Random Matrices and Free Probability

Talk 3 at IAS/TUM
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Talk 3: Free Probability

Friday, October 14, 2011

I. **Asymptotically free random matrices**

II. **Free Probability and free Central Limit Theorem**

III. **Free additive convolution: Analytic approach**

IV. **Free multiplicative convolution: Analytic approach**

V. **Free infinite divisibility**

VI. **From classical to free infinite divisibility via random matrices**

- Two real random variables X_1 and X_2 are **independent** if and only if $\forall$ bounded Borel functions f, g on $\mathbb{R}$

$$\mathbb{E}(f(X_1)g(X_2)) = \mathbb{E}(f(X_1))\mathbb{E}(g(X_2))$$

$$\mathbb{E}\left([f(X_1) - \mathbb{E}(f(X_1)] \, [g(X_2) - \mathbb{E}(g(X_2)]\right) = 0$$

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Some facts about classical independence

- Two real random variables X_1 and X_2 are **independent** if and only if $\forall$ bounded Borel functions f, g on $\mathbb{R}$

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- iff $\forall$ bounded Borel functions f, g on $\mathbb{R}$

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- iff (when distributions of X_1 and X_2 have bounded support) $\forall$ $n, m \geq 1$

$$\mathbb{E}(X_1^n - \mathbb{E}X_1^n)(X_2^m - \mathbb{E}X_2^m) = 0.$$

$$\mathbb{E}X_1^n X_2^m = \mathbb{E}X_1^n \mathbb{E}X_2^m$$

I. Asymptotically free random matrices

Voiculescu (1991)

- For an ensemble of Hermitian random matrices $\mathbf{X} = (X_n)_{n\geq 1}$ define "expectation" τ as the linear functional τ, $(\tau(\mathbf{I}) = 1)$

$$\tau(\mathbf{X}) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\mathrm{tr}(X_n)\right].$$

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- Two Hermitian ensembles $\mathbf{X}_1$ and $\mathbf{X}_2$ are *asymptotically free* if for all integer $r > 0$ and all polynomials $p_i(\cdot)$ and $q_i(\cdot)$ with $1 \leq i \leq r$ and

$$\tau(p_i(\mathbf{X}_1)) = \tau(q_i(\mathbf{X}_2)) = 0,$$

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$$\tau(p_1(\mathbf{X}_1)q_1(\mathbf{X}_2)...p_r(\mathbf{X}_1)q_r(\mathbf{X}_2)) = 0.$$

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- It is not an extension of the concept of *classical independence* to non-commutative set up.
- *Asymptotic freeness is useful to compute joint moments from the moments of* $\mathbf{X}_1, \mathbf{X}_2$

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- Consecutive indices are distinct.
- The definition of $\tau(\mathbf{X})$ and corresponding concept of asymptotic freeness need existence of all moments $\mathbb{E}\left[\mathrm{tr}(X_n^k)\right]$.
- We can drop the expected value in the definition of τ and assume that the spectra of the matrices converges w.p.1. to a nonrandom limit. There is a correspondence concept of *a.s. asymptotic freeness*.

- The pairs of Hermitian random variables $\{\mathbf{X}_1, \mathbf{X}_2\}$ and $\{\mathbf{Y}_1, \mathbf{Y}_2\}$are *asymptotically free* if for all integer $r > 0$ and all polynomials $p_i(\cdot, \cdot)$, $q_i(\cdot)$ in two noncommuting indeterminates with $1 \leq i \leq r$

$$\tau\left(p_1(\mathbf{X}_1, \mathbf{X}_2) q_1(\mathbf{Y}_1, \mathbf{Y}_2) \cdots p_r(\mathbf{X}_1, \mathbf{X}_2) q_r(\mathbf{Y}_1, \mathbf{Y}_2)\right) = 0$$

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- If X_1 and X_2 are independent zero-mean real random variables with nonzero variance, then $\mathbf{X}_1 = X_1 \mathrm{I}$ and $\mathbf{X}_2 = X_2 \mathrm{I}$ are not asymptotically free.

