Moment maps on the unit ball and commuting Toeplitz operators

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For $\lambda > -1$, on the unit ball $\mathbb{B}^n$ and the Siegel domain D_n we have the weighted measures

$$dv_\lambda(z) = c_\lambda(1-|z|^2)^\lambda \, dv(z), \quad dv_\lambda(z) = \frac{c_\lambda}{4}(\mathrm{Im}(z_n)-|z'|^2)^\lambda \, dv(z).$$

The corresponding weighted Bergman spaces and their kernels are

$$\mathcal{A}^2_\lambda(\mathbb{B}^n) = L^2(\mathbb{B}^n, v_\lambda) \cap \mathrm{Hol}(\mathbb{B}^n) \quad \mathcal{A}^2_\lambda(D_n) = L^2(D_n, v_\lambda) \cap \mathrm{Hol}(D_n)$$

$$K_{\mathbb{B}^n,\lambda}(z,w) = \frac{1}{(1 - z \cdot \overline{w})^{\lambda+n+1}} \quad K_{D_n,\lambda}(z,w) = \frac{1}{\left(\frac{z_n - \overline{w}_n}{2i} - z' \cdot \overline{w}'\right)^{\lambda+n+1}}$$

We will use D to denote either $\mathbb{B}^n$ or D_n.

The Toeplitz operator $T_a^{(\lambda)}$ with symbol a is defined by

$$\begin{aligned} T_a^{(\lambda)} : \mathcal{A}^2_\lambda(D) &\to \mathcal{A}^2_\lambda(D) \\ T_a^{(\lambda)}(f)(z) &= \int_D f(w) K_{D,\lambda}(z,w)\, \mathrm{d}v_\lambda(z). \end{aligned}$$

A very important and interesting problem: find and study commutative C^*-algebras generated by Toeplitz operators.
Main strategy: find "nice" spaces of special symbols.

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Some of the first nicest collections of special symbols are given by the maximal Abelian subgroups (MASG) of biholomorphisms on D.

Quasi-elliptic, E(n): $\mathbb{T}^n$-action on $\mathbb{B}^n$

$$t \cdot z = (t_1 z_1, \ldots, t_n z_n).$$

Quasi-parabolic, P(n): $\mathbb{T}^{n-1} \times \mathbb{R}$-action on D_n

$$(t', h) \cdot z = (t' z', z_n + h).$$

Quasi-hyperbolic, H(n): $\mathbb{T}^{n-1} \times \mathbb{R}_+$-action on D_n

$$(t', r) \cdot z = (r^{\frac{1}{2}} t' z', r z_n).$$

Nilpotent, N(n): $\mathbb{R}^n$-action on D_n

$$(b, h) \cdot z = (z' + b, z_n + h + 2iz' \cdot b + i|b|^2).$$

Quasi-nilpotent, N(n,k): $\mathbb{T}^k \times \mathbb{R}^{n-k}$-action on D_n

$$(t, b, h) \cdot z = (tz_{(1)}, z_{(2)} + b, z_n + h + 2iz_{(2)} \cdot b + i|b|^2).$$

We will denote by $L^\infty(D)^G$ the space of essentially bounded G-invariant symbols on D. The group G is some subgroup of biholomorphisms of D.

Theorem (MASG Commutativity Theorem)

Let G be a MASG of the group of biholomorphisms of D. Then, the C^*-algebra $\mathcal{T}^{(\lambda)}(L^\infty(D)^G)$ is commutative for every $\lambda > -1$. Furthermore, there is a unitary map $R : \mathcal{A}^2_\lambda(D) \to L^2(X)$ such that for every $a \in L^\infty(D)^G$

$$RT_a^{(\lambda)}R^* = \gamma_{a,\lambda} I$$

a multiplication operator where

$$\gamma_{a,\lambda}(x) = \textit{nice integral formula for } a.$$

We have learned a few things from this theorem.

- Groups and Lie theory are important.
- The assignment $G \mapsto \mathcal{T}^{(\lambda)}(L^\infty(D)^G)$ yields commutative C^*-algebras for $G \in$ family of MASG.
- If H is a connected Abelian subgroup of biholomorphisms but not a MASG, then $\mathcal{T}^{(\lambda)}(L^\infty(D)^H)$ is not commutative.

