Toda evaluación es contracción + producto tensorial:

*) $\theta \in \mathfrak{X}^*(M)$, $X \in \mathfrak{X}(M)$

$$C_1^1(\theta \otimes X) = \theta(X)$$

*) $A \in \mathcal{T}_s^0(M)$, $X_1, \dots, X_s \in \mathfrak{X}(M)$:

$$A(X_1, \dots, X_s) = \sum A_{j_1 \dots j_s} X^{j_1} \dots X^{j_s} \quad \text{(localmente)}$$

$$= C_1^1 \circ \dots \circ C_1^1 (A \otimes X_1 \otimes \dots \otimes X_s)$$

pues:

$$(A \otimes X_1 \otimes \dots \otimes X_s)^{i_1 \dots i_s}_{j_1 \dots j_s} =$$

$$= A_{j_1 \dots j_s} X^{i_1} \dots X^{i_s}$$

*) $A \in \mathcal{T}_2^1(M)$, $\theta \in \mathfrak{X}^*(M)$, $X, Y \in \mathfrak{X}(M)$:

$$A(\theta, X, Y) = \sum A^i_{jh} \theta_i X^j Y^h$$

$$= C_1^1 \circ C_2^2 \circ C_1^1 (A \otimes \theta \otimes X \otimes Y)$$

Proposición: $\forall A \in \mathcal{T}^r_s(M)$, $\theta^j \in \mathfrak{X}^*(M)$, $Y_h \in \mathfrak{X}(M)$, $j = 1, \dots, r$, $h = 1, \dots, s$:

$$A(\theta^1, \dots, \theta^r, X_1, \dots, X_s) = \widetilde{C}(A \otimes \theta^1 \otimes \dots \otimes X_s)$$

donde $\tilde{C}$ es composición de contracciones.

Reglas de derivación sobre evaluaciones:

Sea $\mathcal{D}$ derivación en M.

*) $\theta \in \mathfrak{X}^*(M)$, $X \in \mathfrak{X}(M)$:

$$\mathcal{D}(\theta(X)) = \mathcal{D}(C_1^1(\theta \otimes X))$$
$$= C_1^1(\mathcal{D}(\theta \otimes X))$$
$$= C_1^1((\mathcal{D}\theta) \otimes X + \theta \otimes (\mathcal{D}X))$$
$$= (\mathcal{D}\theta)(X) + \theta(\mathcal{D}X)$$

$$\therefore (\mathcal{D}\theta)(X) = \mathcal{D}(\theta(X)) - \theta(\mathcal{D}X)$$

Luego $\mathcal{D}_0^0$, $\mathcal{D}_0^1$ determina de manera única a $\mathcal{D}_1^0$

*) $A \in \mathfrak{T}_2^2(M)$, $\theta^1, \theta^2 \in \mathfrak{X}^*(M)$, $X_1, X_2 \in \mathfrak{X}(M)$:

$$\mathcal{D}(A(\theta^1, \theta^1, X_1, X_2)) =$$
$$= \mathcal{D}(\tilde{C}(A \otimes \theta^1 \otimes \theta^2 \otimes X_1 \otimes X_2))$$
$$= \tilde{C}((\mathcal{D}A) \otimes \theta^1 \otimes \theta^2 \otimes X_1 \otimes X_2$$
$$+ A \otimes (\mathcal{D}\theta^1) \otimes \theta^2 \otimes X_1 \otimes X_2$$

$$+ A \otimes \theta^1 \otimes (\mathcal{D}\theta^2) \otimes X_1 \otimes X_2$$
$$+ A \otimes \theta^1 \otimes \theta^2 \otimes (\mathcal{D}X_1) \otimes X_2$$
$$+ A \otimes \theta^1 \otimes \theta^2 \otimes X_1 \otimes (\mathcal{D}X_2))$$

$$= (\mathcal{D}A)(\theta^1, \theta^2, X_1, X_2)$$
$$+ A(\mathcal{D}\theta^1, \theta^2, X_1, X_2) + A(\theta^1, \mathcal{D}\theta^2, X_1, X_2)$$
$$+ A(\theta^1, \theta^2, \mathcal{D}X_1, X_2) + A(\theta^1, \theta^2, X_1, \mathcal{D}X_2)$$

Lo cual da una fórmula para $\mathcal{D}A$ en términos de $\mathcal{D}^0_0, \mathcal{D}^0_1, \mathcal{D}^1_0$.

