{i\in N_j}(1-\overline{\kappa}(i))\overline{\kappa}_\Delta(i)}{\left\{\sum_{i\in N_j}\left[1-\overline{\kappa}_\Delta(i)\right]\right\}^{3/2}}\right) + \sup_{D\subset S_2, |D|\ge |S_2|/2}\frac{\sum_{j\in D}(1-\kappa(j))\kappa_\Delta(j)}{\left\{\sum_{j\in D}\left[1-\kappa_\Delta(j)\right]\right\}^{3/2}}\right]^{1/2}\right).$$

where, for ξ_i' an independent copy of ξ_i, $\overline{\kappa}(i)$ and $\overline{\kappa}_\Delta(i)$ are the jumps associated with $\xi_i - \xi_i'$ and $\kappa(j)$ and $\kappa_\Delta(j)$ are the jumps associated with ξ_j. The implicit constant in $O(\cdot)$ does not depend on F_i, $i = 1,\ldots,n$.

Proof. Let $\delta = \min_{1\le i\le n}\{\delta_i\}$ where $\delta_i > 0$ satisfies $\kappa_\Delta(i) = Q(\xi_i,\delta_i)$, $i = 1,\ldots,n$. If $x \in \mathbb{R}$, we have

$$\mathbb{P}\{\varphi = x\} \le \mathbb{P}\{\varphi \in [x, x+\delta/2]\}.$$

Write $I = [x, x+\delta/2]$, $X = (\xi_i : i \in S_1)$, $Y = (\xi_i : i \in S_2)$ and $X' = (\xi_i' : i \in S_1)$, with X' independent of X and Y, but having the same distribution as X. By Lemma 4,

$$\begin{aligned}\mathbb{P}^2\{\varphi(X,Y)\in I\} &\le \mathbb{P}\left\{\varphi(X,Y)\in I, \varphi(X',Y)\in I\right\}\\ &\le \mathbb{P}\left\{\varphi(X,Y)-\varphi(X',Y)\in[-\delta/2,\delta/2]\right\}.\end{aligned}$$

We can rewrite $\varphi(X,Y) - \varphi(X',Y)$ as

$$\begin{aligned}\varphi(X,Y)-\varphi(X',Y) &= g(X,X') + 2\sum_{j\in S_2}\xi_j\left(\sum_{i\in S_1}c_{ij}\left(\xi_i-\xi_i'\right)\right)\\ &= g(X,X') + 2\sum_{j\in S_2}\xi_j\eta_j,\end{aligned}$$

where $g(X,X') = \sum_{i,j\in S_1} c_{ij}(\xi_i\xi_j - \xi_i'\xi_j')$ and $\eta_j = \sum_{i\in S_1} c_{ij}\left(\xi_i - \xi_i'\right)$.

Let ζ be the number of η_j which are equal to zero. If $J = [-\delta/2, \delta/2]$, we have

$$\mathbb{P}\left\{\varphi(X,Y) - \varphi(X',Y) \in J\right\} \leq \mathbb{P}\left\{\varphi(X,Y) - \varphi(X',Y) \in J, \zeta \leq \frac{|S_2|}{2}\right\} + \mathbb{P}\left\{\zeta > \frac{|S_2|}{2}\right\}.$$

Since $\zeta = \sum_{j\in S_2} \mathbf{1}_{\{\eta_j=0\}}$, using Lemma 3, we have

$$\mathbb{E}\left(\zeta\right) = \sum_{j\in S_2} \mathbb{P}\left\{\eta_j = 0\right\} = \sum_{j\in S_2} \mathbb{P}\left\{\sum_{i\in N_j} c_{ij}\left(\xi_i - \xi_i'\right) = 0\right\} = \sum_{j\in S_2} \mathrm{O}\left(\frac{\sum_{i\in N_j}(1-\overline{\kappa}(i))\overline{\kappa}_\Delta(i)}{\left\{\sum_{i\in N_j}\left[1-\overline{\kappa}_\Delta(i)\right]\right\}^{3/2}}\right),$$

where $\overline{\kappa}(i)$ and $\overline{\kappa}_\Delta(i)$ are the jumps associated with $\xi_i - \xi_i'$. By Markov's inequality, we obtain

$$\mathbb{P}\left\{\zeta > \frac{|S_2|}{2}\right\} \leq \frac{2}{|S_2|}\mathbb{E}\left(\zeta\right) = \frac{1}{|S_2|}\sum_{j\in S_2} \mathrm{O}\left(\frac{\sum_{i\in N_j}(1-\overline{\kappa}(i))\overline{\kappa}_\Delta(i)}{\left\{\sum_{i\in N_j}\left[1-\overline{\kappa}_\Delta(i)\right]\right\}^{3/2}}\right).$$