Question: Is it possible to assign commutative C^*-algebras to connected Abelian subgroups of biholomorphisms? The subgroups are not necessarily MASG.

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A Hermitian metric on a complex manifold M is a Riemannian metric g such that

$$g(J(\cdot), J(\cdot)) = g(\cdot, \cdot).$$

A Kähler manifold (M, g) is a complex manifold M with a Hermitian metric g such that the 2-form $\omega = g(J(\cdot), \cdot)$ is closed. The form ω is called the symplectic form of M.
The Hermitian metric g can be complexified to a sesquilinear tensor g and in this case the symplectic form is given by

$$\omega = -2\mathrm{Im}(g).$$

Our main example (the only one we will need) is given by D.
The Kähler structure of $\mathbb{B}^n$ is given by

$$g_{\mathbb{B}^n} = \sum_{j,k=1}^{n} \frac{(1-|z|^2)\delta_{jk} + \overline{z}_j z_k}{(1-|z|^2)^2}\, \mathrm{d}z_j \otimes \mathrm{d}\overline{z}_k$$

$$\omega_{\mathbb{B}^n} = i \sum_{j,k=1}^{n} \frac{(1-|z|^2)\delta_{jk} + \overline{z}_j z_k}{(1-|z|^2)^2}\, \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

The Kähler structure on D_n is given by

$$g_{D_n} = \frac{1}{(\mathrm{Im}(z_n) - |z'|^2)^2}\Bigg((\mathrm{Im}(z_n) - |z'|^2)\sum_{j=1}^{n-1} \mathrm{d}z_j \otimes \mathrm{d}\overline{z}_j + \frac{1}{4}\,\mathrm{d}z_n \otimes \mathrm{d}\overline{z}_n$$

$$+ \sum_{j,k=1}^{n-1} \overline{z}_j z_k\,\mathrm{d}z_j \otimes \mathrm{d}\overline{z}_k + \frac{1}{2i}\sum_{j=1}^{n-1}(\overline{z}_j\,\mathrm{d}z_j \otimes \mathrm{d}\overline{z}_n - z_j\,\mathrm{d}z_n \otimes \mathrm{d}\overline{z}_j)\Bigg),$$

$$\omega_{D_n} = \frac{i}{(\mathrm{Im}(z_n) - |z'|^2)^2}\Bigg((\mathrm{Im}(z_n) - |z'|^2)\sum_{j=1}^{n-1} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j + \frac{1}{4}\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n$$

$$+ \sum_{j,k=1}^{n-1} \overline{z}_j z_k\,\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k + \frac{1}{2i}\sum_{j=1}^{n-1}(\overline{z}_j\,\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_n - z_j\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_j)\Bigg).$$

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A symplectic manifold (M, ω) is a manifold M together with a non-degenerate closed 2-form ω.
Example: Every Kähler manifold is symplectic.
On a symplectic manifold M, if $f : M \to \mathbb{R}$ is a smooth function, then the Hamiltonian vector field of f is the smooth vector field X_f such that

$$\mathrm{d}f = \omega(X_f, \cdot).$$

Compare with the Riemannian case: On a Riemannian manifold (M, g), if $f : M \to \mathbb{R}$ is a smooth function, then the gradient of f is the smooth vector field ∇f such that

$$\mathrm{d}f = g(\nabla f, \cdot).$$

Conversely, a smooth vector field X on a symplectic manifold M is called Hamiltonian if there is a smooth function $f : M \to \mathbb{R}$ such that

$$\mathrm{d}f = \omega(X, \cdot),$$

i.e. the 1-form $\omega(X, \cdot)$ is exact.
The form $\omega(X, \cdot)$ is not always closed, but for a vector field X the following are equivalent.

- $\omega(X, \cdot)$ is closed.
- $L_X \omega = 0$.
- The local flow of X preserves ω, i.e. acts by symplectomorphisms.