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- If X_1 and X_2 are independent zero-mean real random variables with nonzero variance, then $\mathbf{X}_1 = X_1\mathrm{I}$ and $\mathbf{X}_2 = X_2\mathrm{I}$ are not asymptotically free.
- If two matrices are asymptotically free and they *commute*, then *one of them is deterministic.*

I. Asymptotically free random matrices: Examples

Theorem

Let $\mathbf{X}_1 = (X_1^n/\sqrt{n}), \mathbf{X}_2 = (X_n^2/\sqrt{n})$ *be independent Wigner Ensembles such that* X_n^i *have entries with zero mean, variance 1 and finite moment of all orders. Then* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are (almost surely) asymptotically free.*

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Under appropriate assumptions

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5. If $\mathbf{U}$ and $\mathbf{V}$ are *independent unitary ensembles*, then $\{\mathbf{U}, \mathbf{U}^*\}$ and $\{\mathbf{V}, \mathbf{V}^*\}$ are AF.

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6. If $\mathbf{X}$ and $\mathbf{Y}$ are independent unitarily invariant ensembles, they are AF.
7. If A, B are deterministic ensembles whose ASD have compact support and U is an unitary ensemble, then UAU^* and B are AF.

Definition

A non-commutative **probability space** $(\mathcal{A}, \tau)$ is W^*-**probability space** if $\mathcal{A}$ is a non-commutative von Neumann algebra and τ is a normal faithful trace.

A **family of unital von Neumann subalgebras** $\{\mathcal{A}_i\}_{i \in I} \subset \mathcal{A}$ in a W^*-probability space is **free** if

$$\tau(a_1 a_2 \cdots a_n) = 0$$

whenever

$$\tau(a_j) = 0$$

$a_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2, i_2 \neq i_3, \ldots i_{n-1} \neq i_n$.

Definition

A self-adjoint operator $\mathbf{X}$ *is affiliated with* $\mathcal{A}$ if $f(\mathbf{X}) \in \mathcal{A}$ $\forall$ bounded Borel f on $\mathbb{R}$. $\mathbf{X}$ is a **n***on-commutative random variable*. The *distribution of* $\mathbf{X}$ is the unique measure μ_X satisfying

$$\tau(f(\mathbf{X})) = \int_{\mathbb{R}} f(x)\mu_{\mathbf{X}}(\mathrm{d}x)$$

$\forall$ bounded Borel f on $\mathbb{R}$.
If $\{\mathcal{A}_i\}_{i\in I}$ is a family of free unital von Neumann subalgebras and $\mathbf{X}_i$ is a random variable affiliated with $\mathcal{A}_i$ for each $i \in I$, *the random variables* $\{\mathbf{X}_i\}_{i\in I}$ are said to be *freely independent.*

- **From now on all our non-commutative random variables are self-adjoint, unless it is explicitly mentioned.**

II. Free Central Limit Theorem

- Simplest case

Theorem

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ *be a sequence of independent free random variables with the same distribution with all moments. Assume that* $\tau(X_1) = 0$ *and* $\tau(X_1^2) = 1$*. Then the distribution of*

$$\mathbf{Z}_m = \frac{1}{\sqrt{m}}(\mathbf{X}_1 + \ldots + \mathbf{X}_m)$$

converges to the semicircle distribution.

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- *Idea of proof:* Show that $\tau(\mathbf{Z}_m^k)$ converges to the moments of the semicircle distribution $m_{2k+1} = 0$ and

$$m_{2k} = \frac{1}{k+1}\binom{2k}{k}$$

using combinatorics of noncrossing partitions.

- For k fixed.

$$\tau((\mathbf{X}_1 + \ldots + \mathbf{X}_m)^k) = \sum_{r(i) \in \{1,\ldots,k\}} \tau(\mathbf{X}_{r(1)} \ldots \mathbf{X}_{r(k)}).$$

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- Because free independence and same distribution

$$\tau(\mathbf{X}_{r(1)} \dots \mathbf{X}_{r(k)}) = \tau(\mathbf{X}_{p(1)} \dots \mathbf{X}_{p(k)})$$

whenever

$$r(i) = r(j) \iff p(i) = p(j) \qquad \forall i, j$$

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- Only noncrossing partitions of $\{1, ..., 2k\}$ will contribute to the limit. The number of noncrossing partitions are the Catalan numbers.

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- In the classical case all the partitions will contribute. The number of all partitions of $\{1.,,,.2k\}$ is $\frac{(2k)!}{2^n k!}$; the moments of the Gaussian distribution.