For $M := \{j \in S_2 : \eta_j \neq 0\}$, we note that (i) M is a random set which depends only on X, X' and (ii) $|M| \geq |S_2|/2$ whenever $\zeta \leq |S_2|/2$. Thus for a given realisation x, x' of X, X' respectively, we have

$$\mathbb{P}\left\{\varphi(x,Y) - \varphi(x',Y) \in J \;\middle|\; \zeta \leq \frac{|S_2|}{2}\right\} = \mathbb{P}\left\{2\sum_{j\in S_2} \xi_j\eta_j \in J' \;\middle|\; \zeta \leq \frac{|S_2|}{2}\right\},$$

where $J' = [-g(x,x') - \delta/2, -g(x,x') + \delta/2]$, then by Kolmogorov-Rogozin's inequality

$$\mathbb{P}\left\{\varphi(x,Y) - \varphi(x',Y) \in J \;\middle|\; \zeta \leq \frac{|S_2|}{2}\right\} = \mathrm{O}\left(\frac{\sum_{j\in M(x,x')}(1-\kappa(j))\kappa_\Delta(j)}{\left\{\sum_{j\in M(x,x')}\left[1-\kappa_\Delta(j)\right]\right\}^{3/2}}\right),$$

where $M(x,x')$ is the set M obtained for the realisation x, x' of X, X'. So

$$\begin{aligned}
&\mathbb{P}\left\{\varphi(X,Y)-\varphi(X',Y)\in J \,\middle|\, \zeta\leq\frac{|S_2|}{2}\right\}\\
&\quad=\mathbb{E}\left(\mathbb{P}\left\{\varphi(X,Y)-\varphi(X',Y)\in J \,\middle|\, \zeta\leq\frac{|S_2|}{2},\ X,X'\right\}\right)\\
&\quad=\mathbb{E}\left(\mathrm{O}\left(\sup_{D\subset S_2,|D|\geq|S_2|/2}\frac{\sum_{j\in D}(1-\kappa(j))\kappa_\Delta(j)}{\left\{\sum_{j\in D}\left[1-\kappa_\Delta(j)\right]\right\}^{3/2}}\right)\right)\\
&\quad=\mathrm{O}\left(\sup_{D\subset S_2,|D|\geq|S_2|/2}\frac{\sum_{j\in D}(1-\kappa(j))\kappa_\Delta(j)}{\left\{\sum_{j\in D}\left[1-\kappa_\Delta(j)\right]\right\}^{3/2}}\right).
\end{aligned}$$

Hence

$$\mathbb{P}\left\{\varphi=x\right\}=\mathrm{O}\left(\left[\frac{1}{|S_2|}\sum_{j\in S_2}\left(\frac{\sum_{i\in N_j}(1-\overline{\kappa}(i))\overline{\kappa}_\Delta(i)}{\left\{\sum_{i\in N_j}\left[1-\overline{\kappa}_\Delta(i)\right]\right\}^{3/2}}\right)+\sup_{D\subset S_2,|D|\geq|S_2|/2}\frac{\sum_{j\in D}(1-\kappa(j))\kappa_\Delta(j)}{\left\{\sum_{j\in D}\left[1-\kappa_\Delta(j)\right]\right\}^{3/2}}\right]^{1/2}\right).$$

■

Remark 6. *a) If $\kappa_\Delta(i)<\kappa<1$ for all i, $|S_1|=|S_2|=n/2$ and $|N_j|\geq n^{1-\varepsilon}$ for all j and $\varepsilon>0$,*

$$\mathbb{P}\left\{\varphi=x\right\}=O\left(\left[\frac{\kappa}{\sqrt{(1-\kappa)^3n^{1-\varepsilon}}}\right]^{1/2}\right).$$

b) Lemma 5 holds when s many of the random variables $\xi_1,\ldots,\xi_n$ are degenerate for some $1\leq s<n$, in this situation n is changed by $n-s$. Contribution to the bound of the concentration inequality is only provided by the non-degenerated random variables.

