Sea $\mathfrak{g}$ álgebra de Lie y $\mathcal{D}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]$.

*) $\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ es álgebra de Lie Abeliana.

$$[x+\mathcal{D}(\mathfrak{g}), y+\mathcal{D}(\mathfrak{g})] = [x,y] + \mathcal{D}(\mathfrak{g})$$
$$= \mathcal{D}(\mathfrak{g})$$

*) Si $\mathfrak{h} \trianglelefteq \mathfrak{g} \Rightarrow \mathfrak{g}/\mathfrak{h}$ es Abeliana
$\Longrightarrow \mathfrak{h} \supseteq \mathcal{D}(\mathfrak{g})$.

Sean $x, y \in \mathfrak{g}$ $\therefore$ $[x,y] \in \mathcal{D}(\mathfrak{g})$

$\therefore$ $[x+\mathfrak{h}, y+\mathfrak{h}] = \mathfrak{h}$

$$\| \qquad\qquad \therefore [x,y] \in \mathfrak{h}$$

$[x,y] + \mathfrak{h}$

*) Si $\varphi: \mathfrak{g} \longrightarrow \mathfrak{h}$ es epimorfismo de álgebra de Lie entonces hay una corres_

pondencia biyectiva

$$\{\mathfrak{m} \trianglelefteq \mathfrak{h}\} \longleftrightarrow \left\{ \begin{array}{l} \mathfrak{n} \trianglelefteq \mathfrak{g} \\ \supseteq \end{array} \ \mathfrak{n} \supseteq \ker(\varphi) \right\}$$

$$\mathfrak{m} \longleftrightarrow \varphi^{-1}(\mathfrak{m})$$

*) Aplicamos lo anterior a:

$$\mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathcal{D}(\mathfrak{g})$$

∴ hay una correspondencia biyectiva:

$$\left\{ \begin{array}{l} \mathfrak{h} \trianglelefteq \mathfrak{g} \\ \mathcal{D}(\mathfrak{g}) \subseteq \mathfrak{h} \end{array} \right\} \longleftrightarrow \left\{ \hat{\mathfrak{h}} \trianglelefteq \mathfrak{g}/\mathcal{D}(\mathfrak{g}) \right\} =$$

$$= \left\{ \begin{array}{l} \text{subespacios} \\ \text{de } \mathfrak{g}/\mathcal{D}(\mathfrak{g}) \end{array} \right\}$$

Si $\hat{\mathfrak{h}} \leq \mathfrak{g}/\mathcal{D}(\mathfrak{g})$ es de codim $=1$

$\Rightarrow$ $\pi^{-1}(\hat{\mathfrak{h}}) = \mathfrak{h} \supseteq \mathcal{D}(\mathfrak{g})$ y

$\mathfrak{h}$ es ideal de $\mathfrak{g}$ de codim $=1$.

Dem. de Lema 2.1 página 158:

Se afirma:

$\mathfrak{h} \subseteq \mathfrak{g}$ subálgebra, $\mathfrak{g}$ soluble $\Rightarrow \mathfrak{h}$ soluble.

$$\mathfrak{h} \subseteq \mathfrak{g} \Rightarrow \underbrace{[\mathfrak{h},\mathfrak{h}]}_{\mathcal{D}(\mathfrak{h})} \subseteq \underbrace{[\mathfrak{g},\mathfrak{g}]}_{\mathcal{D}(\mathfrak{g})} \Rightarrow \cdots \Rightarrow \mathcal{D}^r(\mathfrak{h}) \subseteq \mathcal{D}^r(\mathfrak{g})$$

También se puede probar que:

$\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ epimorfismo o $\mathfrak{g}$ soluble $\Rightarrow \mathfrak{h}$ soluble.

Basta ver que:

$\mathcal{D}(\mathfrak{h}) = \varphi(\mathcal{D}(\mathfrak{g}))$ y por inducción

$$\mathcal{D}^r(\mathfrak{h}) = \varphi(\mathcal{D}^r(\mathfrak{g})).$$

Para la dirección $\Leftarrow$ del Lema:

Condición de cadena nos da:

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_{n-1} \supseteq \mathfrak{g}_n = 0$$

con $\mathfrak{g}_{r-1} \trianglelefteq \mathfrak{g}_r$ de codim. 1.

$\therefore$ $\mathfrak{g}_{r-1}/\mathfrak{g}_r$ es Abeliana.

$\therefore$ $\mathcal{D}(\mathfrak{g}_{r-1}) \subseteq \mathfrak{g}_r \quad \forall r$

Luego por inducción:

$$\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}_0) \subseteq \mathfrak{g}_1$$

$$\therefore \mathcal{D}^2(\mathfrak{g}) \subseteq \mathcal{D}(\mathfrak{g}_1) \subseteq \mathfrak{g}_2$$

$$\mathcal{D}^3(\mathfrak{g}) \subseteq \mathcal{D}(\mathfrak{g}_2) \subseteq \mathfrak{g}_3$$

$$\vdots$$

$$\mathcal{D}^r(\mathfrak{g}) \subseteq \mathfrak{g}_r$$

$$\therefore \mathcal{D}^n(\mathfrak{g}) = \mathfrak{g}_n = 0$$

//

Para la demostración del Teorema de Lie:

*) Probar por inducción que:

$$\pi(H)e_p = \lambda(H)e_p \quad \forall H \in h$$

Dem.:

$p=0$ es la elección de e_0.

Supongamos:

$$\pi(H)e_p = \lambda(H)e_p$$

y tenemos:

$$\pi(H)e_{p+1} = \pi(H)\pi(X)e_p$$

$$= \pi([H,X])e_p + \pi(X)\pi(H)e_p$$

(hip. inductiva) $$= \cancel{\lambda([H,X])}^{\,0}e_p + \pi(X)\lambda(H)e_p$$

$$= \lambda(H)\pi(X)e_p = \lambda(H)e_{p+1}.$$

//

Álgebras de Lie nilpotentes:

$\mathfrak{g}$ nilpotente

$\iff$ $\mathrm{ad}_{\mathfrak{g}}(z) : \mathfrak{g} \longrightarrow \mathfrak{g}$ es nilpotente $\forall z \in \mathfrak{g}$.

$\iff$ $\forall z \in \mathfrak{g}$ $\exists$ $n \geq 1$ $\ni$:

$$\underbrace{[z,[z,\cdots[z,}_{n \text{ veces}}X]]\cdots] = 0 \quad \forall X \in \mathfrak{g}$$

Ejemplos:

$N(n)$ = matrices en $M_{n\times n}(K)$ estrictamente triangulares superiores.

$$= \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

es álgebra de Lie nilpotente.
pero:

$$T(n) = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

no es nilpotente.