Definition

Let $\mathbf{X}_1, \mathbf{X}_2$ be free random variables such that $\mu_{\mathbf{X}_i} = \mu_i$. The distribution of $\mathbf{X}_1 + \mathbf{X}_2$ is the *free additive convolution* of μ_1 and μ_2 and it is denoted by

$$\mu_1 \boxplus \mu_2$$

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II. Additive and Multiplicative Convolution

Definition

Let $\mathbf{X}_1, \mathbf{X}_2$ be free random variables such that $\mu_{\mathbf{X}_i} = \mu_i$. The distribution of $\mathbf{X}_1 + \mathbf{X}_2$ is the *free additive convolution* of μ_1 and μ_2 and it is denoted by

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Definition

Let μ_1 have positive support. Then $\mathbf{X}_1$ is a positive self-adjoint operator and $\mu_{\mathbf{X}_1^{1/2}}$ is uniquely determined by μ_1. The distribution of the self-adjoint operator $\mathbf{X}_1^{1/2}\mathbf{X}_2\mathbf{X}_1^{1/2}$ is determined by μ_1 and μ_2. This measure is the *free multiplicative convolution* of μ_1 and μ_2 and it is denoted by

$$\mu_1 \boxtimes \mu_2$$

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- *Cauchy transform of a p.d.*μ, $G_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^-$

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- Analytic function $G = \mathbb{C}^+ \to \mathbb{C}^-$ is Cauchy transform of a p.d. μ iff

$$iyG(iy) \to 1.$$

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- *Voiculescu transform*

$$\phi_\mu(z) = \underline{G}_\mu^{-1}(z) - z, \quad z \in \Gamma^\mu_{\alpha,\beta}.$$

- $(\mu_n)_{n\geq 1}$ converges in distribution to μ if and only if there exist α, β such that $\phi_{\mu_n}(z) \rightarrow \phi_\mu(z)$ in compact sets of $\Gamma_{\alpha,\beta}$.

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- *Free cumulant transform*

$$C_\mu(z) = z\phi_\mu(\frac{1}{z}) = z\underline{G}_\mu^{-1}(\frac{1}{z}) - 1.$$

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$$C_\mu(z) = z\phi_\mu(\frac{1}{z}) = z\underline{G}_\mu^{-1}(\frac{1}{z}) - 1.$$

- The distribution μ can be recovered from the cumulant transform

$$G_\mu(\frac{1}{z}(C_\mu(z)+1)) = z.$$

- $(\mu_n)_{n\geq 1}$ converges in distribution to μ if and only if there exist α, β such that $\phi_{\mu_n}(z) \to \phi_\mu(z)$ in compact sets of $\Gamma_{\alpha,\beta}$.
- *Free cumulant transform*

$$C_\mu(z) = z\phi_\mu(\frac{1}{z}) = z\underline{G}_\mu^{-1}(\frac{1}{z}) - 1.$$

- The distribution μ can be recovered from the cumulant transform

$$G_\mu(\frac{1}{z}(C_\mu(z)+1)) = z.$$

- R-transform

$$R_\mu(z) = \underline{G}_\mu^{-1}(\frac{1}{z}) - \frac{1}{z}$$

III. Free additive convolutions:

The role of the cumulant transforms

- μ p.d. with moments $m_n(\mu)$, $n \geq 1$.

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- Classical cumulants $(c_n)_{n\geq 1}$

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- *Free cumulants* $(\kappa_n)_{n\geq 1}$

$$C_\mu(z) = \kappa_1 z + \kappa_2 z^2 + \ldots + \kappa_n z^k + \ldots$$

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The role of the cumulant transforms

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$$C_\mu(z) = \kappa_1 z + \kappa_2 z^2 + \ldots + \kappa_n z^k + \ldots$$

- Relation between free cumulants $(\kappa_n)_{n\geq 1}$ and moments $m_n(\mu)$, $n \geq 1$, is similar to relation between classical cumulants and moments, but using noncrossing partitions $NC(n)$.