3 Proofs in the Ginibre case

We start with an extension of a result by Slinko [21] who worked the case of a discrete uniform distribution with parameter $1/q$ with $q\in\mathbb{Z}^+$. Throughtout this section all our random variables satisfy

$$\sup_{x\in\mathbb{R}}\mathbb{P}\{X=x\}\leq\kappa_\Delta(X)<\kappa<1.$$

Lemma 6. *Let $k\leq m$ and let $A\in\mathbb{R}^{m\times k}$ be a (deterministic) matrix with rank$(A)=k$. If $b\in\mathbb{R}^m$*

is a random vector whose entries are independent random variables. Then

$$\mathbb{P}\left\{\mathit{rank}(A,b)=k\right\} \leq \kappa^{m-k}.$$

Proof. Since rank$(A)=k$, we can break $[A\ b]$ in the following way

$$A=\begin{pmatrix} A_k & b_k \\ A_{m-k} & b_{m-k} \end{pmatrix},$$

where $A_k \in \mathbb{R}^{k\times k}$, $A_{m-k} \in \mathbb{R}^{(m-k)\times k}$, $b_k \in \mathbb{R}^k$ and $b_{m-k} \in \mathbb{R}^{m-k}$. We note A_k is an invertible matrix. We have there exists a random matrix $\Delta \in \mathbb{R}^k$ such that $A_k\Delta = b_k$ and $A_{m-k}\Delta = b_{m-k}$, then $A_{m-k}A_k^{-1}b_k = b_{m-k}$. So

$$\begin{aligned}
\mathbb{P}\left\{r(A,b)=k\right\} &\leq \mathbb{P}\left\{A_{m-k}A_k^{-1}b_k = b_{m-k}\right\} \\
&= \mathbb{E}\left\{\mathbb{P}\left\{A_{m-k}A_k^{-1}b_k = b_{m-k}\,\middle|\,A_{m-k}A_k^{-1}b_k\right\}\right\} \\
&\leq \kappa^{m-k},
\end{aligned}$$

the last line is due to the independence of every entry in b_{m-k}. ■

Lemma 7. *Let $k \leq m$ and let $A \in \mathbb{R}^{m\times k}$ be a random matrix (whose entries are independent random variables). Then*

$$\mathbb{P}\left\{\mathit{rank}(A)<k\right\} < \frac{\kappa}{1-\kappa}\kappa^{m-k}.$$

Proof. We note that if $A=[a_1|\cdots|a_k]$, $a_i \in \mathbb{R}^m$ $i=1,\ldots,k$, then

$$\begin{aligned}
\mathbb{P}\{\text{rank}(A)=k\} &= \mathbb{P}\{a_1 \notin \{0\}, a_2 \notin \text{span}\{a_1\},\ldots,a_k \notin \text{span}\{a_1,a_2,\ldots,a_{k-1}\}\} \\
&= \mathbb{P}\{a_1 \notin \{0\}\}\prod_{i=2}^{k}\mathbb{P}\{E_i\}
\end{aligned}$$

where we use the notation span$\{\cdot\}$ for the space generated for some vectors and

$$E_i = \{a_i \notin \text{span}\{a_1,a_2,\ldots,a_{i-1}\}\,|a_1 \notin \{0\}, a_2 \notin \text{span}\{a_1\},\ldots,a_{i-1} \notin \text{span}\{a_1,a_2,\ldots,a_{i-2}\}\}.$$

Hence by Corollary 6 and the Weierstrass product inequality

$$\mathbb{P}\{\mathrm{rank}(A) = k\} \geq \prod_{i=0}^{k-1}(1-\kappa^{m-i}) \geq 1 - \sum_{i=0}^{k-1}\kappa^{m-i} = 1 - \frac{\kappa}{1-\kappa}\kappa^{m-k}.$$

■

We consider the following concept used by Komlós [3]. Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors. Let us define the strong rank of S, denoted $sr(S)$, to be n if S is a set of vectors linearly independent; and k if any k of the v_i's are linearly independent but some $k+1$ of the vectors are linearly dependent. For a matrix A we denote the strong rank of the system of columns and the strong rank of the system of rows by $sr_c(A)$ and $sr_r(A)$, respectively.