We will denote by $\mathcal{X}(M, \omega)$ the Lie algebra of all vector fields on M satisfying these conditions. The elements of $\mathcal{X}(M, \omega)$ are called symplectic vector fields.

For f, g smooth functions on the symplectic manifold M we define their Poisson brackets by

$$\{f, g\} = \omega(X_f, X_g).$$

Then, $(C^\infty(M), \{\cdot, \cdot\})$ is Lie algebra and the map

$$\begin{aligned} C^\infty(M) &\to \mathcal{X}(M, \omega) \\ f &\mapsto X_f \end{aligned}$$

is an anti-homomorphism of Lie algebras: $[X_f, X_g] = -X_{\{f,g\}}$.

Let H be a connected Lie group acting by symplectomorphisms on (M, ω):

$$\omega(\mathrm{d}h(\cdot), \mathrm{d}h(\cdot)) = \omega(\cdot, \cdot),$$

for all $h \in H$.
For every $X \in \mathfrak{h}$ we consider the vector field $X^\sharp$ on M given by

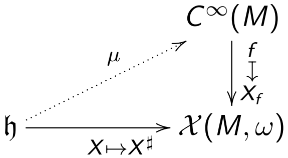
$$X^\sharp_z = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \exp(sX)z.$$

In particular, $X^\sharp \in \mathcal{X}(M, \omega)$.
It is easy to see that the map

$$\begin{aligned} \mathfrak{h} &\to \mathcal{X}(M, \omega) \\ X &\mapsto X^\sharp \end{aligned}$$

is an anti-homomorphism of Lie algebras.

The previous discussion leads us to consider the diagram



where we want consider the existence of $\mu : \mathfrak{h} \dashrightarrow C^\infty(M)$ so that this diagram commutes. In other words

$$X_{\mu(X)} = X^\sharp$$

for all $X \in \mathfrak{h}$. Such a map is equivalent to maps

- $M \times \mathfrak{h} \to \mathbb{R}$,
- $M \to \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the vector space dual of $\mathfrak{h}$.

It is customary and convenient to use the last realization.

Definition

If H is a Lie group acting by symplectomorphisms on (M, ω), then a moment map for the H-action is a smooth map $\mu : M \to \mathfrak{h}^*$ such that

1. For every $X \in \mathfrak{h}$, the smooth function $\mu_X : M \to \mathbb{R}$ defined by
$$\mu_X(z) = \langle \mu(z), X \rangle$$
has Hamiltonian vector field given $X^\sharp$: $X^\sharp = X_{\mu_X}$.
2. For every $h \in H$ we have $\mu \circ h = \mathrm{Ad}^*(h) \circ \mu$.

If H is Abelian, the second condition is just H-invariance:
$$\mu \circ h = \mu$$
for all $h \in H$.

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We are interested in the connected Abelian groups of biholomorphisms of D.
If G is such a MASG, then $\mathfrak{g} = \mathbb{R}^n$ and so we have a natural identification $\mathfrak{g}^* = \mathbb{R}^n$.
If H is a connected Abelian group of biholomorphisms, up to conjugacy, we can assume $H \subset G$, where G is some MASG. Hence, $\mathfrak{h} \subset \mathbb{R}^n$ and so $\mathfrak{h}^* = \mathfrak{h}$.

The moment maps for the MASG are the following.

E(n): $$\mu(z) = -\frac{1}{1-|z|^2}(|z_1|^2, \dots, |z_n|^2),$$

P(n): $$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(2|z_1|^2, \dots, 2|z_{n-1}|^2, 1),$$

H(n): $$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(2|z_1|^2, \dots, 2|z_{n-1}|^2, \mathrm{Re}(z_n)),$$

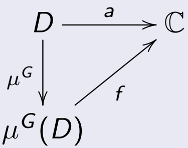
N(n): $$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(-4\mathrm{Im}(z'), 1),$$

N(n,k): $$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(2|z_1|^2, \dots, 2|z_k|^2, -4\mathrm{Im}(z_{(2)}), 1).$$

Corollary (Q-Sanchez-Nungaray)

Let G be a MASG and $\mu^G : D \to \mathbb{R}^n$ its moment map, then the following conditions are equivalent for a symbol $a \in L^\infty(D)$.