- *Analytic definition of free additive convolution* $\mu_1 \boxplus \mu_2$*: For* μ_1 and μ_2 p.d. on $\mathbb{R}$, $\mu_1 \boxplus \mu_2$ is the unique p.d. with

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$$

equivalently to

$$R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z)$$

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- **If $(X_n^1)_{n\geq 1}, (X_n^2)_{n\geq 1}$ are asymptotically free random matrices with ASD μ_1 and μ_2, then $(X_n^1 + X_n^2)_{n\geq 1}$ has ASD $\mu_1 \boxplus \mu_2$.**

- *Semicircle distribution* w_{m,σ^2} *on* $(-2\sigma, 2\sigma)$ *centered at m*

$$w_{m,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}1_{[m-2\sigma,m+2\sigma]}(x).$$

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- *Cauchy transform: :*

$$G_{\mathrm{w}_{m,\sigma^2}}(z) = \frac{1}{2\sigma^2}\left(z - \sqrt{(z-m)^2 - 4\sigma^2}\right),$$

III. Free additive convolutions: Examples

Wigner or semicircle distribution

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- $\boxplus$*-convolution of Wigner distributions is a Wigner distribution*:

$$\mathrm{w}_{m_1,\sigma_1^2} \boxplus \mathrm{w}_{m_2,\sigma_2^2} = \mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2}.$$

III. Free additive convolutions: Examples

Marchenko-Pastur distribution

- $c > 0$

$$\mathrm{m}_c(\mathrm{d}x) = (1 - c)_+\delta_0 + \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\, 1_{[a,b]}(x)\mathrm{d}x.$$

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$$G_{\mathrm{m}_c} = \frac{1}{2} - \frac{\sqrt{(z-a)(z-b)}}{2z} + \frac{1-c}{2z}$$

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- $\boxplus$-*convolution of M-P distributions is a MP distribution*:

$$\mathrm{m}_{c_1} \boxplus \mathrm{m}_{c_2} = \mathrm{m}_{c_1+c_2}$$

- $\lambda > 0$,Cauchy distribution

$$c_\lambda(\mathrm{d}x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + x^2}\mathrm{d}x$$

- $\lambda > 0$,Cauchy distribution

$$\mathrm{c}_{\lambda}(\mathrm{d}x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + x^2}\mathrm{d}x$$

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$$G_{\mathrm{c}_{\lambda}}(z) = \frac{1}{z + \lambda i}$$

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- $\boxplus$-*convolution of Cauchy distributions is a Cauchy distribution*

$$c_{\lambda_1} \boxplus c_{\lambda_2} = c_{\lambda_1+\lambda_2}.$$

III. Free additive convolutions: Examples

Pathological example

What is $\mathrm{b} \boxplus \mathrm{b}$ if b is the symmetric Bernoulli distribution

$$\mathrm{b}(\mathrm{d}x) = \frac{1}{2}\left(\delta_{\{-1\}}(\mathrm{d}x) + \delta_{\{1\}}(\mathrm{d}x)\right)?.$$

Cauchy transform:

$$G_b(z) = \frac{z}{z^2 - 1}.$$

Free cumulant transform:

$$C_{\mathrm{b}}(z) = \frac{1}{2}(\sqrt{1 + 4z^2} - 1)$$

Then

$$C_{\mathrm{b} \boxplus \mathrm{b}}(z) = \sqrt{1 + 4z^2} - 1$$

Solving for

$$G_{\mathrm{b} \boxplus \mathrm{b}}\left(\frac{1}{z}(C_\mu(z) + 1)\right) = z$$

III. Free additive convolutions: Examples

Pathological example

- Solving for

$$G_{\mathrm{b}\boxplus\mathrm{b}}(\frac{1}{z}(\sqrt{1+4z^2}) = z$$

-

$$G_{\mathrm{b}\boxplus\mathrm{b}}(z) = \frac{1}{\sqrt{z^2-4}},$$

which is the Cauchy transform of the arcsine distribution

$$\mathrm{a}(\mathrm{d}x) = \frac{1}{\pi\sqrt{1-x^2}} 1_{(-1,1)}(x)\mathrm{d}x.$$

- Then

$$\mathrm{b} \boxplus \mathrm{b} = \mathrm{a}.$$

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$$M_{\mu_X}(z) = \mathbb{E}_{\mu_X}\left[X^{z-1}\right] = \int_{\mathbb{R}} x^{z-1}\mu_X(\mathrm{d}x), \quad z \in \mathbb{C}$$

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- We call μ_{XY} the **classical multiplicative convolution** of μ_X and μ_Y

IV. Free multiplicative convolution

Classical multiplicative convolution of random variables

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- We call μ_{XY} the **classical multiplicative convolution** of μ_X and μ_Y
- *An important problem in classical probability is the infinite divisibility of the "mixture" XY.*
- Analogous in free probability?