Remark 7. *(a) Let A be an $m \times n$ random matrix with all entries being independent random variables. It follows immediately from Lemma 7, than*

$$\mathbb{P}\left\{sr_c(A) < k\right\} \leq \binom{n}{k}\frac{\kappa}{1-\kappa}\kappa^{m-k}$$

(b) For every κ and $0 < \alpha \leq 1$ there exists $\beta > 0$ which satisfies

$$\frac{h(\beta)}{\log_2 \kappa} + \beta < \alpha < 1, \tag{10}$$

where $h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the entropy function. Indeed, let

$$g(x) = \frac{h(x)}{\log \kappa} + x$$

and since the function g is a continuous and $g(0) = 0$, so there exists a positive number $\beta > 0$ such that $g(\beta) < \alpha < 1$.

c) We note from (a) and (b) that if $m = \lfloor \alpha n \rfloor$ and $k = \lceil \beta n \rceil$, then

$$\mathbb{P}\left\{rank(A) < \lceil \beta n \rceil\right\} < \binom{n}{\lceil \beta n \rceil}\frac{\kappa}{1-\kappa}\kappa^{\lfloor \alpha n \rfloor - \lceil \beta n \rceil} < \frac{\kappa}{1-\kappa}2^{n(h(\beta)-(\alpha-\beta)\log_2(\kappa))} < \frac{\kappa}{1-\kappa}2^{-n\gamma_\kappa},$$

where we use $\binom{n}{\beta n} < 2^{nh(\beta)}$ and γ_κ is a postive constant which depends on κ.

Lemma 8. *Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^m$ be (deterministic) linearly independent vectors. Let $B = [v_1|\ldots|v_k]$ and $sc_r(B) = s$. Then for a random vector $a \in \mathbb{R}^m$, whose entries are independent* *random variables,*

$$\mathbb{P}\left\{rank(v_1, v_2, \ldots, v_k, a) = k\right\} < C_1 \kappa^{m-k} s^{-1/2}.$$

Proof. Let $b_1, b_2, \ldots, b_m$ are the rows of B. Without loss of generality we assume that $b_1, b_2, \ldots, b_k$ are linearly independent and that all other rows are linear combination of them. We have

$$\sum_{i=1}^{k} \beta_i^{(r)} b_i = b^{(r)}$$

for $r = k+1, \ldots, m$. As $sc_r(B) = s$, at least s of the coefficients $\beta_1^{(r)} \ldots, \beta_k^{(r)}$ are nonzero.

Now, since we consider the event $[\text{rank}(v_1, v_2, \ldots, v_k, a) = k]$, we have

$$\sum_{j=1}^{k} \alpha_j v_j = a$$

for some $\alpha_1 \ldots, \alpha_k$, all non-zero. In particular $\sum_{j=1}^{k} \alpha_j v_{k+1,j} = a_{k+1}$, where a_{k+1} is the $(k+1)$-th entry of a. But

$$a_{k+1} = \sum_{j=1}^{k} \alpha_j v_{k+1,j} = \sum_{j=1}^{k} \alpha_j \left(\sum_{i=1}^{k} \beta_i^{(k+1)} v_{i,j} \right) = \sum_{i=1}^{k} \beta_i^{(k+1)} \left(\sum_{j=1}^{k} \alpha_j v_{i,j} \right) = \sum_{i=1}^{k} \beta_i^{(k+1)} a_i.$$

