Si $\alpha : T_zM \longrightarrow \mathbb{C}$ es $\mathbb{R}$-lineal, entonces:

$$\alpha^{\mathbb{C}}(v_1 + i v_2) = \alpha(v_1) + i\alpha(v_2)$$

$\forall\, v_1, v_2 \in T_zM$.

Antes probamos:

$$dx_j\left(\frac{\partial}{\partial x_k}\right) = \delta_{jk}$$

y se probó llevando el cálculo a $\mathbb{R}^n$.

Algunas propiedades extra de $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$:

$$\frac{\partial z_j}{\partial z_k} = \delta_{jk}:$$

$$\frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)(x_j + i y_j) =$$

$$= \frac{1}{2}\left(\frac{\partial x_j}{\partial x_k} + \frac{\partial y_j}{\partial y_k}\right) + i\,0 = \delta_{jk}$$

Similarmente:

$$\frac{\partial \bar{z}_j}{\partial \bar{z}_h} = \delta_{jh} \ , \ \frac{\partial z_j}{\partial \bar{z}_h} = 0 \quad (\text{C.-R.})$$

$$\frac{\partial \bar{z}_j}{\partial z_h} = 0 \quad (\text{C.-R.} (?))$$

Por otro lado:

$$f : U \subseteq M \longrightarrow \mathbb{C}$$

$$(U, \varphi = (z_1, \ldots, z_n))$$

carta holomorfa

$$\overline{\frac{\partial f}{\partial z_h}} = \frac{\partial \bar{f}}{\partial \bar{z}_h}$$

Pues:

$$\overline{\frac{\partial f}{\partial z_h}} = \overline{\frac{1}{2}\left(\frac{\partial}{\partial x_h} - i \frac{\partial}{\partial y_h}\right)(u + iv)}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x_h} + i \frac{\partial}{\partial y_h}\right)(u - iv) = \frac{\partial \bar{f}}{\partial \bar{z}_h}$$

En particular:

$$\frac{\partial f}{\partial \bar{z}_h} = \overline{\frac{\partial \bar{f}}{\partial z_h}}$$

Luego:

f holomorfa

$$\overset{\text{def.}}{\iff} \quad \overline{f} \text{ anti-holomorfa}$$

$$\iff \frac{\partial f}{\partial \overline{z}_k} = 0 \quad \forall k$$

$$\iff \frac{\partial \overline{f}}{\partial z_k} = 0 \quad \forall k$$

Por eso a veces se escribe:

$$a(z, \overline{z})$$

(un tanto informalmente).

Hay un caso especial:

$$z^\alpha \overline{z}^\beta = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n} \overline{z}_1^{\beta_1} \cdot \ldots \cdot \overline{z}_n^{\beta_n}$$

$\alpha, \beta \in \mathbb{N}^n$ (polinomios) ó $\mathbb{Z}^n$.

Para (M, g) variedad Hermitiana, la matriz de g_z complexificada se ve como sigue:

$$
\begin{array}{c|cc}
 & \overline{T_z}^{1,0}M & \overline{T_z}^{0,1}M \\
\hline
\overline{T_z}^{1,0}M & 0 & A \\
\overline{T_z}^{0,1}M & A^T & 0
\end{array}
= g_z^{\mathbb{C}}
$$

con A no-singular.

$$
\begin{tikzcd}
 & \overline{T_z}^{1,0}M \\
\overline{T_z}M \arrow[ur] \arrow[dr] & \\
 & \overline{T_z}^{0,1}M
\end{tikzcd}
$$