From Folland's book:

Theorem 2.28:

Let $f:[a,b] \longrightarrow \mathbb{R}$ be bounded.

a) If f is Riemann integrable, then f is Lebesgue integrable and

$$\underset{\text{(Riemann)}}{\int_a^b f(x)\,dx} = \underset{\text{(Lebesgue)}}{\int_{[a,b]} f\,dm}$$

b) f is Riemann integrable iff:

$$m\left(\left\{x \in [a,b] \;\middle|\; \begin{matrix} f \text{ discontinuous} \\ \text{at } x \end{matrix}\right\}\right) = 0.$$

Proof:

a):

For $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ a partition of $[a,b]$ we define:

$$|P| = \max_{j=1,\dots,n} \{ |t_j - t_{j-1}| \}$$

and:

$$S_P f = \sum_{j=1}^{n} M_j (t_j - t_{j-1})$$

$$s_P f = \sum_{j=1}^{n} m_j (t_j - t_{j-1})$$

where

$$M_j = \sup f([t_{j-1}, t_j])$$
$$m_j = \inf f([t_{j-1}, t_j])$$

$j = 1, \dots, n$.

We also define:

$$\overline{I}_a^b f = \inf \left\{ S_P f \;\middle|\; P \text{ partition of } [a,b] \right\}$$

$$\underline{I}_a^b f = \sup \left\{ s_P f \;\middle|\; P \text{ partition of } [a,b] \right\}$$

Since f is Riemann integrable we have (by definition):

$$\int_a^b f(x)\,dx = \bar{I}_a^b f = \underline{I}_a^b f.$$

For any partition P and with the previous notation let:

$$G_P = \sum_{j=1}^{n} M_j \chi_{(t_{j-1}, t_j]}$$

$$g_P = \sum_{j=1}^{n} m_j \chi_{(t_{j-1}, t_j]}$$

which are both simple and satisfy:

$$S_P f = \int_{[a,b]} G_P \, dm, \quad s_P f = \int_{[a,b]} g_P \, dm.$$

Choose a sequence $(P_k)_{k=1}^{+\infty}$ of partitions satisfying the following properties:

1) $P_{k+1} \subseteq P_k \quad \forall k$

2) $|P_k| \xrightarrow[k \to +\infty]{} 0$

3) $$\begin{cases} \displaystyle\int_{[a,b]} G_{P_k}\, dm = S_{P_k} f \xrightarrow[k \to +\infty]{} \int_a^b f(x)\, dx \\ \displaystyle\int_{[a,b]} g_{P_k}\, dm = s_{P_k} f \xrightarrow[k \to +\infty]{} \int_a^b f(x)\, dx \end{cases}$$

1) implies that $(G_{P_k})_k$ is decreasing and $(g_{P_k})_k$ is increasing.

Hence, the limits:

$$G = \lim_{k \to +\infty} G_{P_k}$$

$$g = \lim_{k \to +\infty} g_{P_k}$$

exist pointwise and are measurable.

By definition is clear that:

$$\inf f([a,b]) \leq g \leq f \leq G \leq \sup f([a,b])$$

This implies that all the functions coasider belong to $L^1([a,b], m)$ and are bounded on $[a,b]$ which has finite measure.

Hence we can apply the Dominated Convergence Theorem to conclude that

$$\int_{[a,b]} G \, dm = \lim_{k\to+\infty} \int_{[a,b]} G_{p_k} \, dm = \int_a^b f(x) \, dx$$

$$\int_{[a,b]} g \, dm = \lim_{k\to+\infty} \int_{[a,b]} g_{p_k} \, dm = \int_a^b f(x) \, dx$$

In particular

$$G - g \geq 0 \quad \text{and} \quad \int_{[a,b]} (G-g) \, dm = 0$$

and this implies:

$$G = g \quad m\text{-a.e.}$$

$$\Rightarrow \quad f = G \quad m\text{-a.e.}$$

This shows that f is Lebesgue measurable and since it is bounded with $m([a,b]) = 0$ we conclude that:

$$f \in L^1([a,b], m)$$

Furthermore, we have:

$$\int_{[a,b]} f \, dm = \int_{[a,b]} G \, dm = \int_a^b f(x) \, dx$$

<u>b)</u>: This is proved in Exercise 23. (See another set of notes). //