From the above and the independence of entries of a

$$\begin{aligned}
\mathbb{P}\left\{\text{rank}(v_1, v_2, \ldots, v_k, a) = k\right\} &\leq \mathbb{P}\left\{ \sum_{i=1}^{k} \beta_i^{(r)} a_i = a_r, r = k+1, \ldots, m \right\} \\
&= \mathbb{E}\left\{ \mathbb{P}\left\{ \sum_{i=1}^{k} \beta_i^{(r)} a_i = a_r, r = k+1, \ldots, m \mid a_1, \ldots, a_k \right\} \right\} \\
&= \mathbb{E}\left\{ \mathbb{P}\left\{ \sum_{i=1}^{k} \beta_i^{(m)} a_i = a_m \mid a_1, \ldots, a_k \right\} \prod_{l=k+1}^{m-1} \mathbb{P}\left\{ \sum_{i=1}^{k} \beta_i^{(l)} a_i = a_l \mid a_1, \ldots, a_k \right\} \right\} \\
&\leq \mathbb{E}\left\{ \kappa^{m-k-1} \mathbb{P}\left\{ \sum_{i=1}^{k} \beta_i^{(m)} a_i = a_m \mid a_1, \ldots, a_k \right\} \right\} \\
&= \kappa^{m-k-1} \mathbb{P}\left\{ \sum_{i=1}^{k} \beta_i^{(m)} a_i = a_m \right\} \\
&\leq C_1 \kappa^{m-k} s^{-1/2},
\end{aligned}$$

the last line is due to Kolomogorov-Rogozin Inequality. ■

Proof of Theorem 3.a. Let $\alpha \in (0,1)$ and $\beta > 0$ as in the equation (10) and let $n_0 = \lfloor \alpha n \rfloor$. Let B be the $n_0 \times n$ matrix whose columns are the first n_0 columns of G_n.

Since

$$\mathbb{P}\{\text{rank}(G_n) = n\} = \mathbb{P}\{\text{rank}(G_n) = n, sr_r(B) < \beta n\} + \mathbb{P}\{\text{rank}(G_n) = n, sr_r(B) \geq \beta n\},$$

by Lemma 7 and Remark 8, we have

$$\mathbb{P}\{\text{rank}(G_n) = n\} \geq \prod_{i=1}^{n-n_0} \left(1 - C_1(\beta n)^{-1/2}\kappa^i\right) \geq 1 - \frac{C_1}{1-\kappa}(\beta n)^{-1/2},$$

which proves Theorem 3.a. ■

Proof of Proposition 1. Let F_1 a distribution function whose biggest jump is κ_1, we take $m_n = 1$ and $\delta_1 = \kappa_1/2$, then $\mathbb{P}\{G_{m_1}$ has full rank $\} > 1 - \delta_1$. Now, let F_n a distribution function whose biggest jump is κ_n, by the Lemma 2 in [16] there is $m_n \geq m_{n-1}$ and $\delta_n \leq 1/n \leq$ for $n > 1$ such that

$$\mathbb{P}\{G_{m_n} \text{ has full rank } \} > 1 - \delta_n$$

where the entries of G_{m_n} have the same distribution and $\delta_n \to 0$ as $n \to \infty$. ■

4 Proofs in the Wigner case

Following the terminology introduced in Costello, Tao and Vu [8], given n vectors $\{v_1, \ldots, v_n\}$, a linear combination of the v_i's is a vector $v = \sum_{i=1}^n c_i v_i$, where the c_i are real numbers. We say that a linear combination vanishes if v is the zero vector. A vanishing linear combination has degree k if exactly k among the c_i are nonzero.

A singular $n \times n$ matrix is called *normal* if its row vectors do not admit a non-trivial vanishing linear combination with degree less than $n^{1-\varepsilon}$ for a given $\varepsilon \in (0,1)$. Otherwise it is said that the matrix is *abnormal*. Furthermore, a row of an $n \times n$ non-singular matrix is called *good* if its exclusion leads to an $(n-1) \times n$ matrix whose column vectors admit a non-trivial vanishing linear combination with degree at least $n^{1-\varepsilon}$ (in fact, there is exactly one such combination as the rank of this $(n-1) \times n$ matrix is $n-1$). A row is said to be *bad* otherwise. Finally, an $n \times n$ non-singular matrix A is *perfect* if every row in A is good row. If a non-singular matrix is not perfect, it is called *imperfect*.

For the proof of Theorem 3.b, we first present three lemmas which generalize results in [8] for

Wigner matrices $W_n = (\xi_{ij})$ with independent entries which need not be identically distributed and the appropriate estimates in these new cases are found in terms of the size of the biggest jump of the distribution functions governing the entries under the hypothesis $\kappa_\Delta(i) < \kappa < 1$. We also obtain a better rate of convergence which is universal. The proofs we give follow ideas in [8] but also take into account the size of the biggest jump.