- The Ψ_μ transform of a general probability distribution μ

$$\Psi_\mu(z) = \frac{1}{z} G_\mu\left(\frac{1}{z}\right) - 1$$

IV. Free multiplicative convolution: The S-transform

For distributions with nonnegative support: Bercovici & Voiculescu (93)

- The Ψ_μ transform of a general probability distribution μ

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- μ in $\mathcal{M}^+$: There exists $\chi_\mu : \Psi_\mu(i\mathbb{C}_+) \to i\mathbb{C}_+$ inverse function of Ψ_μ.

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- Multiplicative convolution of μ_1, μ_2 in $\mathcal{M}^+(\neq \delta_0) : \mu_1 \boxtimes \mu_2$ in $\mathcal{M}^+$

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) S_{\mu_2}(z).$$

- **If** $(X_n)_{n\geq 1}, (Y_n)_{n\geq 1}$ **are asymptotically free nonnegative definite random matrices with ASD** μ_1 **and** μ_2, **then the product** $(X_n^{1/2} Y_n X_n^{1/2})_{n\geq 1}$ **has ASD** $\mu_1 \boxtimes \mu_2$.

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- Arizmendi and PA (2008): *Analytic approach*, μ_1, μ_2 with *unbounded support*, $\mu_1 \in \mathcal{M}^+$, μ_2 *symmetric.*

IV. Free multiplicative convolution: The S-transform

For symmetric distributions: Arizmendi-PA (2009).

- μ in $\mathcal{M}_s$ symmetric p.d., $Q(\mu) = \mu^2$ p-d. in $\mathcal{M}^+$ induced by $t \to t^2$,

$$G_\mu(z) = zG_{\mu^2}(z^2), z \in \mathbb{C} \backslash \mathbb{R}_+$$
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- If $\mu \neq \delta_0$, Ψ_μ, there are disjoint sets H, $\widetilde{H}$, Ψ_μ has unique inverse $\chi_\mu : \Psi_\mu(H) \to H$ and unique inverse $\widetilde{\chi}_\mu : \Psi_\mu(\widetilde{H}) \to \widetilde{H}$.

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- There are two S-transforms

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z} \text{ and } \widetilde{S}_\mu(z) = \widetilde{\chi}_\mu(z)\frac{1+z}{z}$$
$$S_\mu^2(z) = \frac{1+z}{z}S_{\mu^2}(z) \text{ and } \widetilde{S}_\mu^2(z) = \frac{1+z}{z}S_{\mu^2}(z).$$

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- If μ_1 in $\mathcal{M}^+$ and μ_2 in $\mathcal{M}_s$

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z) = S_{\mu_1}(z)\widetilde{S}_{\mu_2}(z).$$

IV. Examples of S-transforms

- w Wigner distribution on $(-2, 2)$

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- a arcsine distribution

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- Notice that $\mathrm{bs} = \mathrm{m}_c \otimes \mathrm{a}$. This shows that if W and U are independent Wishart and Univariant ensembles, respectively, then bs is the asymptotic spectral distribution of $W^{1/2}(U + U^*)W^{1/2}$.

V. Free Infinite Divisibility

- A d. f. μ is **infinitely divisible** with respect to free convolution $\boxplus$ iff $\forall n \geq 1$, $\exists$ p.m. $\mu_{1/n}$ and

$$\mu = \mu_{1/n} \boxplus \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}$$

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- Iff $\mathbf{X}$, $\mathcal{L}(\mathbf{X}) = \mu$, there are n free independent random variables $\mathbf{X}_1, ..., \mathbf{X}_n$ with
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- If and only if Lévy-Khintchine representation:

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$$\int_{\mathbb{R}} \min(1, x^2) \rho(\mathrm{d}x) < \infty.$$

V. Free Infinite Divisibility

- A d. f. μ is **infinitely divisible** with respect to free convolution $\boxplus$ iff $\forall n \geq 1$, $\exists$ p.m. $\mu_{1/n}$ and

$$\mu = \mu_{1/n} \boxplus \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}$$

- Iff $\mathbf{X}$, $\mathcal{L}(\mathbf{X}) = \mu$, there are n free independent random variables $\mathbf{X}_1, ..., \mathbf{X}_n$ with

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- Notation: $I^{\boxplus}$ (I^*) class of all free (classical) infinitely divisible distributions on $\mathbb{R}$.