Proposición: Toda derivación $\mathcal{D}$ satisface Leibniz "respecto de la evaluación". En particular:

$A \in \mathcal{T}^r_s(M)$:

$$(\mathcal{D}A)(\theta^1, \dots, \theta^r, X_1, \dots, X_s) =$$

$$= \mathcal{D}(A(\theta^1, \dots, \theta^r, X_1, \dots, X_s))$$
$$- \sum_{i=1}^{r} A(\theta^1, \dots, \mathcal{D}\theta^i, \dots, \theta^r, X_1, \dots, X_s)$$
$$- \sum_{j=1}^{s} A(\theta^1, \dots, \theta^r, X_1, \dots, \mathcal{D}X_j, \dots, X_s)$$

Corolario:
Si $\mathcal{D}$, $\mathcal{D}'$ son derivaciones tensoriales $\ni$:
$$\mathcal{D}^0_0 = \mathcal{D}'^0_0 \;, \quad \mathcal{D}^1_0 = \mathcal{D}'^1_0$$
$$\Rightarrow \quad \mathcal{D}^r_s = \mathcal{D}'^r_s \quad \forall r, s \geq 0.$$

Teorema: Sea M variedad.
Si $V \in \mathfrak{X}(M)$ es dado junto con una transformación $\mathbb{R}$-lineal $\delta : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ y se satisface:
$$\delta(fX) = V(f)X + f\delta(X)$$
$\forall f \in C^\infty(M)$ y $X \in \mathfrak{X}(M)$, entonces $\exists!$ derivación tensorial $\mathcal{D}$ en M $\ni$ $\mathcal{D}^0_0 = V$, $\mathcal{D}^1_0 = \delta$.

Idea de la dem.:

Se definen:
$$\mathcal{D}^0_0 = V \;, \quad \mathcal{D}^1_0 = \delta$$
$$\mathcal{D}^0_1(\theta)(X) = V(\theta(X)) - \theta(\delta(X))$$
$$(\mathcal{D}^r_s A)(\theta^1, \ldots, X_s) =$$
$$= V(A(\theta^1, \ldots, X_s)) - A(\mathcal{D}^0_1\theta^1, \ldots, X_s)$$

$- \cdots - A(\theta^1, \dots, \partial(x_s)))$
(fórmula anterior) //

Definición: Si $V \in \mathfrak{X}(M)$, entonces la derivada de Lie L_V es la única derivación tensorial tal que:

$$L_V : C^\infty(M) \longrightarrow C^\infty(M)$$
$$L_V(f) = V(f)$$

$$L_V : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$
$$L_V(X) = [V, X]$$

Lo anterior tiene sentido pues:

$$[V, fX] = V(f)X + f[V, X]$$

(se checa aplicando a g)

Podemos calcular:

$$(L_V A) = \cdots \qquad A \in \mathcal{T}^r_s(M).$$

Observación: (ver Kobayashi-Nomizu):
Sea $V \in \mathfrak{X}(M)$ con flujo local φ_t
Entonces: $\forall A \in \mathcal{T}^r_s(M)$:

$$L_V A = 0 \iff \varphi_t \text{ preserva } A.$$

(Dem.:

$$L_v A = \frac{d}{dt}\Big|_{t=0} \varphi_{-t}(A).)$$

$\longrightarrow$) Ver propiedades de formas bilineales en el libro de O'Neill.

$\mathbb{R}^n_\nu$: $\langle u, v\rangle = \langle u, v\rangle_\nu$

$$= -\sum_{j=1}^{\nu} u_j v_j + \sum_{j=\nu+1}^{n} u_j v_j$$

$$O(\mathbb{R}^n_\nu) = O(\nu, n-\nu) \quad (= O(p,q))$$

$$= \left\{ T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \;\middle|\; \begin{array}{c} T \text{ lineal} \\ \langle Tu, Tv\rangle_\nu = \langle u, v\rangle_\nu \\ \forall u, v \end{array} \right\}$$

$$= \{ A \in M_{n\times n}(\mathbb{R}) \mid A^T I_{\nu, n-\nu} A = I_{\nu, n-\nu} \}$$

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \qquad p = \nu, \; q = n-\nu$$

signatura!