Lemma 9. *Let $\varepsilon \in (0,1)$, then for all n large*

$$\mathbb{P}\left\{W_n \textit{ is singular and abnormal}\right\} \leq \kappa^{(n-n^{1-\varepsilon})/2} \tag{11}$$

and

$$\mathbb{P}\left\{W_n \textit{ is non-singular and imperfect}\right\} \leq \kappa^{(n-n^{1-\varepsilon})/2}. \tag{12}$$

Proof. If W_n is singular and abnormal the rows vectors of W_n admit a non-trivial vanishing linear combination with degree at most $N := n^{1-\varepsilon}$. For $i = 1, \ldots, N$, we have that if $i = 1$, there is a row of W_n that contains only zeros, and if $i > 1$, the i-th row is a linear combination of the first $i-1$ rows of W_n that are linearly independent. We denote by $D(n,i)$ this last event and by T_{i-1} the upper triangular part of W_n until the row $i-1$ (included). The linear dependence of the i-th row of W_n with the $i-1$ rows of W_n is determined only by its last $n-i+1$ entries. Then by the stochastic independence of T_{i-1} with the last $n-i+1$ entries of the row i

$$\begin{aligned}\mathbb{P}\left\{W_n \text{ is singular and abnormal}\right\} &\leq \sum_{i=1}^{N}\binom{n}{i}\mathbb{P}\left\{D(n,i)\right\} \leq \sum_{i=1}^{N}\binom{n}{i}\mathbb{E}\left\{\mathbb{P}\left\{\left. D(n,i)\right| T_{i-1}\right\}\right\} \\ &\leq \sum_{i=1}^{N} n^N \kappa^{n-N+1} = N n^N \kappa^{n-N+1},\end{aligned}$$

and for all n large,

$$\mathbb{P}\left\{W_n \text{ is singular and abnormal}\right\} \leq \kappa^{\frac{3}{4}(n-n^{1-\varepsilon})} \leq \kappa^{\frac{1}{2}(n-n^{1-\varepsilon})}.$$

Now, we consider the case when W_n is non-singular and imperfect. We can suppose that the last row of W_n is the bad row. The $(n-1)\times n$-matrix obtained has rank $n-1$, hence there is a unique column that admit a non-trivial vanishing linear combination with degree at most $n^{1-\varepsilon}$, then the last $n-k-1$ of this column is completely determined by its k first entries and k linear independent columns, for $1 \leq k \leq n^{1-\varepsilon}$. Since we can choose this bad row, we have as above for n

large

$$\mathbb{P}\left\{W_n \text{ is non-singular and imperfect}\right\} \leq n\kappa^{\frac{3}{4}(n-1-(n-1)^{1-\varepsilon})} \leq \kappa^{\frac{1}{2}(n-n^{1-\varepsilon})}.$$

■

Lemma 10. *Let A be a deterministic $n \times n$ singular normal matrix, then*

$$\mathbb{P}\left\{\mathit{rank}(W_{n+1}) - \mathit{rank}(W_n) < 2 \,|W_n = A\right\} = O_\varepsilon\left(\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right).$$

Proof. Since $r := \text{rank}(A) < n$, without loss of generality it is possible to suppose that the first r rows of A are linearly independent. If $v_1, \ldots, v_r$ are the first rows of A, then $v_n = \sum_{i=1}^r \alpha_i v_i$, and as A is normal, the numbers of coefficients in this linear combination is at least $n^{1-\varepsilon}$. If it does not hold that $\xi_n = \sum_{i=1}^r \alpha_i \xi_i$, where ξ_i are entries of the last column of W_{n+1}, by symmetry of W_{n+1} we have $\text{rank}(W_{n+1}) = \text{rank}(A) + 2$. Hence

$$\begin{aligned}\mathbb{P}\left\{\text{rank}(W_{n+1}) - \text{rank}(W_n) < 2 \,|W_n = A\right\} &\leq \mathbb{P}\left\{\xi_n = \sum_{i=1}^r \alpha_i \xi_i\right\} \\ &= \mathrm{O}_\varepsilon\left(\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right).\end{aligned}$$

The last expression follows from Lemma 3. ■

Lemma 11. *Let A be a deterministic $n \times n$ non-singular perfect symmetric matrix, then*

$$\mathbb{P}\left\{\mathit{rank}(W_{n+1}) = n \,|W_n = A\right\} = O_\varepsilon\left(\left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right]^{1/2}\right).$$

