Para la demostración de la Prop. 6.1, página 131:

$$\mathfrak{a} \trianglelefteq \mathfrak{g}$$

$X \in \mathfrak{a}^\perp$, $Y \in \mathfrak{g}$: ¿$[X,Y] \in \mathfrak{a}^\perp$?

Sea $Z \in \mathfrak{a}$:

$$B([X,Y],Z) = -B(X,\underbrace{[Z,Y]}_{\text{en } \mathfrak{a}}) = 0$$

$\therefore$ $[X,Y] \in \mathfrak{a}^\perp$.

Se escoge $\mathfrak{b}$ subespacio de $\mathfrak{g}$ $\ni$:

$$\mathfrak{g} = \mathfrak{a} \cap \mathfrak{a}^\perp \oplus \mathfrak{b}$$

$Z \in \mathfrak{g}$, $T \in \mathfrak{a} \cap \mathfrak{a}^\perp$

$$\operatorname{ad}(T)\operatorname{ad}(Z) = ?$$

$$\mathrm{ad}(T)\,\mathrm{ad}(Z):\ \begin{array}{c} \\ \mathfrak{a}\cap\mathfrak{a}^\perp \\ \mathfrak{b}\end{array}\begin{array}{c} \begin{array}{cc}\mathfrak{a}\cap\mathfrak{a}^\perp & \mathfrak{b}\end{array} \\ \left(\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right)\end{array}$$

$$\mathrm{ad}(Z)(\mathfrak{a}\cap\mathfrak{a}^\perp) \subseteq \mathfrak{a}\cap\mathfrak{a}^\perp$$

$$\mathrm{ad}(T)(\mathfrak{a}\cap\mathfrak{a}^\perp) = 0$$

($\mathfrak{a}\cap\mathfrak{a}^\perp$ Abeliana)

$$\mathrm{ad}(Z)(\mathfrak{b}) \subseteq \mathfrak{g}$$

$$\mathrm{ad}(T)\,\mathrm{ad}(Z)(\mathfrak{b}) \subseteq \mathrm{ad}(T)(\mathfrak{g}) \subseteq \mathfrak{a}\cap\mathfrak{a}^\perp$$

$\therefore\ B(T,Z) = \mathrm{tr}(\mathrm{ad}(T)\,\mathrm{ad}(Z)) = 0$

$\forall\, T \in \mathfrak{a}\cap\mathfrak{a}^\perp,\ Z \in \mathfrak{g}.$

$\therefore\ \mathfrak{a}\cap\mathfrak{a}^\perp \subseteq \mathfrak{g}^\perp = 0$

$\therefore\ \mathfrak{a}\cap\mathfrak{a}^\perp = 0$

Concluimos que: (álgebra lineal)

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$$

$$B_{\mathfrak{g}} = B_{\mathfrak{g}}|_{\mathfrak{a}\times\mathfrak{a}} \oplus B_{\mathfrak{g}}|_{\mathfrak{a}^{\perp}\times\mathfrak{a}^{\perp}}$$

$$= B_{\mathfrak{a}} \oplus B_{\mathfrak{a}^{\perp}}$$

$B_{\mathfrak{g}}$ no degenerada implica, por la última ecuación que:

$B_{\mathfrak{a}}, B_{\mathfrak{a}^{\perp}}$ no degeneradas.

Corolario 6.2:

$\mathfrak{g}$ semisimple $\Rightarrow Z(\mathfrak{g}) = 0$.

Si $Z(\mathfrak{g}) \neq 0$, entonces $Z(\mathfrak{g})$ es ideal semisimple, pero:

$$B_{Z(\mathfrak{g})} \equiv 0$$

porque $Z(\mathfrak{g})$ es Abeliana.