1. *The function a is G-invariant.*
2. *There exists a function $f : \mu^G(D) \to \mathbb{C}$ such that the diagram*



commutes.

Definition

Let H be a connected Abelian group of biholomorphisms of D and let $\mu^H : D \to \mathfrak{h}$ be a moment map function for the H-action. A symbol $a \in L^\infty(D)$ is called a moment map function or a μ^H-function if there is a function f such that $a = f \circ \mu^H$. The space of such symbols is denoted by $L^\infty(D)^{\mu^H}$.

Corollary (Q-Sanchez-Nungaray)

For a MASG G of biholomorphisms of D

$$L^\infty(D)^G = L^\infty(D)^{\mu^G}.$$

Theorem (Q-Sanchez-Nungaray)

If H is a connected Abelian group of biholomorphisms of D, then for every $\lambda > -1$ the C^*-algebra $\mathcal{T}^{(\lambda)}(L^\infty(D)^{\mu^H})$ is commutative.

Idea of the proof.

If H is contained in the MASG G, then μ^H is G-invariant, and so we have

$$\mathcal{T}^{(\lambda)}(L^\infty(D)^{\mu^H}) \subset \mathcal{T}^{(\lambda)}(L^\infty(D)^G).$$

□

Proposition (Q-Sanchez-Nungaray)

The assignment

$$H \mapsto \mathcal{T}^{(\lambda)}(L^\infty(D)^{\mu^H})$$

maps connected Abelian groups of biholomorphisms of D into commutative C^*-algebras. This assignment preserves inclusions.

Compare the previous result with the fact that $H_1 \subset H_2$ implies

$$\mathcal{T}^{(\lambda)}(L^\infty(D)^{H_2}) \subset \mathcal{T}^{(\lambda)}(L^\infty(D)^{H_1}).$$

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We can describe specific types of symbols by describing explicitly the moment maps of connected Abelian groups.
Some general facts:

- Every connected Abelian group H of biholomorphisms of D is contained in a MASG G.
- There is a one-to-one correspondence between the connected subgroups of a MASG G and subspaces $\mathfrak{h}$ of $\mathbb{R}^n$ given by

$$\mathfrak{h} \mapsto \exp(\mathfrak{h})$$

where exp is the exponential map of G.
- To introduce coordinates, consider linearly independent sets $\beta \subset \mathbb{R}^n$. Hence, there is a correspondence (onto only)

$$\beta \mapsto \exp(\mathbb{R}\langle\beta\rangle)$$

where $\mathbb{R}\langle\beta\rangle$ denotes the subspace generated by β.

Proposition

Let G be a MASG of biholomorphisms of D and H a connected Abelian subgroup of G. Let $\beta = \{v_1, \dots, v_m\}$ be an orthogonal basis of the Lie algebra $\mathfrak{h}$ of H. Then, the moment map μ^H for the H-action on D is given by

$$\mu^H(z) = \sum_{j=1}^{m} \frac{\langle \mu^G(z), v_j \rangle}{\langle v_j, v_j \rangle} v_j.$$

Corollary

If $\beta = \{v_1, \dots, v_m\}$ is an arbitrary basis of $\mathfrak{h}$, then the moment map functions are precisely those of the form $a(z) = f(a_1(z), \dots, a_m(z))$, where

$$a_j(z) = \langle \mu^G(z), v_j \rangle$$

for $j = 1, \dots, m$.

Definition

The symbols from the Corollary are called β-symbols. The essentially bounded β-symbols are denoted by $L^\infty(D)_\beta$.

Corresponding to the choice of G we have β-quasi-elliptic, β-quasi-parabolic, β-quasi-hyperbolic, β-nilpotent and β-quasi-nilpotent symbols.