Proof. If $\text{rank}(W_{n+1}) = n$, then $\det(W_{n+1}) = 0$, and we have

$$0 = \det(W_{n+1}) = (\det A)\xi_{n+1} + \sum_{i=1}^n \sum_{j=1}^n c_{ij}\xi_i\xi_j,$$

where ξ_i are entries of the last column of W_{n+1} and its transpose, and c_{ij} are cofactors of A. Since A is perfect, when we eliminate the i-th row of A, the columns of the matrix thus obtained admit a vanishing linear combination of degree at least $n^{1-\varepsilon}$. When the column j is selected, where j is the index of a non-zero coefficient in this linear combination, we obtain an $(n-1)\times(n-1)$ non-singular matrix. Since there are at least $n^{1-\varepsilon}$ indices i such that there are at least $n^{1-\varepsilon}$ indices j with $c_{i,j} \neq 0$.

Taking the partion of $\{1,2,\ldots,n\}$ as $S_1=\{1,2,\ldots,\lfloor n/2\rfloor\}$ and $S_2=\{1,2,\ldots,n\}-S_1$, by Remark 6

$$\begin{aligned}
\mathbb{P}\left\{\operatorname{rank}(W_{n+1})=n\,|W_n=A\right\} &\le \mathbb{P}\left\{(\det A)\xi_{n+1}+\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}\xi_i\xi_j=0\right\}\\
&=\mathbb{E}\left(\mathbb{P}\left\{(\det A)\xi_{n+1}+\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}\xi_i\xi_j=0\,|\,\xi_{n+1}\right\}\right)\\
&=\mathbb{E}\left(\mathrm{O}_\varepsilon\left(\left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right]^{1/2}\right)\right)\\
&=\mathrm{O}_\varepsilon\left(\left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right]^{1/2}\right).
\end{aligned}$$

■

Now we consider the discrete stochastic process

$$X_n=\begin{cases}0 & \text{if } \operatorname{rank}(W_n)=n\\ \left(\kappa^{-1/8}\right)^{n-\operatorname{rank}(W_n)} & \text{if } \operatorname{rank}(W_n)<n,\end{cases}$$

for which we can prove the following result.

Proposition 3.

$$\mathbb{E}\left(X_n\right)=O_\varepsilon\left(\left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right]^{1/2}\right).$$

Proof. For $j=0,\ldots,n$, write $A_j=\{\operatorname{rank}(W_n)=n-j\}$ and let $1+\gamma=\kappa^{-1/8}$. We have

$$\begin{aligned}
\mathbb{E}\left(X_n\right)&=\sum_{j=1}^{n}(1+\gamma)^j\mathbb{P}\left\{A_j\right\}\\
&=\sum_{j=1}^{n}(1+\gamma)^j\mathbb{P}\left\{A_j,\, W_n \text{ normal}\right\}+S_1,
\end{aligned}$$

where

$$S_1=\sum_{j=1}^{n}(1+\gamma)^j\mathbb{P}\left\{A_j,\, W_n \text{ abnormal}\right\}.$$

By Lemma 9,

$$\begin{aligned}
S_1 &\le \sum_{j=1}^{n} (1+\gamma)^j \kappa^{(n-n^{1-\varepsilon})/2} \\
&\le \kappa^{(n-n^{1-\varepsilon})/2} \sum_{j=1}^{n} (1+\gamma)^j \\
&\le \frac{1-(\kappa^{-1/8})^{n+1}}{1-\kappa^{-1/8}} \kappa^{(n-n^{1-\varepsilon})/2} \\
&= C\kappa^{(3n-4n^{1-\varepsilon})/8}
\end{aligned}$$

for some constant $C > 0$.

