Proposition 2.11

The following implications are valid if and only if μ is complete.

a) If f is measurable and $f = g$ μ-a.e., then g is measurable.

b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \longrightarrow f$ μ-a.e., then f is measurable.

Proof:

The result will be considered for functions $X \longrightarrow Y$ with Y measurable arbitrary.

μ complete $\Longrightarrow$ a), b):

For a) choose N null such that:

$$f = g \quad \text{in } X \setminus N$$

Let $A \subseteq Y$ be measurable. Then:

$$g^{-1}(A) = (g^{-1}(A) \cap (X \setminus N)) \cup (g^{-1}(A) \cap N)$$

$$= (g|_{X \setminus N})^{-1}(A) \cup (g^{-1}(A) \cap N)$$

$$= (f|_{X \setminus N})^{-1}(A) \cup (g^{-1}(A) \cap N)$$

Since μ is complete and N null, $g^{-1}(A) \cap N$ is measurable and so is $g^{-1}(A)$.
This proves a).

For b), choose N null such that:

$$\lim_{n \to +\infty} f_n(x) = f(x) \quad \forall x \in X \setminus N.$$

(this limit requires functions $X \longrightarrow Y$ where Y has convergence, e.g. topological spaces, and we use the Borel σ-algebra in Y).

For every $n \in \mathbb{N}$ define:

$$\hat{f}_n(x) = \begin{cases} f_n(x) & x \in X \setminus N \\ y_0 & x \in N \end{cases}$$

where $y_0 \in Y$ is any fixed point such that $\{y_0\}$ is closed (or Borel, we need here e.g. a Hausdorff space)

Then, $\hat{f}_n$ is measurable and:

$$\hat{f}(x) = \lim_{n \to +\infty} \hat{f}_n(x) \quad x \in X$$

is a well defined and measurable function (limit of measurable functions).

Since $\hat{f} = f$ on $X \setminus N$, by (a) it follows that f is measurable. This proves (b).

(a) $\Longrightarrow$ μ complete:

Let N be a null measurable set and $A \subseteq N$. We need to show that A is measurable. But:

$$\chi_{X \setminus A} = 1 \quad \text{on } X \setminus N$$

$\therefore \quad \chi_{X \setminus A} = 1 \quad \mu$-a.e.

and so $\chi_{X \setminus A}$ is measurable by (a). This implies A is measurable.

(b) $\Rightarrow$ μ complete:

Again let $A \subseteq N$ with N null. We prove that A is measurable.

Let $f_n = \chi_{X \setminus N}$ $\forall n$, $f = \chi_{X \setminus A}$.

Then

$$\lim_{n \to +\infty} f_n = \chi_{X \setminus N} = \chi_{X \setminus A} = f$$

on $X \setminus N$

$$\therefore \quad f = \lim_{n \to +\infty} f_n \quad \mu\text{-a.e.}$$

$\therefore$ f is measurable by (b)

This implies that A is measurable. //

Proposition 2.12:

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, then there is an $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-almost everywhere.

Proof:

If $f = \chi_E$ with $E \in \overline{\mathcal{M}}$, then we can write:

$$E = A \cup F$$

with $F \in \mathcal{M}$ and $F \subseteq N \in \mathcal{M}$ such that $\mu(N) = 0$. Hence, $\overline{\mu}(F) = 0$ and so:

$$\chi_E = \chi_A \quad \overline{\mu}\text{-a.e.}$$

and the conclusion holds in this case.

Let f be measurable (real, complex or $[0, +\infty]$-valued)

By Theorem 2.10 there is a sequence $(\varphi_n)_n$ of $\overline{\mathcal{M}}$-measurable functions such that:

$$\lim_{n\to+\infty} \varphi_n(x) = f(x)$$

$\forall x \in X$.

By the first part, $\exists (\psi_n)_n$ $\mathcal{M}$-measurable $\ni \forall n$ $\exists F_n$ $\overline{\mu}$-null such that:

$$\varphi_n = \psi_n \text{ on } X \setminus F_n$$

Let $F = \bigcup_{n=1}^{+\infty} F_n$, which is $\overline{\mu}$-null.

Hence, $\exists N$ $\mathcal{M}$-measurable μ-null $\ni F \subseteq N$.

It follows that:

$$g(x) = \lim_{n\to+\infty} (\chi_{X\setminus N} \psi_n)(x)$$

exists $\forall x \in X$ and defines a $\mathcal{M}$-measurable function.

Furthermore,

$$f = g \text{ on } X \setminus N$$

and so $f = g \bar{\mu}$ - a.e. //