Every currently known family of symbols whose Toeplitz operators generate commutative C^*-algebras on every weighted Bergman space of D is a set of β-symbols for some β.
For example, quasi-radial symbols corresponding to a partition $k \in \mathbb{N}^m$ of n are precisely the β-quasi-elliptic symbols for β that consists of the rows of the matrix

$$A(\beta) = \begin{pmatrix} 1_{k_1} & 0 & \cdots & 0 \\ 0 & 1_{k_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{k_m} \end{pmatrix},$$

where $1_{k_j} = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ with entries 1 exactly at indices corresponding to k_j.
Similar constructions recover many other special symbols.

For each of the five types of MASG there is a linearly independent set $\beta \subset \mathbb{R}^n$ such that the β-symbols cannot be realized by the currently known special symbols in the literature.

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The use of moment map and their induced coordinates can be used to simplify the known spectral integral formulas for the MASGs.

Theorem (Q-Sanchez-Nungaray)

For every $\lambda > -1$ there exists a unitary map $R : \mathcal{A}^2_\lambda(\mathbb{B}^n) \to \ell^2(\mathbb{N}^n)$ such that for every essentially bounded quasi-elliptic symbol $a : \mathbb{B}^n \to \mathbb{C}$ of the form $a(z) = f(-\mu^{\mathbb{T}^n}(z))$ we have $RT^{(\lambda)}_a R^* = \gamma_{a,\lambda} I$, a multiplication operator, where $\gamma_{a,\lambda}$ is given by

$$\gamma_{a,\lambda}(p) = \frac{\Gamma(\lambda + |p| + n + 1)}{p!\Gamma(\lambda + 1)} \int_{\mathbb{R}^n_+} \frac{f(u)u^p}{(1 + |u|)^{\lambda+|p|+n+1}} \, du,$$

for every $p \in \mathbb{N}^n$.

Corresponding formulas are easy to obtain for the β-symbols.

Corollary (Q-Sanchez-Nungaray)

Let $\beta = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ be a linearly independent set. Then, for every $\lambda > -1$ there exists a unitary map $R : \mathcal{A}^2_\lambda(\mathbb{B}^n) \to \ell^2(\mathbb{N}^n)$ such that for every essentially bounded β-quasi-elliptic symbol $a : \mathbb{B}^n \to \mathbb{C}$ of the form $a(z) = f(-A(\beta)\mu^{\mathbb{T}^n}(z)^\top)$ we have $RT_a^{(\lambda)}R^* = \gamma_{a,\lambda} I$, a multiplication operator, where $\gamma_{a,\lambda}$ is given by

$$\gamma_{a,\lambda}(p) = \frac{\Gamma(\lambda + |p| + n + 1)}{p!\Gamma(\lambda + 1)} \int_{\mathbb{R}^n_+} \frac{f(A(\beta)u^\top)u^p}{(1 + |u|)^{\lambda+|p|+n+1}} \, \mathrm{d}u,$$

for every $p \in \mathbb{N}^n$. Here $A(\beta)$ denotes the matrix whose rows are the elements of β.

Theorem (Q-Sanchez-Nungaray)

For every weight $\lambda > -1$ there exists a unitary transformation $R : \mathcal{A}^2_\lambda(D_n) \to L^2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$ such that for every essentially bounded nilpotent symbol $a : D_n \to \mathbb{C}$ of the form $a(z) = f(-\mu^{\mathbb{R}^n}(z))$ we have $RT_a^{(\lambda)}R^* = \gamma_{a,\lambda} I$, a multiplication operator, where $\gamma_{a,\lambda}$ is given by

$$\gamma_{a,\lambda}(y',\xi) = \frac{\xi^{\lambda+\frac{n+1}{2}}}{2^{n-1}\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)} \int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} \frac{f(u)e^{-\frac{\xi}{u_n}-\left|-\frac{\sqrt{\xi}u'}{2u_n}+y'\right|^2} \, du' \, du_n}{u_n^{\lambda+n+1}},$$

for every $y' \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}_+$.

For general β-nilpotent symbols it is enough to replace $f(u)$ with $f(A(\beta)u^\top)$.

References

Quiroga-Barranco, Raul and Sanchez-Nungaray, Armando: Moment maps of Abelian groups and commuting Toeplitz operators acting on the unit ball. arxiv preprint: http://arxiv.org/abs/2009.12448