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Some facts

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  - But also $\mathrm{a} = \mathrm{b} \boxplus \mathrm{b}$ with

$$\mathrm{b}(\mathrm{d}x) = \frac{1}{2}\left(\delta_{\{-1\}}(\mathrm{d}x) + \delta_{\{1\}}(\mathrm{d}x)\right).$$

    and b is not free infinitely divisible.

V. Relation between classical and free infinite divisibility

- Classical Lévy-Khintchine representation $\mu \in I^*$

$$C_\mu^*(t) = \log \mathcal{F}_\mu(t) = \eta t - \frac{1}{2}at^2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - tx1_{[-1,1]}(x)\right) \rho(\mathrm{d}x), \ t$$

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- Λ preserves convolutions (and weak convergence)

$$\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$$

- *Free Gaussian:* For classical Gaussian distribution γ_{m,σ^2},

$$\mathrm{w}_{m,\sigma^2} = \Lambda(\gamma_{m,\sigma^2})$$

is Wigner distribution on $(m - 2\sigma, m + 2\sigma)$ with free cumulant transform

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- *Free Poisson:* For classical Poisson distribution p_c, $c > 0$,

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is the M-P distribution with free cumulant transform

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- *Free Cauchy:* $\Lambda(\mathrm{c}_\lambda) = \mathrm{c}_\lambda$ for the Cauchy distribution

$$\mathrm{c}_\lambda(\mathrm{d}x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + x^2}\mathrm{d}x$$

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$$S^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical stable}\}.$$

IV. Examples of free infinitely divisible distributions

Images of classical i.d. distributions under Bercovici-Pata bijection

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$$GGC^{\boxplus} = \{\Lambda(\mu); \mu \text{ is classical } GGC\}$$

V. Multiplicative convolutions with free Poisson

PA & Sakuma (2011)

Let m_1 be Marchenko-Pastur distribution and $\tau \in \mathcal{M}_+$ or $\tau \in \mathcal{M}_s$. Then $\mu = \mathrm{m}_1 \boxtimes \tau$ is always $\boxplus$-infinitely divisible. Moreover, $\mathrm{m}_1 \boxtimes \tau$ is the free compound Poisson distribution with free cumulant transform

$$\mathcal{C}_\mu(z) = c \int_{\mathbb{R}} \left(\frac{1}{1 - zx} - 1 \right) \tau(\mathrm{d}x) \quad z \in \mathbb{C}^-, c > 0.$$

Under the Bercovici-Pata bijection Λ, it corresponds to the distribution which is randomization of X, $\mathcal{L}(X) = \tau$:

$$\Lambda^{-1}(\mathrm{m}_1 \boxtimes \tau) = \mathcal{L}(\sum_{i=1}^{N} X_i)$$

where $N, X_1, X_2, \ldots$ are independent classical r.v. $\mathcal{L}(X_i) = \tau$ and N has Poisson distribution of mean one.

V. Multiplicative convolutions with Wigner distribution

PA & Sakuma (2011)

- Let w be the Wigner distribution on $(-2,2)$ and $\overline{\tau} \in \mathcal{M}_+$. Then

$$\mu = \overline{\tau} \boxtimes \mathrm{w}$$

is $\boxplus$-infinitely divisible iff

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for a d.f. $\lambda \in I_+^* = I^* \cap \mathcal{M}_+$.

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$$\mathcal{C}_\mu(z) = \mathcal{C}_\tau(z^2), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

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Theorem

For $\mu \in I^*$ there is an ensemble of unitary invariant random matrices $(M_d)_{d \geq 1}$, and w.p.1. its ESD converges in distribution to $\Lambda(\mu) \in I^{\boxplus}$.

- **Final remarks:**

VI. Random Matrix Approach to Bercovici-Pata Bijection

Benachy-Georges (2005)

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- For each $d \geq 1$, the distribution μ^{d} of the $d \times d$ matrix M_d is infinitely divisible in the space of matrices $\mathbb{M}_d(\mathbb{C})$.

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- The Lévy measure of M_d is concentrated in matrices of rank one.

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