Por tanto: $\mathfrak{g}$ semisimple

$$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

es inyectiva. Luego:

$$\mathfrak{g} \cong \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Int}(\mathfrak{g}))$$

Para la Proposición 6.4, página 132:

$\mathfrak{g}$ semisimple

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \underbrace{\mathrm{ad}(\mathfrak{g}) \subseteq \partial(\mathfrak{g})}$$

derivaciones

$$T \in \mathfrak{gl}(\mathfrak{g})$$

$$T([X,Y]) = [TX, Y] + [X, TY]$$

$\therefore$ $T = \mathrm{ad}(Z)$ cumple esta condición (Jacobi)

Si $\mathfrak{g} = \mathbb{R}^n$ Abeliana:

$$\mathrm{ad}(\mathfrak{g}) = 0, \ \partial(\mathfrak{g}) = \mathfrak{gl}(\mathbb{R}^n)$$

Proposición 6.4 dice que:

$\mathfrak{g}$ semisimple $\Rightarrow \partial(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$

Se llega a probar que

$$\mathfrak{a} = \mathrm{ad}(\mathfrak{g})^{\perp} = 0$$

en $\partial(\mathfrak{g})$

Pero

$$\mathfrak{a} \cap \mathrm{ad}(\mathfrak{g}) = 0 \Rightarrow \partial(\mathfrak{g}) = \mathfrak{a} \oplus \mathrm{ad}(\mathfrak{g})$$

$$\therefore \ \mathrm{ad}(\mathfrak{g}) = \partial(\mathfrak{g}).$$

Corolario 6.5

$\mathfrak{g}$ semisimple real

$\Longrightarrow$ $\mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}_0(\mathfrak{g}) \leftarrow$

$\mathrm{Int}(\mathfrak{g})$ cerrado topológico en $\mathrm{Aut}(\mathfrak{g})$.

Sabemos que:

$$\mathrm{Lie}(\mathrm{Int}(\mathfrak{g})) = \mathrm{ad}(\mathfrak{g})$$

$$\mathrm{Lie}(\mathrm{Aut}_0(\mathfrak{g})) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \partial(\mathfrak{g})$$

Como $\mathrm{ad}(\mathfrak{g}) = \partial(\mathfrak{g})$, la unicidad de subgrupo conexo con subálgebra especificada implica:

$$\mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}_0(\mathfrak{g})$$

En la página 133 se usa la siguiente afirmación:

$$H \leq GL(V) \quad \dim V < +\infty$$

H subgrupo compacto

$\Rightarrow \exists \langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}$
bilineal simétrica definida positiva tal que

$$\langle hu, hv \rangle = \langle u, v \rangle \quad \forall u, v \in V,\ h \in H$$

$$\therefore \quad H \leq O(V, \langle\ ,\ \rangle)$$

La probaremos usando la medida de Haar.

$h \in H \Rightarrow h \in O(V, \langle\ ,\ \rangle)$

$\Rightarrow h^T h = I$ en matrices respecto de una base ortonormal

$\therefore \quad X \in \mathfrak{h}$

$$\exp(tX)^T \exp(tX) = I$$

derivando:

$$X^T + X = 0$$

En la demostración de la Proposición 6.6:

$$\mathrm{Int}(\mathfrak{g}) \text{ preserva } Q$$

$$\mathfrak{g}' = \mathfrak{z}^\perp \text{ respecto de } Q$$

$\forall \sigma \in \mathrm{Int}(\mathfrak{g})$

$$\sigma(\mathfrak{z}) = \mathfrak{z}$$

$$\sigma(\mathfrak{g}') = \mathfrak{g}'$$

$$\therefore \quad e^{ad(X)}(\mathfrak{g}') = \mathfrak{g}'$$

$$\therefore \quad e^{t\,ad(X)}(u) \in \mathfrak{g}' \quad \forall u \in \mathfrak{g}'$$

derivando:

$$\operatorname{ad}(X)(u) \in \mathfrak{g}' \quad \forall u \in \mathfrak{g}'$$

$\forall X \in \mathfrak{g}$

$\therefore$ $\mathfrak{g}'$ es ideal de $\mathfrak{g}$ y tenemos:

$$\mathfrak{g} = \zeta \oplus \mathfrak{g}'$$

Por probar que $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.