So

$$\mathbb{E}\left(X_n\right) = \sum_{j=1}^{n} (1+\gamma)^j \mathbb{P}\left\{A_j,\, W_n \text{ normal}\right\} + \mathrm{O}_\varepsilon\left(\kappa^{(3n-4n^{1-\varepsilon})/8}\right). \tag{13}$$

On the other hand,

$$\mathbb{E}\left(X_{n+1}\right) = S_2 + S_3 + S_4 + S_5,$$

where

$$\begin{aligned}
S_2 &= \mathbb{E}\left(X_{n+1}\,|A_0, W_n \text{ perfect}\right)\mathbb{P}\left\{A_0,\, W_n \text{ perfect}\right\} \\
S_3 &= \mathbb{E}\left(X_{n+1}\,|A_0, W_n \text{ imperfect}\right)\mathbb{P}\left\{A_0,\, W_n \text{ imperfect}\right\} \\
S_4 &= \sum_{j=1}^{n} \mathbb{E}\left(X_{n+1}\,|A_j, W_n \text{ normal}\right)\mathbb{P}\left\{A_j,\, W_n \text{ normal}\right\} \\
S_5 &= \sum_{j=1}^{n} \mathbb{E}\left(X_{n+1}\,|A_j, W_n \text{ abnormal}\right)\mathbb{P}\left\{A_j,\, W_n \text{ abnormal}\right\}.
\end{aligned}$$

By Lemma 11 and $\mathrm{rank}(W_n) = n$

$$\begin{aligned}
S_2 &\le (\kappa^{-1/8})^{n+1-n}\mathbb{P}\{\mathrm{rank}(W_{n+1}) = n\,|W_n \text{ is perfect and non-singular}\} \\
&= \mathrm{O}_\varepsilon\left(\left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right]^{1/2}\right).
\end{aligned}$$

On the other hand, Lemma 9 and definition of X_{n+1} give

$$S_3 \le (\kappa^{-1/8})^{n+1}\kappa^{(n-n^{1-\varepsilon})/2} = \mathrm{O}_\varepsilon\left(\kappa^{(3n-4n^{1-\varepsilon})/8}\right).$$

Using again Lemma 9 and definition of A_j

$$S_5 \leq \sum_{j=1}^{n} (\kappa^{-1/8})^{j+1} \kappa^{(n-n^{1-\varepsilon})/2} = \mathrm{O}_\varepsilon \left(\kappa^{(3n-4n^{1-\varepsilon})/8} \right).$$

If $\mathrm{rank}(W_n) = n - j$ then $\mathrm{rank}(W_{n+1})$ is equal to $n - j + 2$ or $n - j$ since W_{n+1} is a symmetric matrix. By Lemma 10 and for n sufficiently large

$$\begin{aligned}
\mathbb{E}\left(X_{n+1} \,|A_j, W_n \text{ normal}\right) &= (1+\gamma)^{j+1} \mathbb{P}\{\mathrm{rank}(W_{n+1}) = \mathrm{rank}(W_n) \,|W_n \text{ normal and singular}\} \\
&\quad + (1+\gamma)^{j-1} \\
&= (1+\gamma)^j \left((1+\gamma)^{-1} + \mathrm{O}_\varepsilon \left(\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}} \right) \right) \\
&\leq \alpha (1+\gamma)^j
\end{aligned}$$

for some $\alpha < 1$.

Then we have

$$\mathbb{E}(X_{n+1}) = \alpha \sum_{j=1}^{n} (1+\gamma)^j \mathbb{P}\{A_j, W_n \text{ normal}\} + \mathrm{O}_\varepsilon\left(f(\kappa, n)\right),$$

where

$$f(\kappa, n) := \frac{\kappa^{\frac{3}{8}n - \frac{1}{2}n^{1-\varepsilon}}}{\kappa(1-\kappa)} + \left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}} \right]^{1/2}.$$

Using (13)

$$\mathbb{E}(X_{n+1}) \leq \alpha \mathbb{E}(X_n) + \mathrm{O}_\varepsilon\left(f(\kappa, n)\right),$$

so

$$\mathbb{E}(X_{n+1}) \leq \alpha^n \mathbb{E}(X_1) + \mathrm{O}_\varepsilon\left(f(\kappa, n)\right).$$

This proves the proposition. ∎

Proof of Theorem 3.b. By Markov's inequality,

$$\begin{aligned}\mathbb{P}\{\operatorname{rank}(W_n) < n\} &= \mathbb{P}\{X_n \geq 1\} \\ &\leq \mathbb{E}(X_n) \\ &= \mathrm{O}_\varepsilon\left(\left[\frac{\kappa}{\sqrt{n^{1-\varepsilon}(1-\kappa)^3}}\right]^{1/2}\right), \end{aligned} \tag{14}$$

where we have used Proposition 3. ■

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